Lecture 13, Fall 04/05

- Short review of last class
- NP hardness
- coNP and coNP completeness
- Additional reductions and NP complete problems
- Decision, search, and optimization problems
- Coping with NP completeness (1):
  Approximation

- Sipser, chapter 7 and section 10.1
  (some material not covered in book)

NP-Completeness (reminder)

A language $\mathcal{B}$ is NP-complete if it satisfies

- $\mathcal{B} \in NP$, and
- For every $\mathcal{A}$ in NP, $\mathcal{A} \leq_P \mathcal{B}$

coNP-Completeness (analog)

A language $\mathcal{C}$ is coNP-complete if it satisfies

- $\mathcal{C} \in coNP$ (namely its complement is in NP, and
- For every $\mathcal{D}$ in coNP, $\mathcal{D} \leq_P \mathcal{C}$

NP Hardness

A language $\mathcal{B}$ is NP hard if for every $\mathcal{A}$ in NP, $\mathcal{A} \leq_P \mathcal{B}$.

Difference from NP completeness: $\mathcal{B} \in NP$ is not required.

In homework assignment 5, asked to show that A_{TM} is NP hard . Clearly A_{TM} is not NP-complete (why?).

The Language SAT (reminder)

Definition: A Boolean formula is in conjunctive normal form (CNF) if it consists of terms, connected with $\wedge$s.

For example

$(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6})$

Definition:

SAT $= \{\langle\phi\rangle \mid \phi$ is a satisfiable CNF formula$\}$

3SAT (reminder)

Definition: A Boolean formula is in 3CNF form if it is a CNF formula, and all terms have three literals.

$$(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6} \vee x_4)$$

Define

$$3SAT = \{\langle\phi\rangle \mid \phi \text{ is satisfiable 3CNF formula}\}$$

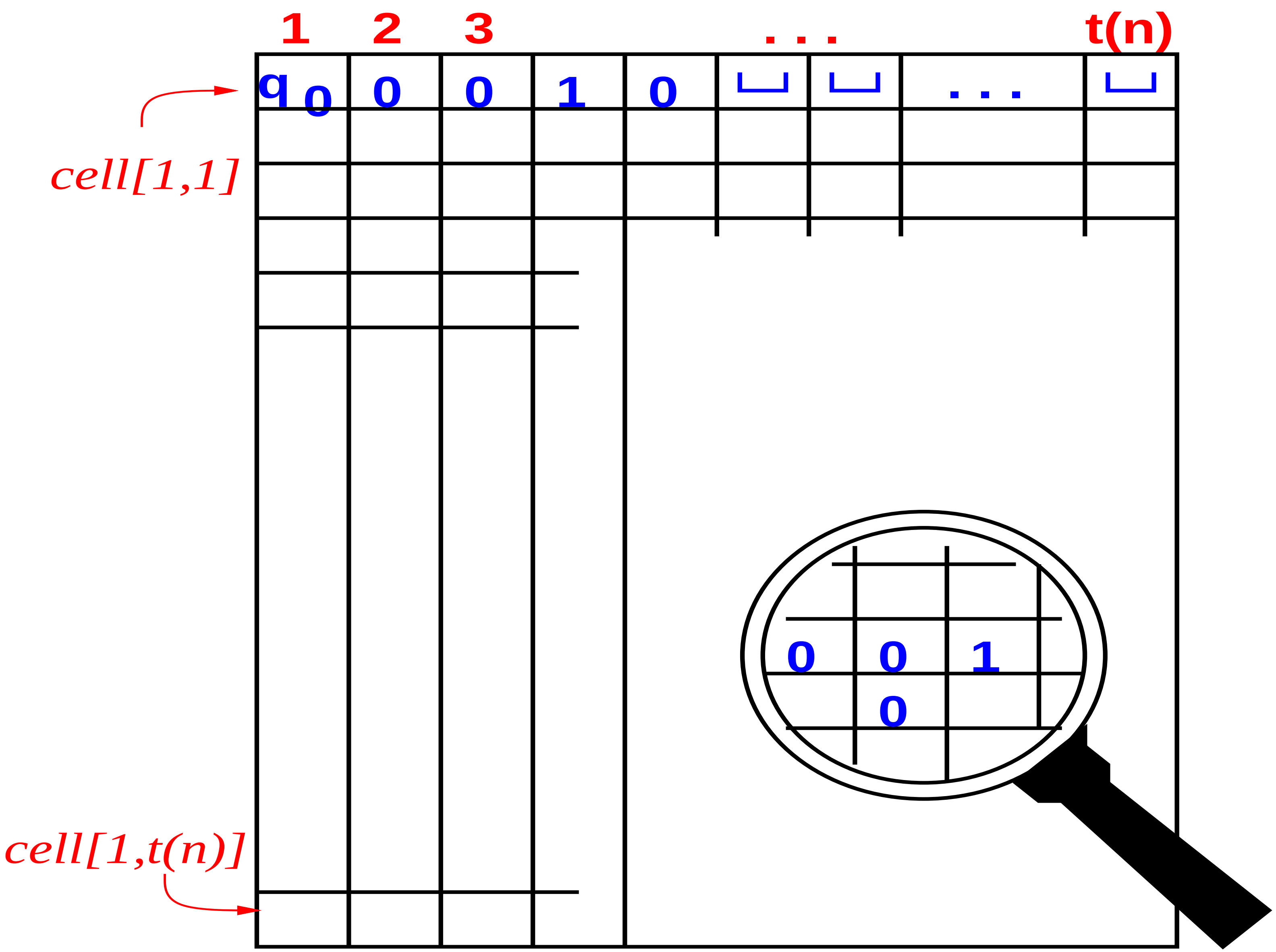
Clearly, if ϕ is a satisfiable 3CNF formula, then for any satisfying assignment of ϕ, every clause must contain at least one literal assigned 1.

