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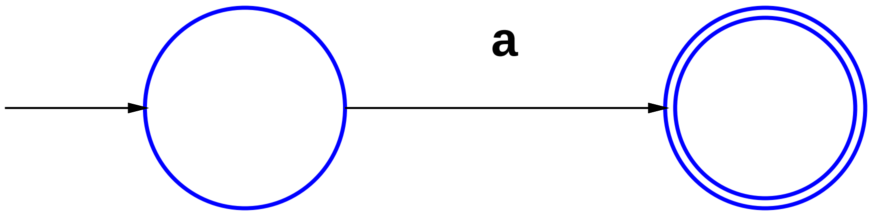
$\Longleftarrow$ Given a regular language, L, construct an equivalent regular expression.

NFA Accepting a Regular Expression, R $(\Longrightarrow)$

1. $R = a$, for some $a \in \Sigma$



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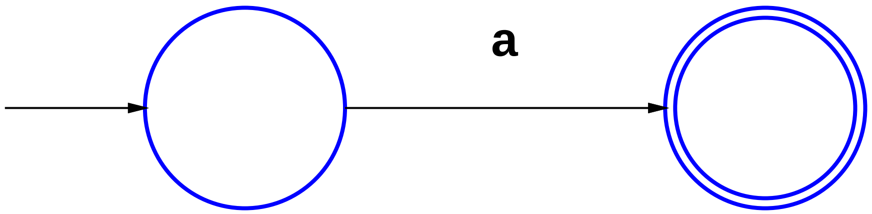

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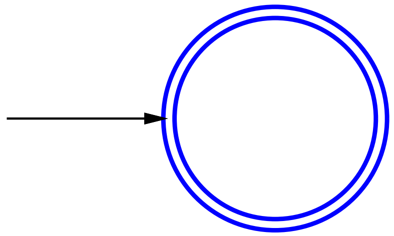

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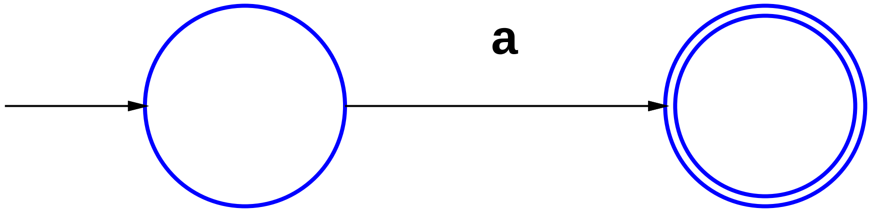

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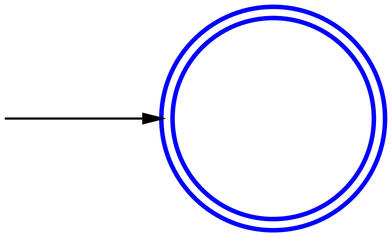

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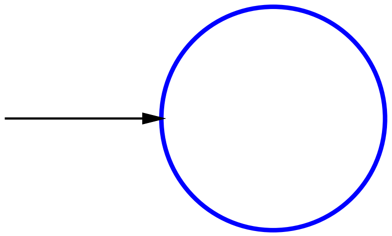

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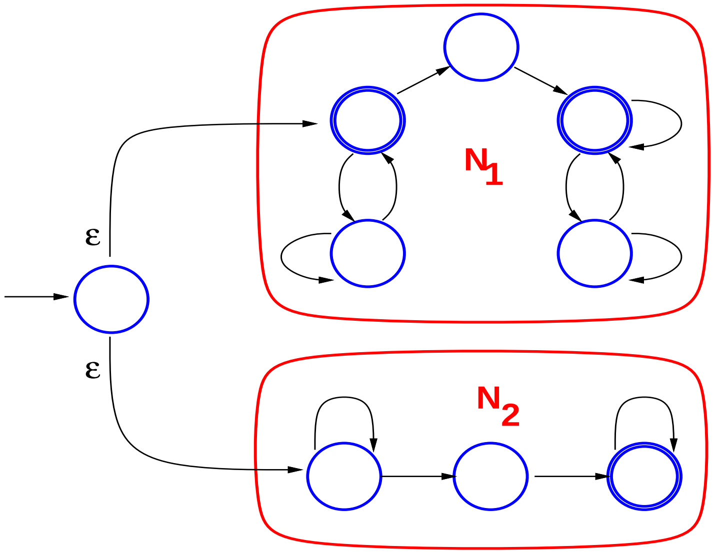

