Leftovers from Lecture 3

Implementing GF(2^k)

Multiplication: Polynomial multiplication, and then remainder modulo the defining polynomial f(x):

```
> g(x):=(x^4+x^3+x+1)*(x^3+x+1);
```

$$g(x) := (x^4+x^3+x+1)(x^3+x+1)$$

```
> f(x):=x^5+x^4+x^3+x+1;
```

$$f(x) := x^5 + x^4 + x^3 + x + 1$$

```
> rem(g(x),f(x),x);
```

$$1 + 3x^4 + x^3 + 2x$$

```
> % mod 2;
```

$$1 + x^4 + x^3$$

(1,1,0,1,1) *(0,1,0,1,1)

= (1,1,0,0,1)

For small size finite field, a lookup table is the most efficient method for implementing multiplication.

Implementing GF(2^5) in XMAPLE

Irreducible polynomial

```
> G32:=GF(2,5,x^5+x^4+x^3+x+1):
> a := G32[ConvertIn](x);
```

$$a := x$$

```
> b := G32[`^`](a,8):              # colon at end of
  statement supresses printing
  c := G32[`^`](a,9):
  G32[ConvertOut](b);      # canonical
  representation, higher momonials to the left
  G32[ConvertOut](c);
```

$$x^3 + x^2 + x + 1$$
$$x^4 + x^3 + x^2 + x$$

More GF(2^5) Operations in XMAPLE

```
> d := G32[`+`](b,c):
  G32[ConvertOut](d);
```

$$x^4 + 1$$

```
> G32[isPrimitiveElement](d);
```

$$true$$

```
> e:=G32[`^`](a,-1):
  G32[ConvertOut](e);
```

$$x^4 + x^3 + x^2 + 1$$

```
> G32[`*`](a,e);
```

$$1$$

```
> for i from 1 to 32 do
    f:= G32[`^`](a,i):
    print(f, G32[isPrimitiveElement](f))
    end do:
```

$$x, true$$
$$x^2, true$$
$$x^3, true$$
$$x^4, true$$
$$1 + x + x^3 + x^4, true$$
$$1 + x^2 + x^3, true$$
$$x + x^3 + x^4, true$$

Addition: b+c

test primitive element

e <--inverse of a
Multiplication: a*e

Loop for finding primitive elements

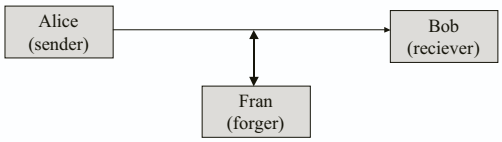
LECTURE 4

Data Integrity & Authentication

Message Authentication Codes (MACs)

Goal

Ensure integrity of messages, even in presence of an active adversary who sends own messages.



Remark: Authentication is orthogonal to secrecy, yet systems often required to provide both.

Definitions

- Authentication algorithm - A
- Verification algorithm - V ("accept"/"reject")
- Authentication key – k
- Message space (usually binary strings)
- Every message between Alice and Bob is a pair $(m, A_k(m))$
- $A_k(m)$ is called the authentication tag of m

Definition (cont.)

- Requirement – $V_k(m,A_k(m))$ = "accept"
  - The authentication algorithm is called MAC (Message Authentication Code)
  - $A_k(m)$ is frequently denoted $MAC_k(m)$
  - Verification is by executing authentication on m and comparing with $MAC_k(m)$

Properties of MAC Functions

- Security requirement – adversary can't construct a new legal pair $(m, MAC_k(m))$ even after seeing $(m_i, MAC_k(m_i))$ $(i=1,2,\ldots,n)$
- Output should be as short as possible
- The MAC function is *not* 1-to-1

Adversarial Model

- Available Data:
  - The MAC algorithm
  - Known plaintext
  - Chosen plaintext
- Note: chosen MAC is unrealistic
- Goal: Given n legal pairs $(m_1, MAC_k(m_1)), \ldots, (m_n, MAC_k(m_n))$ find a new legal pair $(m, MAC_k(m))$

Adversarial Model

We will say that the adversary succeeded even if the message Fran forged is "meaningless". The reason is that it is hard to predict what has and what does not have a meaning in an unknown context, and how will Bob, the reciever, react to such successful forgery.

Efficiency

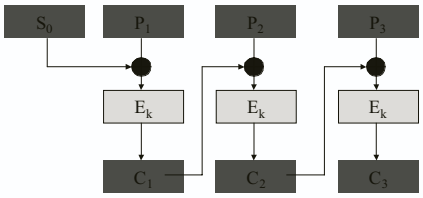
- Adversary goal: given n legal pairs $(m_1, MAC_k(m_1)), \ldots, (m_n, MAC_k(m_n))$ find a new legal pair $(m, MAC_k(m))$ efficiently and with non negligible probability.
- If n is large enough then n pairs $(m_i, MAC_k(m_i))$ determine the key k uniquely (with high prob.). Thus a non-deterministic machine can guess k and verify it. But doing this deterministically should be computationally hard.

MACs Used in Practice

We describe a MAC based on CBC Mode Encryption, and a MAC based on cryptographic hash functions.

Reminder: CBC Mode Encryption

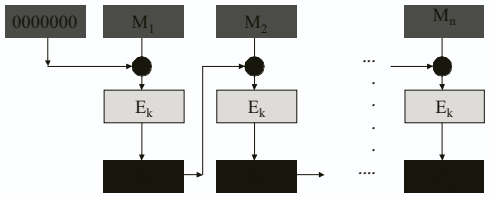
(Cipher Block Chaining)



Previous ciphertext is XORed with current plaintext before encrypting current block.
An initialization vector S_0 is used as a "seed" for the process.
Seed can be "openly" transmitted.