Cook-Levin Theorem (reminder)

Theorem: SAT is NP complete.

- Must show that every NP problem reduces to SAT in poly-time.
- **Proof Idea**: Suppose $\mathcal{L} \in \mathcal{NP}$, and M is an NTM that accepts $\mathcal{L}$.
- On input w of length n, M runs in time $t(n) = n^c$.
- We consider the n^c-by-n^c tableau that describes the computation of M on input w.

The Tableau



Saw a Few Reductions

- SAT $\leq_P$ 3SAT ($\Rightarrow$ 3SAT is NP-complete)
- 3SAT $\leq_P$ Clique ($\Rightarrow$ Clique is NP-complete)
- 3SAT $\leq_P$ Clique ($\Rightarrow$ Clique is NP-complete)
- Clique $\leq_P$ Vertex Cover ($\Rightarrow$ VC is NP-complete)

- HamPath $\leq_P$ HamCircuit
- HamCircuit $\leq_P$ TSP

- Will now show 3SAT $\leq_P$ HamPath, thus establishing NP-completeness of HamPath, HamCircuit, and TSP.

Hamiltonian Path

For any 3CNF formula ϕ,

- we construct a graph G
- with vertices s and t
- such that ϕ is satisfiable iff there is a Hamiltonian path from s to t.

Hamiltonian Path

Here is a 3CNF formula ϕ:

$$(a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \cdots (a_k \vee b_k \vee c_k) \wedge$$

where

- each a_i, b_i, c_i is x_i or $\overline{x_i}$
- the ℓ clauses are $C_1, \ldots, C_\ell$,
- the k variables are $x_1, \ldots, x_k$.

HamPath: NP Completeness Proof

Turn to a separate, postscript presentation

Integer Programming (IP)

- Definition: A linear inequality has the form

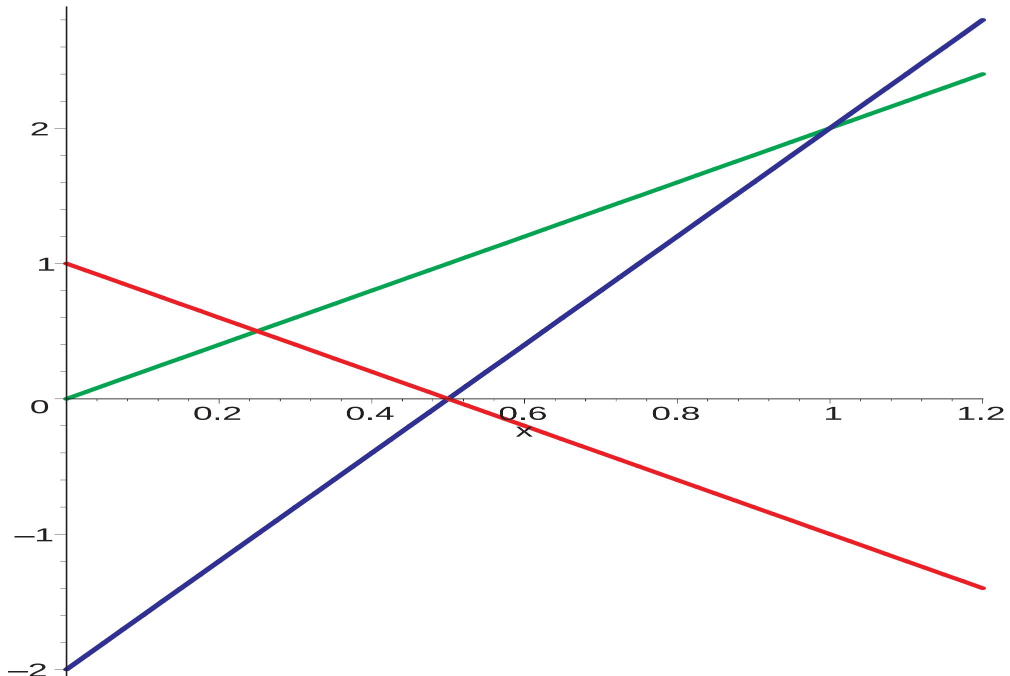
$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b$$

  where $a_1, \ldots, a_n, b$ are real numbers, and $x_1, \ldots, x_n$ are real variables.
- The Integer Programming (IP) problem:
- Input: A set of m linear inequalities with integer coefficients (a_i, b) in n variables $x_1, x_2, \ldots, x_n$.
- The language IP is the collection of all systems of linear inequalities that have a solution where all x_i are integers.

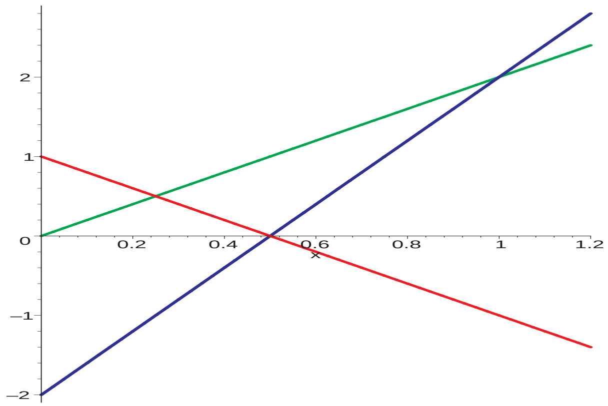
Integer Programming: Example

Consider the following system of linear inequalities

$$
\begin{array}{rcll}
y & \leq & 2x & \text{green line} \\
-2x+1 & \leq & y & \text{red line} \\
4x-2 & \leq & y & \text{purple line} \\
0 \leq & x & \leq 1 & \\
0 \leq & y & \leq 2 &
\end{array}
$$



Integer Programming: Example



This set does have a unique solution: the right hand corner of the solid triangle, $(1, 2)$.