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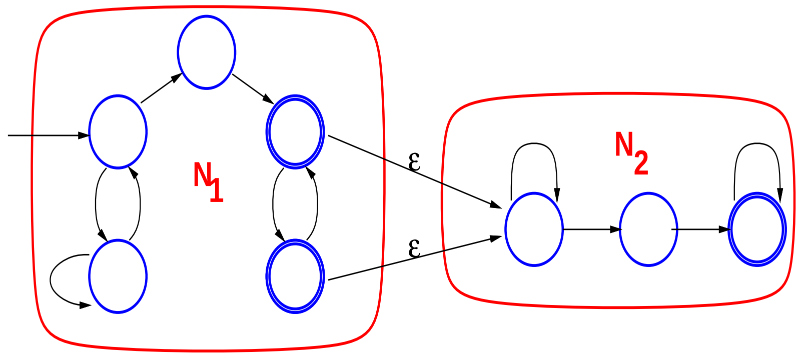

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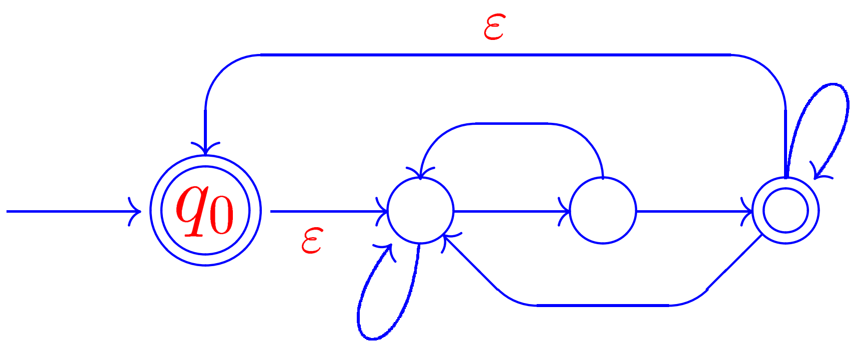
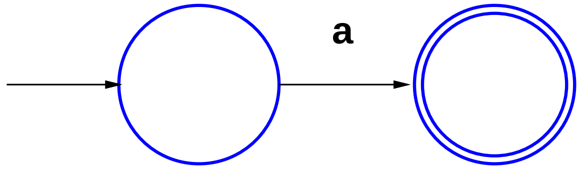


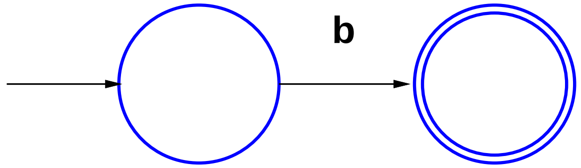
♣

Example

a



b



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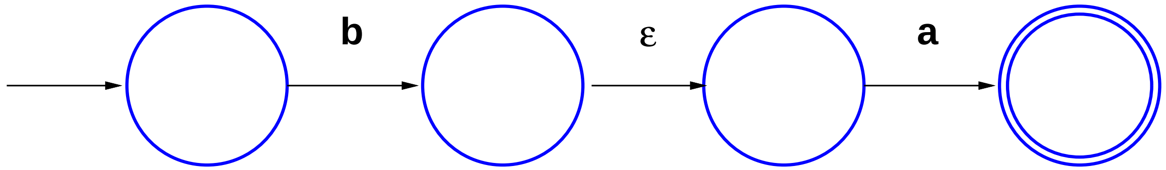
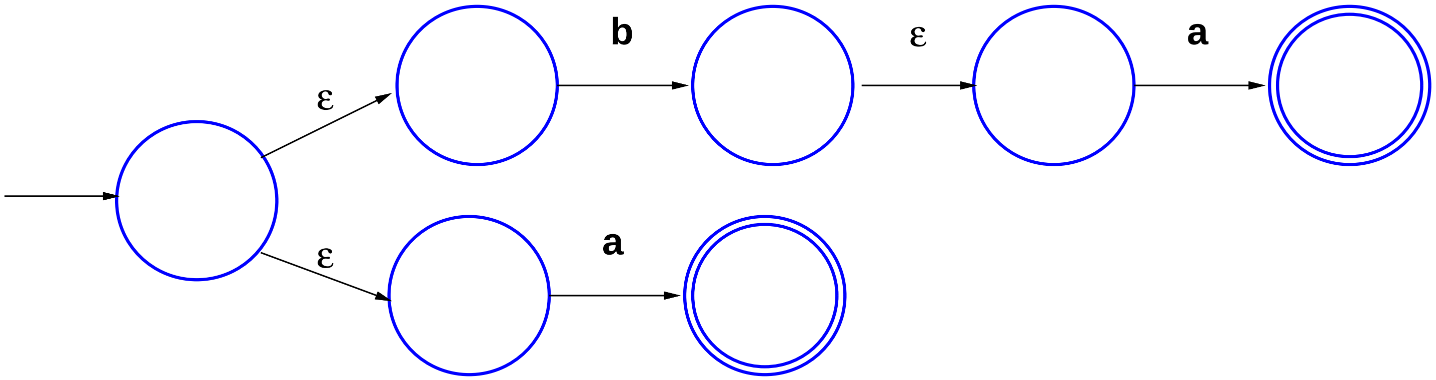


