Jumping and Escaping: Modular Termination and the Abstract Path Ordering☆

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Abstract

Combinatorial commutation properties for reordering a sequence consisting of two kinds of steps, and for separating the well-foundedness of their combination into well-foundedness of each, are investigated. A weak commutation property, called "jumping", along with a weakened version of the lifting property, called "escaping" and requiring only an *eventual* lifting, are used for proving well-foundedness of a generic, abstract version of the recursive path orderings.

Keywords: Termination, Commutation, Path orderings, Well-founded orderings

***For Yoshihito Toyama on the happy occasion of his* kanreki.**

At the foundation of well-founded beliefs lies belief that is not well-founded.

—Ludwig Wittgenstein, *On Certainty*, §253

1. Motivation

One of the many topics Toyama investigated is the preservation of termination (strong normalization) of combinations of term-rewriting systems [31, 32]. Here we are interested in sufficient conditions for preservation of termination of combinations of abstract relations, that is, for the union of two binary relations to be terminating when each of the two is terminating on its own.

We apply combinatorial properties of combinations of relations to demonstrate well-foundedness of orderings typically used for proving termination of

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rewriting systems. The various "path orderings" [9] provide a convenient and popular method of proving termination, particularly of term-rewriting systems. We set out to prove the well-foundedness of a very abstract ordering that includes the usual path orderings as special cases. Abstract versions of the path orderings have already been proposed in [18, 6]; we aim for a more general version yet, and develop new proof techniques for this purpose. In particular, we make use of a remarkably fruitful condition of Doornbos, Backhouse, and van der Woude [13], denominated here "jumping" (a "lazy" version of commutation), which guarantees modular termination.

Path orderings are defined by recursion on subterms, usually building a term ordering from a symbol ordering, called a "precedence". To this end, the multiset ordering of unordered tuples [12] is used in the original multiset path ordering [8], as is a lexicographic comparison of ordered tuples in its generalizations [21]. But there is a noticeable difference between these two types of sequence orderings. Given an infinite descending sequence in the multiset ordering, one can be sure that there is an infinite descending sequence of elements of the underlying set that begins with some element of the very first multiset. One says that membership "lifts" (the element ordering) to the level of multisets. In the lexicographic case, on the other hand, one only knows that some tuple along the way has a component that initiates an infinite descending sequence in the underlying set of elements. This distinction leads us to investigate sequences with the latter, weaker property, which we call "escaping".

As the prime application of jumping and escaping, we explain in the next section why the method developed here demonstrates that a quite general abstraction of the path orderings is in fact well-founded. Then, in Section 3, after agreeing on notations and some terminology, we collect some basic observations on modularity. In particular, we explain the method of constriction, pioneered by Plaisted [27], of which we make crucial use in our proofs. Section 4 considers various more powerful commutation properties that are applicable to sequences mixing two kinds of steps. A weaker notion than jumping, defined in Section 5 and dubbed "selection", suffices to prove the main modularity result. Section 6 discusses the lifting of termination of one relation to that of the other in the presence of selection; escaping is also explained and used with jumping for the same purpose. A hint of future possibilities concludes this article.

2. The Path Ordering

The relations we care about in this work are binary; whenever we speak of a "relation", it should be understood to be binary. In what follows, the underlying (finite or infinite) set of "points" V is fixed, with relations A, B, R, S, etc., connecting pairs of points via (directed) "steps" that are "colored" A, B, etc. Points may be connected by steps of more than one color.

The *immortal* elements for a relation E over V are those points $v \in V$ that initiate an infinite E-chain of (not necessarily distinct) points in V: $v \; E \; v' \; E$

$v'' \; E \cdots$.[1] *Termination* of E means that all points are mortal; we will also say in this case that E is *well-founded*, regardless of whether E is a partial order.

2.1. Main Result

This article explores properties of the (non-disjoint) union of two relations. Throughout, we will regularly use E to denote the union $A + B$ of the two relations, A and B, under consideration, and, when we speak of "mortality", without elaboration, we will mean mortality with respect to this E.

The two main properties of a pair of relations that we need for the well-foundedness of the path ordering are jumping and escaping. The first gives modularity, making the union E terminating if both A and B are. The second then reduces termination of B to that of A. Since the path ordering will be composed of A- and B-steps, this means that it is well-founded as long as A is.

Definition 1 (Jumping [13, 14]). Relation A *jumps over* relation B if

$$BA \quad \subseteq \quad A(A+B)^* + B \; .$$

We are using juxtaposition for composition of binary relations, the plus sign for union, and the star operation for reflexive-transitive closure. So this condition means that, if there is a path $s \; B \; t \; A \; u$ from point s to point u via some point t, by means of a B-step followed by an A-step, then either there is an alternate route from s to u beginning with an A-step followed by any mix of A- and B-steps or else one can get directly from s to u by a lone B-step.