But if we change the constraint on y to $0 \leq y \leq 1$, then we'd have no solution with integer coordinates, even though there are many solutions with rational, or real, coordinates.

Will now show IP is NP complete.

Membership in NP easy (why?)

SAT $\leq_P$ IP

SAT $= \{\langle\phi\rangle \mid \varphi$ is a satisfiable CNF formula$\}$

For example, the following formula is in SAT:
$(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6})$

Let φ be a CNF formula with m clauses and n variables $x_1, \ldots, x_n$ (either $x_i, \bar{x}_i$, or both, can appear in φ).

Will reduce φ to an IP instance with $2n$ variables $x_1, y_1, \ldots, x_n, y_n$ and $m + 2m$ linear inequalities, and n linear equalities (???).

SAT $\leq_P$ IP

- Each x_i in φ corresponds to the variable x_i in IP.
- Each $\bar{x}_i$ in φ corresponds to the variable y_i in IP.

- For each i, we add the inequalities $x_i \geq 0$, $y_i \geq 0$, and the equality $x_i + y_i = 1$
(what do these three express?)
- For each clause k, we add the inequality
$\sum_{z_j \in Clause_k} z_j \geq 1$
(what does this inequality express?)
- For example, $(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4)$ is translated to
$x_1 + y_2 + y_3 + x_4 \geq 1$.

SAT $\leq_P$ IP: Example

$$\varphi = (x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6})$$

translates to

$$\begin{aligned}
x_1 + y_2 + y_3 + x_4 &\geq 1 \\
x_3 + y_5 + x_6 &\geq 1 \\
x_3 + x_6 &\geq 1 \\
x_1 \geq 0,\ y_1 \geq 0,\ x_1 + y_1 &= 1 \\
x_2 \geq 0,\ y_2 \geq 0,\ x_2 + y_2 &= 1 \\
x_3 \geq 0,\ y_3 \geq 0,\ x_3 + y_3 &= 1 \\
x_4 \geq 0,\ y_4 \geq 0,\ x_4 + y_4 &= 1 \\
x_5 \geq 0,\ y_5 \geq 0,\ x_5 + y_5 &= 1 \\
x_6 \geq 0,\ y_6 \geq 0,\ x_6 + y_6 &= 1
\end{aligned}$$

SAT $\leq_P$ IP: Validity (sketch)

Should show

(a) Reduction g is poly-time computable

(b) $\varphi \in \text{SAT} \implies g(\varphi) \in IP$

(c) $g(\varphi) \in IP \implies \varphi \in SAT$.

- Poly time: easy (verify details!).
- Suppose $\varphi \in \text{SAT}$. Take a satisfying assignment. If $x_i = 1$ assign $x_i = 1, y_i = 0$ in IP. If $x_i = 0$ assign $x_i = 0, y_i = 1$ in IP.
- So "sanity check" constraints satisfied. "Clause constraints" are satisfied due to at least one literal satisfied in each clause., implying $g(\varphi) \in IP$.
- $g(\varphi) \in IP \implies \varphi \in SAT$ is similar. ♣