ab ∪ a



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Really? How? . . .

Converting DFA to Regular Expression $(\Longleftarrow)$

Strategy – sequence of equivalent transformations

- given a k-state DFA

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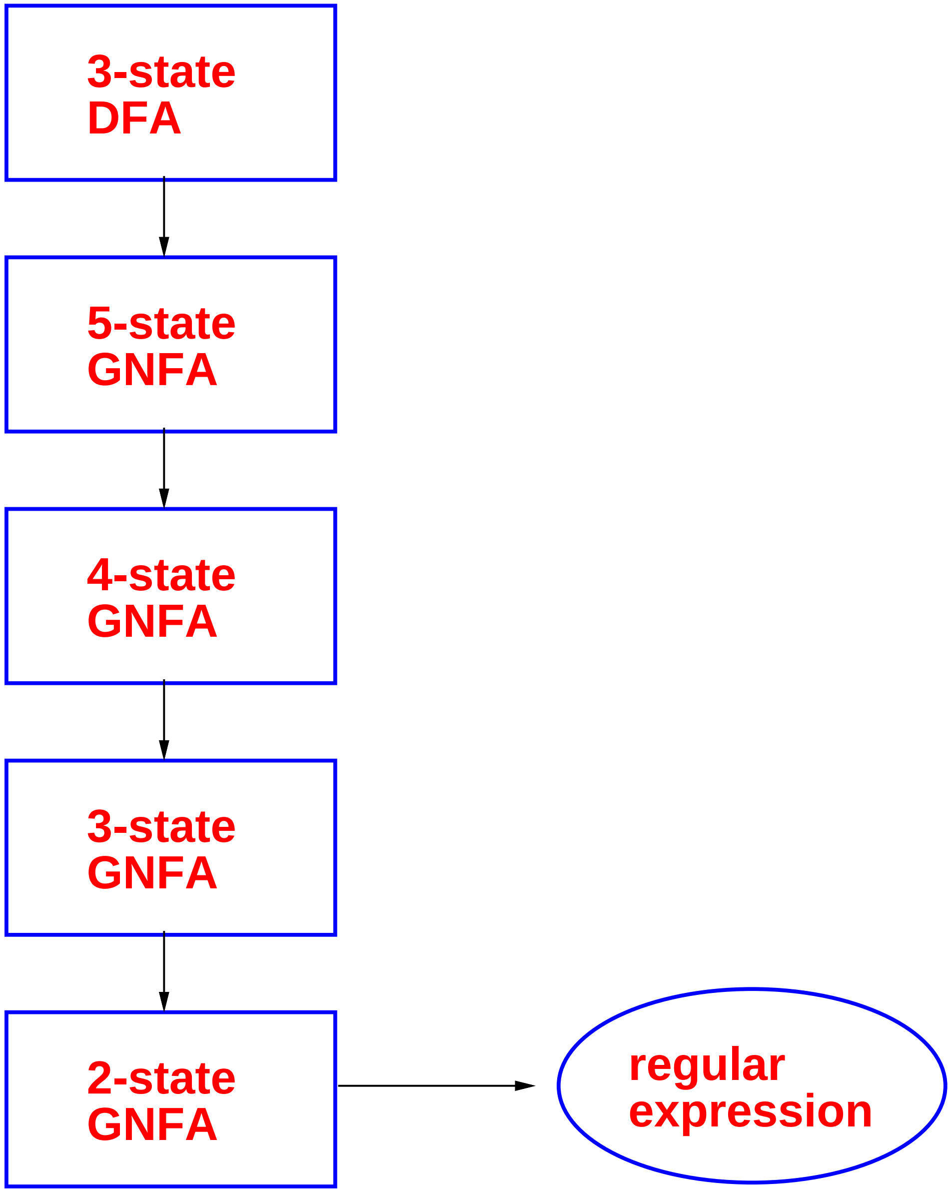
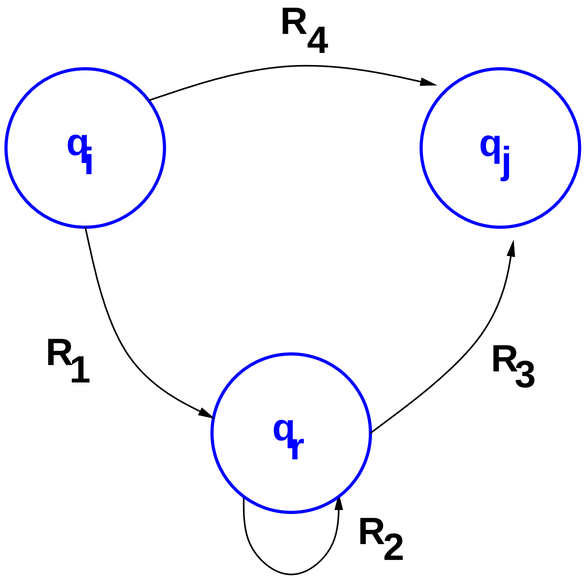
- label on single transition is the desired regular expression.