CBC Mode MACs

- Start with the all zero seed.
- Given a message consisting of n blocks $M_1,M_2,\ldots,M_n$, apply CBC (using the secret key k).



- Produce n "cipertext" blocks $C_1,C_2,\ldots,C_n$, discard first n-1.
- Send $M_1,M_2,\ldots,M_n$ & the authentication tag $MAC_k(M)=C_n$.

Security of CBC MAC [BKR]

- Claim: If E_k is a pseudo random function, then CBC MACis resilient to forgery.
- Proof outline: Assume CBC MAC can be forged efficiently. Transform the forging algorithm into an algorithm distinguishing E_k from random function efficiently.

Combined Secrecy & MAC

- Given a message consisting of n blocks $M_1,M_2,\ldots,M_n$, apply CBC (using the secret key k1) to produce $MAC_{k1}(M)$.

- Produce n cipertext blocks $C_1,C_2,\ldots,C_n$ under a different key, k2.
- Send $C_1,C_2,\ldots,C_n$ & the authentication tag $MAC_{k1}(M)$.

Hash Functions

- Map large domains to smaller ranges
- Example *h:* $\{0,1,\ldots,p^2\} \rightarrow \{0,1,\ldots,p-1\}$ defined by $h(x) = ax+b \bmod p$
- Used extensively for searching (hash tables)
- Collisions are resolved by several possible means – chaining, double hashing, etc.

Collision Resistance

- A hash function *h: D → R* is called *weakly collision resistant* for x∈D if it is hard to find x'≠x such that h(x')=h(x)
- A function *h: D→R* is called *strongly collision resistant* if it is hard to find x, x' such that x'≠x but h(x)=h(x')

The Birthday Paradox

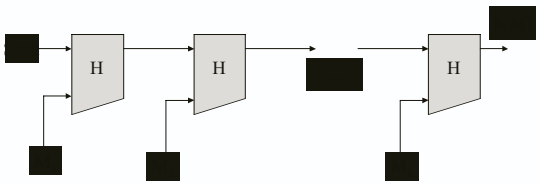
- If 23 people are chosen at random the probability that two of them have the same birth-day is greater than 0.5
- More generally, let h:D→R be any mapping. If we chose $1.17|R|^{1/2}$ elements of D at random, the probability that two of them are mapped to the same image is greater than 0.5.

Cryptographic Hash Functions

Cryptographic hash functions are hash functions that are strongly collision resistant.

- Notice: No secret key.
- Should be very fast to compute, yet hard to find coliding pairs (impossible if P=NP).
- Usually defined by:
  - Compression function mapping n bits (e.g. 512) to m bits (e.g 160), m < n.

Extending to Longer Strings



H : D --> R (fixed sets, typically $\{0,1\}^n$ and $\{0,1\}^m$)

Extending the Domain (cont.)

- The seed is usually constant
- Typically, padding (including text length of original message) is used to ensure a multiple of n.
- Claim: if the basic function H is collision resistant, then so is its extension.

Lengths

- Input message length should be arbitrary. In practice it is usually up to 2^{64}, which is good enough for all practical purposes.
- Block length is usually 512 bits.
- Output length should be at least 160 bits to prevent birthday attacks.

Real-World Hash Functions

- MD family ("message digest")
  - MD-2
  - MD-4 (full description in Stinson's book)
  - MD-5
- SHA and SHA-1 (secure hash standard, 160 bits)
  (www.itl.nist.gov/fipspubs/fip180-1.htm)
- RIPE-MD
- SHA-256, 384 and 512 (proposed standards, longer digests)

Basing MACs on Hash Functions

- First goal: combine message and secret key, hash and produce MAC
- Second goal: work with any cryptographic hash function

- First attempt: $MAC_k(m)=h(k,m)$
- Second attempt: $MAC_k(m)=h(m,k)$

HMAC

- Proposed in 1996 by [BCK]
- Receives as input a message m, a key k and a hash function h
- Outputs a MAC by:
  - $HMAC_k(m,h)= h(k\oplus opad, h(k\oplus ipad,m))$
- Theorem [BCK]: HMAC can be forged if and only if the underlying hash function is broken (collisions found).

HMAC in Practice

- SSL / TLS
- WTLS
- IPSec:
  - AH
  - ESP

Back to Number Theory

Quadratic Residues

- An element x is a *quadratic residue* modulo n if there exists y such that $y^2 \equiv x \bmod n$
- If x is a quadratic residue then so is $-x \bmod n$
- If p is prime there are exactly (p-1)/2 quadratic residues
- If p is prime, and g is a generator of the multiplicative group, the quadratic residues are even powers of g.

One-Way Functions

- A function *f: D→R* is called *one-way* if:
  - Computing f(x) is "easy"
  - Computing $f^{-1}(y)$ for almost all the images is "hard"
- Given the "real-world" definition of "hard" a one-way function may be a single function (e.g. SHA-1)
- Given the theoretical definition, we refer to a family of *one-way* functions

Example

- The Domain is all the pairs of prime numbers.
- The function is f(p,q) = pq
- Multiplication is easy – naïve algorithm is $O(n^2)$
- Factoring is difficult – simple algorithm is $O(2^{n/2})$. NFS and ECM are better but not polynomial.
- The function f(p,q) = pq maintains length