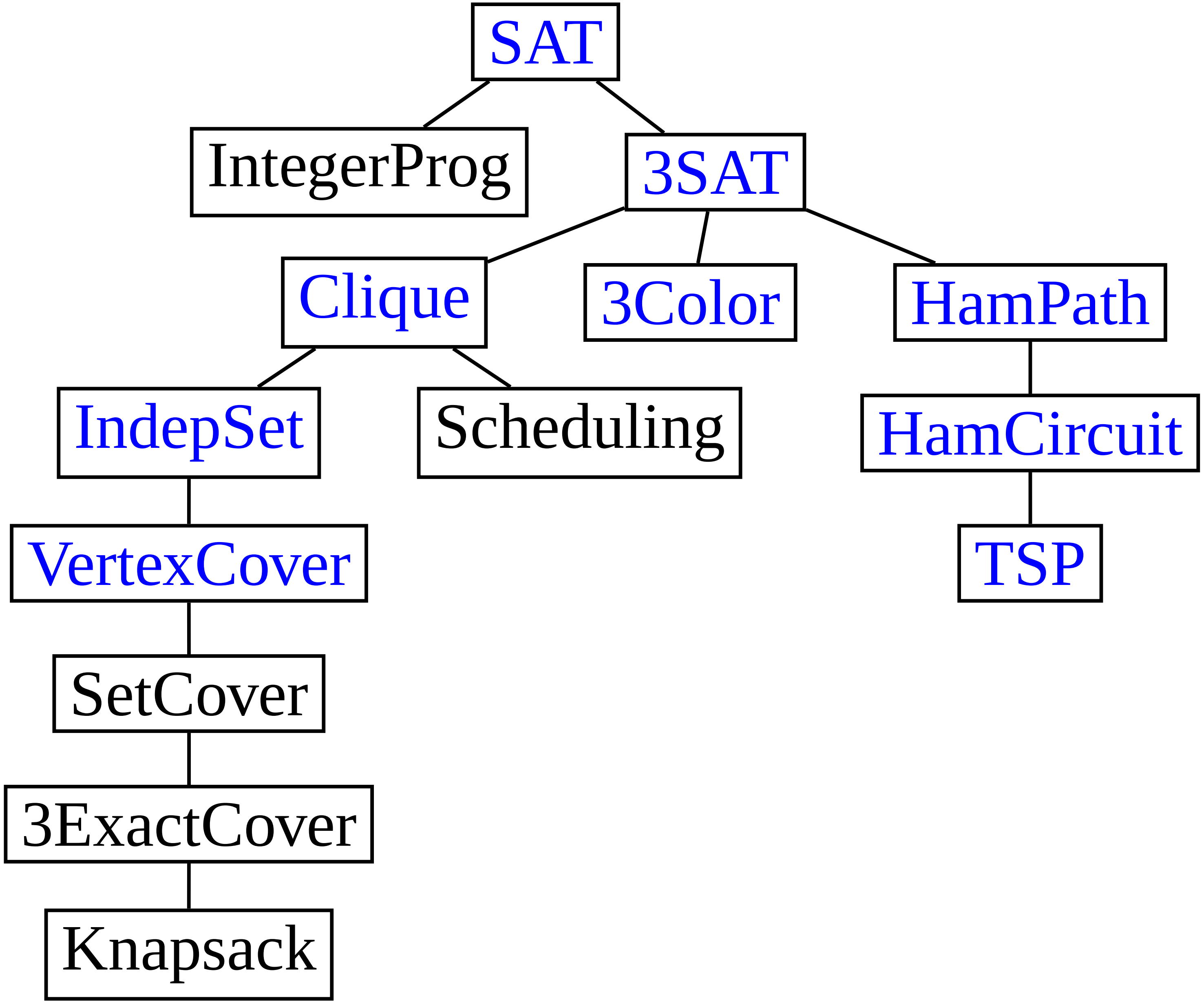
More Intractable Problems

- Bounded A_{TM}: Given encoding $\langle M \rangle$ of non-deterministic TM, an input w, time bound 1^k in unary, does M have an accepting computation of w in k steps or less?
- Bounded A_{TM} is NP complete, via a “generic” reduction.
- Bounded tiling: Given a set of colored, rectangular tiles, initial tiling (part of first row), and a bound k in unary (*i.e.* 1^k). Is there a legal extension that fills up the k-by-k square?
- Bounded tiling is NP complete, via a “generic” reduction (some modifications regarding final states wrt unbounded case).
- Blackboard, chalk and dust proof for both problems.

Yet More Intractable Problems

- Subgraph isomorphism is NP complete.
- Graph isomorphism is in NP, seems not to be in P, but we got many good reasons to believe it is not NP complete.

Chains of Reductions: NPC Problems



On Search, Decision, and Optimization

Let $R(\cdot, \cdot)$ be a poly time computable predicate.

- Decision Problem: Given input x, decide if there is some y satisfying $R(x, y)$?
- Using the "certificate" characterization of languages in NP, the decision problem is the same as deciding membership $x \in L$ for $L \in NP$.
- Search Problem: Given input x, find some y satisfying $R(x, y)$, or declare that none exist.
- The search problem seems harder to solve than the decision problem.

On Search, Decision, and Optimization

- Search Problem: Given input x, find some y satisfying $R(x, y)$, or declare that none exist.
- The search problem seems harder to solve than the decision problem.
- Turns out that for NP complete languages, search and decision have the same difficulty.
- Specifically, given access to an oracle for L (the decision problem), we can solve the search problem in poly time.
- Examples: SAT and Clique (on board).

Coping with NP-Completeness

- Approximation algorithms for hard optimization problems.
- Randomized (coin flipping) algorithms.
- Fixed parameter algorithms.

Approximation Algorithms

In this course, we deal with three kinds of problems

- Decision problems: is there a solution (yes/no answer)?
- Search problems: if there is a solution, find one.
- Optimization problems: find a solution that optimizes some objective function.
- Optimization comes in two flavours
  - maximization
  - minimization

Approximation Problems

A maximization (minimization) problem consists of

- Set of feasible solutions
- Each feasible solution A has a cost $c(A)$
- Suppose solution with max (min) cost OPT is optimal.

Definition: An ε-approximation algorithm A is one that satisfies

$$c(A)/OPT \geq 1 - \varepsilon \quad \text{(maximization)}$$

$$c(A)/OPT \leq 1 + \varepsilon \quad \text{(minimization)}$$

Note that $0 \leq \varepsilon$, and for maximization problems $\varepsilon \leq 1$.

Approximation

Question: What is the smallest ε for which a given NP-complete problem has a polynomial-time ε-approximation?

Not all NP-complete problems are created equal.

NP-complete problems may have

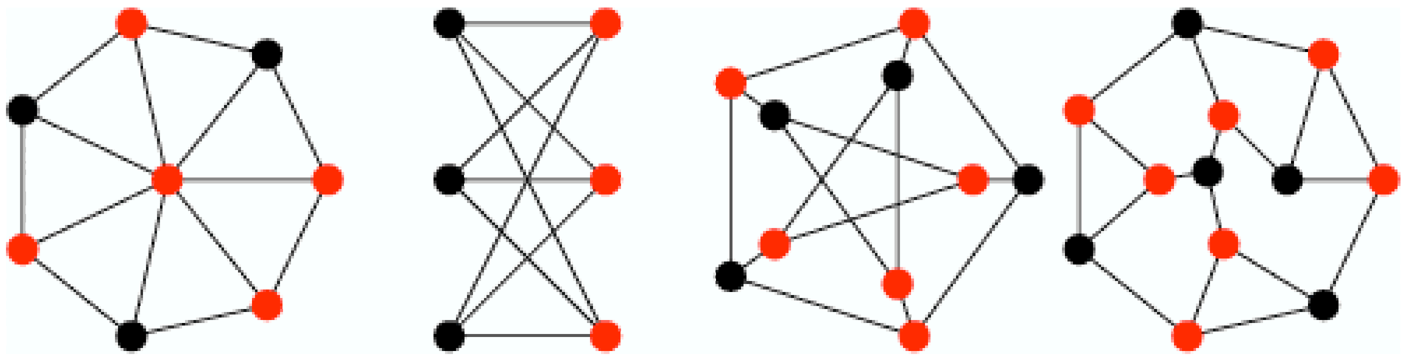
- no ε-approximation, for any ε.
- an ε-approximation, for some ε.
- an ε-approximation, for every ε.