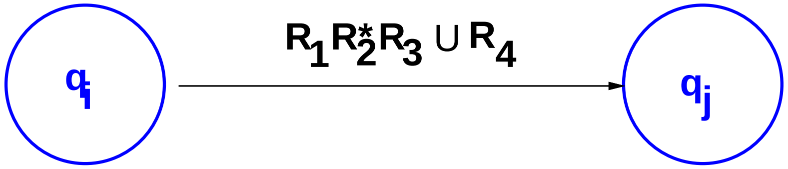
Converting Strategy ($\Longleftarrow$)



Removing One State

We remove one state q_r, and then repair the machine by altering regular expression of other transitions.





Formal Treatment – GNDA Definition

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If $\delta(q_i, q_j) = R$, then arrow from q_i to q_j has label R.

Formal Definition

A generalized deterministic finite automaton (GDFA) is $(Q, \Sigma, \delta, q_s, q_a)$, where

- Q is a finite set of states,
- Σ is the alphabet,
- $\delta : (Q - \{q_a\}) \times (Q - \{q_s\}) \to \mathcal{R}$ is the transition function.
- $q_s \in Q$ is the start state, and
- $q_a \in Q$ is the unique accept state.

A Formal Model of GNFA Computation

A GNFA accepts a string $w \in \Sigma^*$ if there exists a *parsing* of w, $w = w_1 w_2 \cdots w_k$, where each $w_i \in \Sigma^*$, and there exists a sequence of states $q_0, \ldots, q_k$ such that

- $q_0 = q_s$, the start state,
- $q_k = q_a$, the accept state, and
- for each i, $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$.
- (namely w_i is an element of the language described by the regular expression R_i.)

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- Denote the resulting $k - 1$ states GNFA by G'.

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Three steps:

1. If G accepts w, then so does G'.
2. If G' accepts w, then so does G.
3. Therefore G and G' are equivalent.

Step One

Claim: If G accepts w, then so does G':

- If G accepts w, then there exists a "path of states" $q_s, q_1, q_2, \ldots, q_a$ traversed by G on w, leading to the accept state q_a.

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- Either way, the claim holds.

Steps Two and Three

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Proof: Each transition from q_i to q_j in G' corresponds to a transition in G, either directly or through q_r. Thus if G' accepts w, then so does G.

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- This completes the proof of the claim that $L(G) = L(G')$.
- Combined with the induction hypothesis, this shows that G and the regular expression $\text{CONVERT}(G)$ accept the same language.
- This, in turn, proves our *remarkable claim*: A language, L, is described by a regular expression, R, if and only if L is regular. ♣

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- $D = \{w | w$ has an equal number of occurrences of 01 and 10 substrings$\}$

Consider B:

- DFA must "remember" how many 0's it has seen
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The others are exactly the same.

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Question: Is this a proof?

Answer: No, D is regular!??? (see problem set 1)

Pumping Lemma

We will show that all regular languages have a special property.

- Suppose L is regular.
- If a string in L is longer than a certain critical length ℓ (the pumping length),
- then it can be "pumped" to a longer string by repeating an internal substring any number of times.
- The longer string must be in L too.
- This is a powerful technique for showing that a language is not regular.