Remark: Polynomial reductions do not necessarily preserve approximations.

Example: Vertex Cover

Given a graph (V, E)

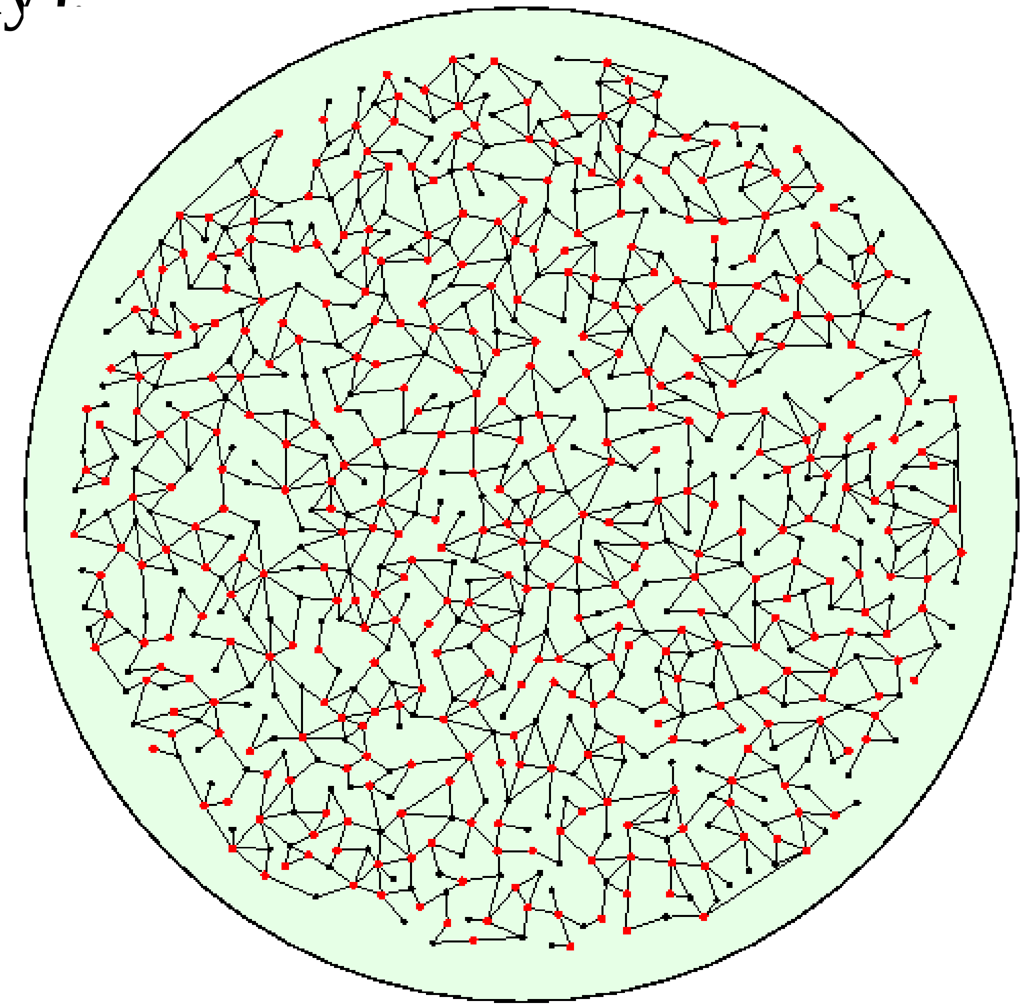
- find the smallest set of vertices C
- such that for each edge in the graph,
- C contains at least one endpoint.



(figure from www.cc.ioc.ee/jus/gtglossary/assets/vertex_cover.gif)

Vertex Cover

The decision version of this problem is NP-complete by a reduction from IS (a fact you should be able to prove easily).



(figure from http://wwwbrauer.in.tum.de/gruppen/theorie/hard/vc1.png)

A Greedy Heuristic

Remark: A node with high degree looks promising for inclusion in cover. This intuition leads to following greedy algorithm:

- $C := \emptyset$
- while there are edges in G
  - choose node $v \in G$ with highest degree
  - add it to C
  - remove it and all edges incident to it from G

- **Question:** How are we doing?

The Greedy Heuristic

Question: How are we doing?

Answer: Poorly!

This greedy algorithm is not an $1+\varepsilon$-approximation algorithm for any constant ε. There are instances where

$c(A)/OPT \geq \Omega(\log|V|)$, implying

$OPT/c(A) \not\geq 1+\varepsilon$ for any constant ε.

Another Greedy Algorithm (Gavril '74)

- $C := \emptyset$
- while there are edges in G
  - choose any edge (u, v) in G
  - add u and v to C
  - remove them from G

- **Claim:** This algorithm is a 1-approximation algorithm for vertex cover.
- Meaning C is at most twice as large as a minimum vertex cover.

Gavril's Approximation Algorithm

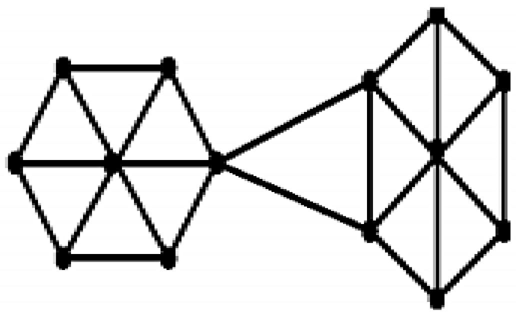
Claim: This is a 1-approximation algorithm.

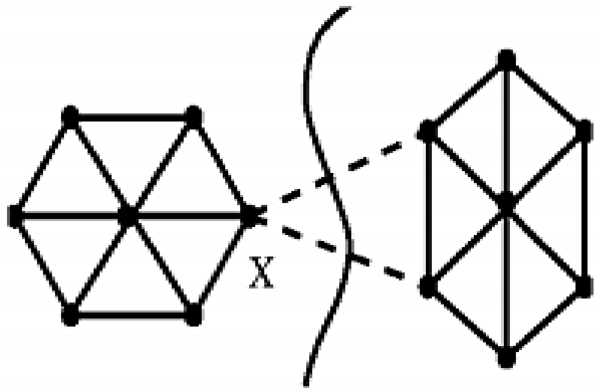
- Cover C constructed from $|C|/2$ edges of G
- no two edges of these share a vertex
- any vertex cover, including the optimum,
- contains at least one node from each of these edges (otherwise an edge would not be covered).
- It follows that $OPT(G) \geq |C|/2$ (so $c(A)/OPT/ \geq 1 + 1$)

- **Remark:** Despite simplicity and time, this is the best approximation ratio for vertex cover known todate.

Cuts in Graphs

Definition Let $G = (V, E)$ be an undirected graph. For any partition of the nodes of into two sets, S and $V - S$, the set of edges between S and $V - S$ is called a cut .





(pictures from http://www.cs.sunysb.edu/∼algorith/files/edge-vertex-connectivity.shtml)

Cuts in Graphs

For cuts, both optimization problems make sense (in different contexts):

1. Min Cut: Find a partition that minimizes the number of edges between S and $V - S$.
2. Max Cut: Find a partition that maximizes the number of edges between S and $V - S$.

The two optimization problems have very different complexities:

1. Min Cut is tightly related to network flow, and has polynomial time algorithms.
2. Max Cut is NP-complete.

Max Cut Algorithm

Consider the following local improvement strategy

- Pick any partition S and $V - S$
- If the cut can be improved by moving any vertex from $V - S$ to S, or vice-versa, do so.
- Quit when no improvement is possible (local maximum reached).