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The third condition is technical and useful occasionally.

Pumping Lemma – Proof

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$$\begin{array}{ccccccccccccccccc}
 & s_1 & & s_2 & & s_3 & & s_4 & & s_5 & & s_6 & & \ldots & & s_n & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
q_1 & & q_{20} & & q_9 & & q_{17} & & q_{12} & & q_{13} & & q_9 & & q_2 & & q_5 \in F
\end{array}$$

Pumping Lemma – Proof

Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA that accepts L.

Let ℓ be $|Q|$, the number of states of M.

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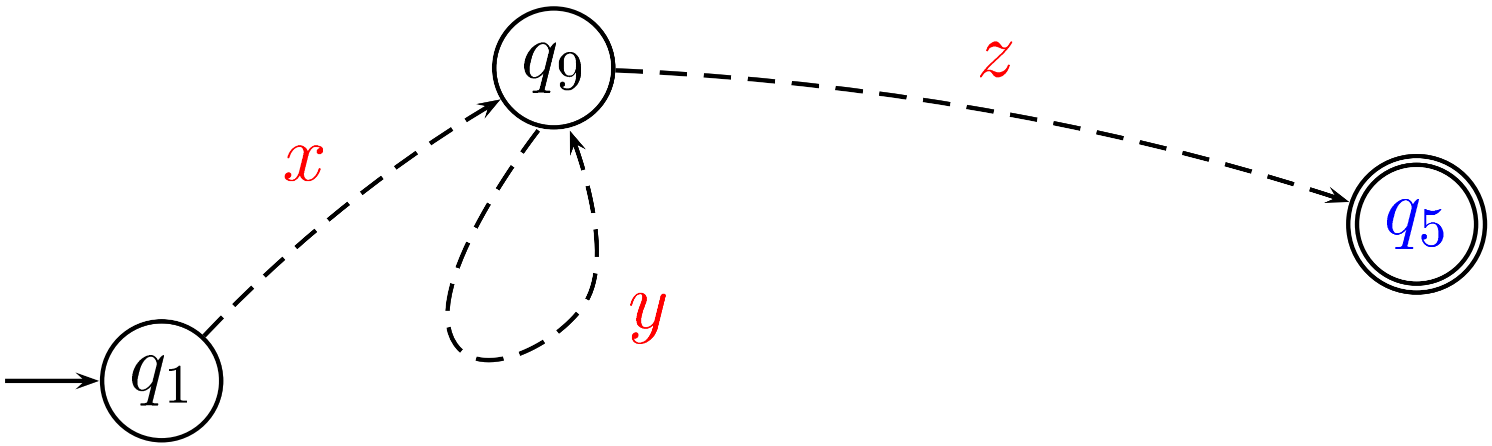
\end{array}$$

Since the sequence of states is of length $|s| + 1 > \ell$, and there are only ℓ different states in Q, at least one state is repeated (by the pigeonhole principle).

Pumping Lemma – Proof (cont.)

Write down $s = xyz$



- By inspection, M accepts xy^kz for every $k \geq 0$.

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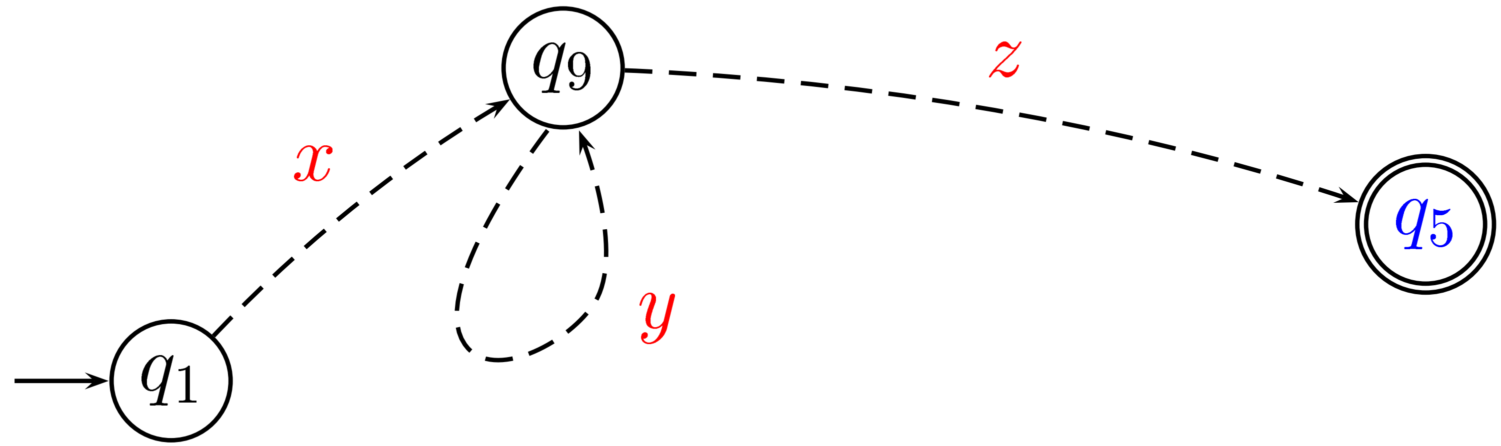