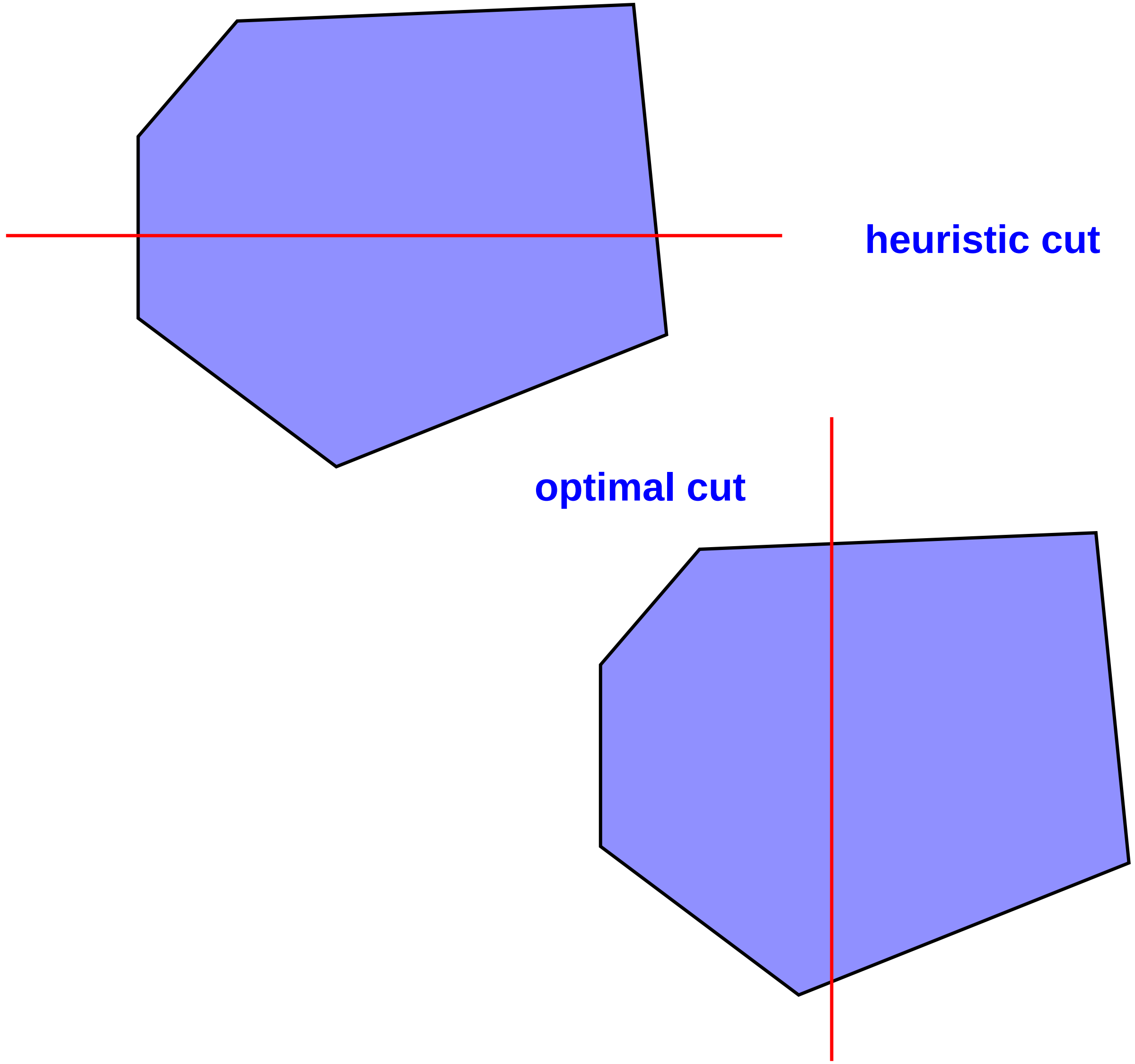
Running time

- Any cut has at most $|E|$ edges,
- thus at most $|E|$ improvements possible,

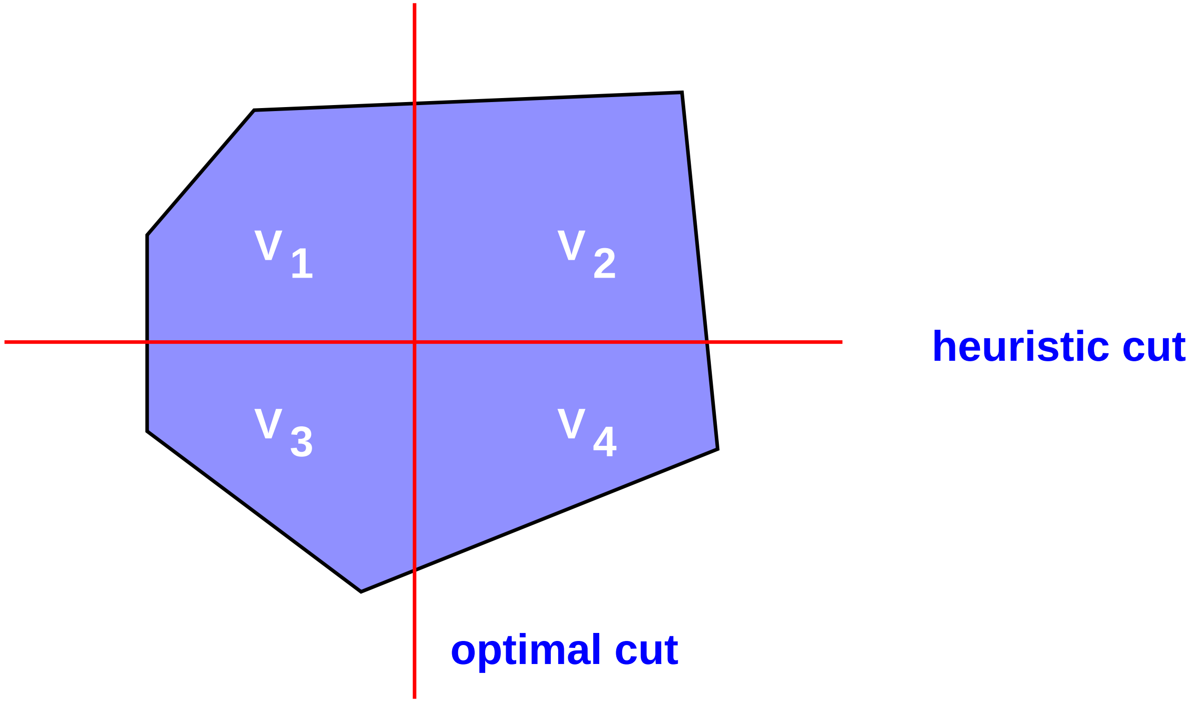
$\Longrightarrow$ algorithm is polynomial time.

Claim: This is a $\frac{1}{2}$-approximation algorithm.

Max Cut Algorithm

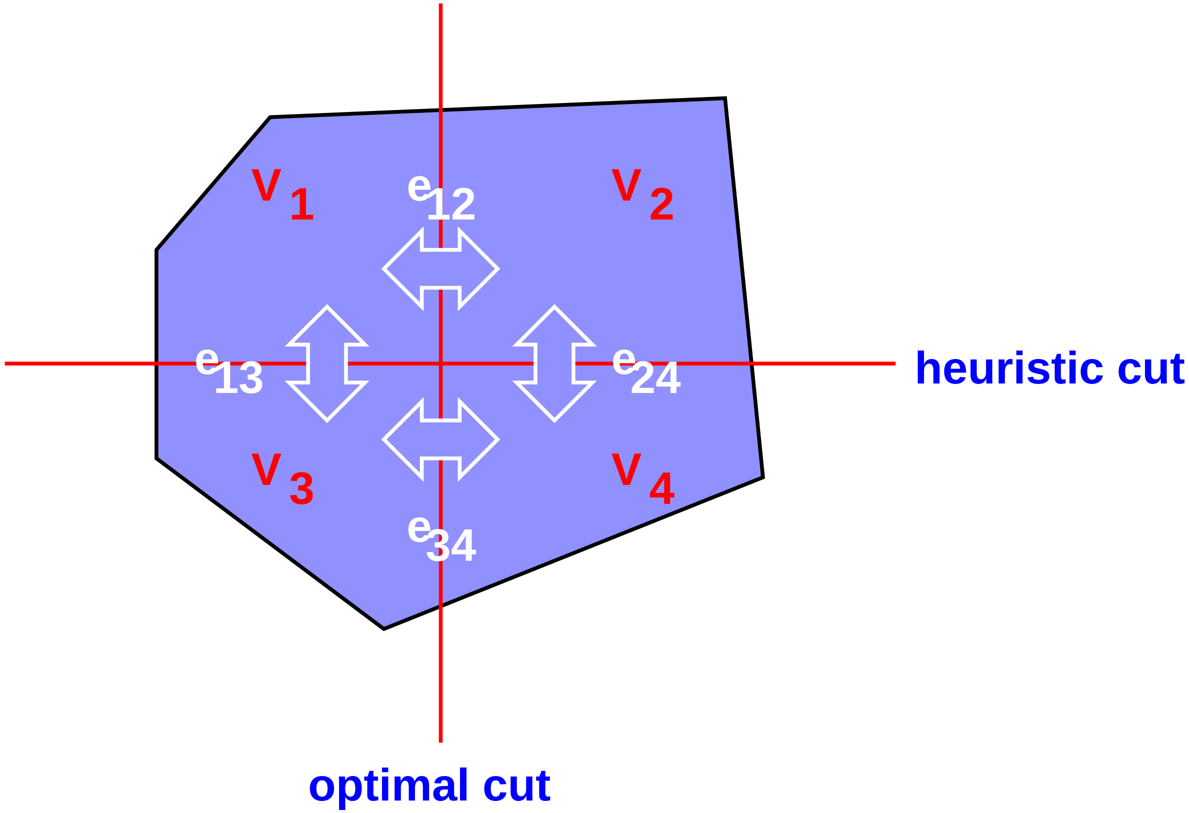


Max Cut Algorithm



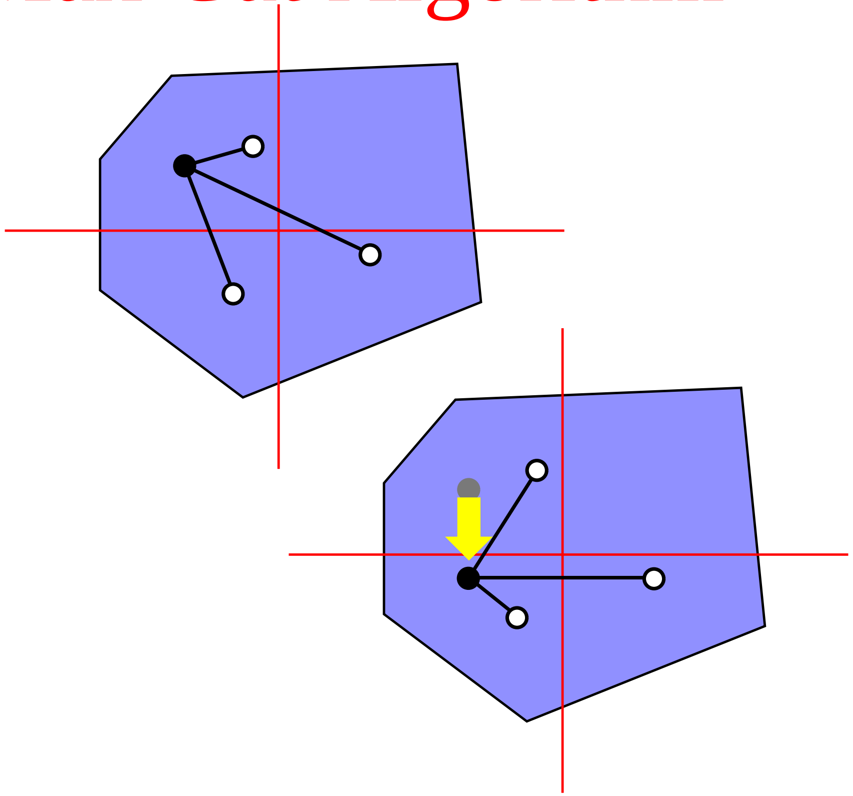
- Heuristic yields $V_1 \cup V_2$, $V_3 \cup V_4$
- Optimal yields $V_1 \cup V_3$, $V_2 \cup V_4$

Max Cut Algorithm



- Every cut partitions the edges into cut edges, E_C, and non-cut edges, E_N.
- Let c_v be the number of cut edges from node v.
- Let n_v be the number of non-cut edges from v.

Max Cut Algorithm



- When algorithm termnates, for every node v, the number of cut edges is greater or equal than the number of non-cut edges, $c_v \geq n_v$.
- Otherwise, switching the node v would increase the size of the cut produced by the algorithm.
- Summing over all nodes in V: $\sum_v c_v \geq \sum_v n_v$

Max Cut Algorithm

- Summing over all nodes in V: $\sum_v c_v \geq \sum_v n_v$.
- But $\sum_v c_v = 2|E_C|$, $\sum_v n_v = 2|E_N|$ (each edge is counted twice).
- Thus $|E_C| \geq |E_N|$.
- $\Longrightarrow 2|E_C| \geq |E_N| + |E_C| = |E|$
- So $|E_C| \geq |E|/2$.
- Clearly $|E| \geq OPT$ (any cut is set of edges).
- Thus $c(\mathcal{A}) \geq OPT/2$, *i.e.* algorithm is $\frac{1}{2}$-MaxCut approximation $(c(A)/OPT \geq 1 - 1/2)$. ♣