- By inspection, M accepts $xy^k z$ for every $k \geq 0$.
- $|y| > 0$ because the state (q_9 in figure) is repeated.

Pumping Lemma – Proof (cont.)

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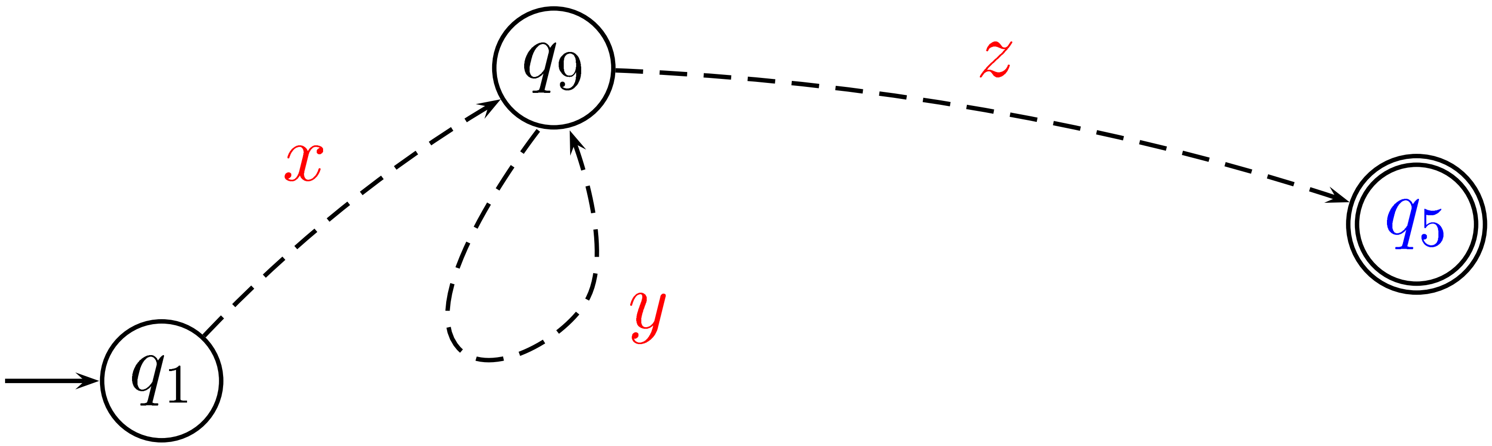

- By inspection, M accepts xy^kz for every $k \geq 0$.
- $|y| > 0$ because the state (q_9 in figure) is repeated.
- To ensure that $|xy| \leq \ell$, pick first state repetition, which must occure no later than $\ell + 1$ states in sequence.

An Application

Theorem: The language $B = \{0^n 1^n | n > 0\}$ is not regular.

Proof: By contradiction. Suppose B is regular, accepted by DFA M. Let ℓ be the pumping length.

- Consider the string $s = 0^\ell 1^\ell$.

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- If y is mixed, then $xy^k z$ is not of right form. ♣

Another Application

Theorem: The language $C = \{w|w \text{ has an equal number of 0's and 1's}\}$ is not regular.

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♣

Algorithms for NDA's

Given an NDA, N, and a string s, is $s \in L(N)$?

Answer: Construct the DFA equivalent to N and run it on w.

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Is $L(N) = \emptyset$?

Answer: This is a reachability question in graphs: Is there a path in the states' graph of N from the start state to some accepting state. There are simple, efficient algorithms for this task.

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Given N_1 and N_2, is $L(N_1) = L(N_2)$?

Answer: Check if $L(N_1) \subseteq L(N_2)$ and $L(N_2) \subseteq L(N_1)$.