It is shown in [13] (Theorem 53 below) that jumping guarantees that the union $A + B$ is terminating whenever both A and B are, individually.

The second property concerns infinite B-paths.

Definition 2 (Escaping). Relation A *escapes from* relation B if there is some point in every infinite B-chain from which an A-step leads to a point that is immortal in the union $A + B$.

This escaping property will be expressed more formally in Section 3.4.

Consider, for example, $>$ on the natural numbers (which is well-founded), plus $>$ of the negative integers (which is not), but no relation between positive and negative. Let B be $>$ combined with a lexicographic comparison of pairs of integers, and let A give either component. Then, in any infinite B-sequence of pairs, either the first component of the initial pair is immortal (i.e. negative), or else—from some point on—the second is immortal. So A escapes from B.

Our main result (Theorem 60 below) is the following:

Main Theorem. *If relation A jumps over relation B and escapes from B, then their union $A + B$ is well-founded if, and only if, A is.*

[1] These are the non-strongly-normalizing elements. The term "immortal" was suggested by Philip K. Hooper ("The undecidability of the Turing machine immortality problem", *J. of Symbolic Logic* **31**(2), June 1966, pp. 219–234) for instantaneous descriptions and is much more succinct and perspicacious than "non-strongly normalizing".

Proof Sketch. Call an immortal point *constricted* if all its A-neighbors (one A-step away) are mortal. If A is well-founded, there is such a point s and it must have immortal B-neighbors. Among those, consider a t that is minimal with respect to A. It must likewise be constricted, because were t to have an immortal A-neighbor u ($s\ B\ t\ A\ u$), then, by jumping, u would be a smaller B-neighbor of s (constricted s has no immortal A-neighbors) than is t. In this way, every constricted point initiates an infinite B-chain of constricted points, whence there can be no escape. □

2.2. The Abstract Ordering

We apply the main theorem to an abstract version of the path orderings. What distinguishes the various path orderings in the literature is the use of a surface ordering, like a precedence or numeric interpretation, together with a recursive comparison of subterms.

Suppose the set V of points (think terms, if you wish) is equipped with a well-founded partial ordering $\rhd$ that provides some structure on V (think subterms, or "special" subterms of some sort, or something more elaborate). This will provide our A steps. Suppose further that we have some kind of "surface" relation $\gg$ on V (think precedences, lexicographically ordered tuples, limited forms of rewriting, and the like). Let's call t' a *descendant* of V if $t \rhd t'$ and say that u *follows* t if $t \gg u$. We define t to be bigger than u in the *(abstract) path ordering*, written here as $t \succ u$ ($t, u \in V$), if either of two cases holds, in a "two-dimensional" recursive way. The second case will be our B-steps; descent ($\rhd$) will be our A-steps.

Definition 3 (Path Ordering). The *(abstract) path ordering (apo)* is a relation $\succ$ (not necessarily transitive) on some set V, parameterized by three other abstract relations, $\gg$, $\vartriangleright$, and well-founded $\rhd$. It is the smallest relation $\succ$ such that

$$t \succ u \quad \text{if} \quad \begin{cases} t \vartriangleright u \text{ and } t \rhd^{+} \succ^{*} u, \text{ or} & \text{(a)} \\ t \gg u \text{ and } t\ (\rhd^{+} \succ^{*} + \succ)/\rhd\ u\,. & \text{(b)} \end{cases}$$

The division-like notation is defined, for binary relations R and S, as follows:

$$R/S = \{\langle x, y\rangle : \forall z.\ ySz \Rightarrow xRz\}\ .$$

In other words, $x(R/S)y$ means that xRz for all z such that ySz. (Division is the upper adjoint [residual mapping] of joining with $\rhd$ with respect to inclusion.)

Strictly speaking, this abstract path ordering need not be an ordering, as it can be non-transitive. It is a generalization of the abstract ordering of [18].

Loosely speaking, u is smaller than t if (a) it is smaller than a descendant of t or (b) it follows t and each of its own descendants is smaller than some descendant of t. The purpose of $\vartriangleright$ is to optionally constrain the first case, so the ordering need not have the *abstract* "subterm" property $\rhd \subseteq \succ$. For the defined relation to be useful, we will soon connect the followed-by relation $\gg$ with the bigger-than relation $\succ$ on descendants via $\rhd$, in a fashion that guarantees well-foundedness.

Lemma 4. *For the path ordering, relation $\rhd$ jumps over $\sqsupset$, where $\sqsupset$ is case* (b) *of the path ordering of Definition 3.*

PROOF. By the division in (b), $\sqsupset\,\rhd \subseteq \rhd^+\succ^* + \succ$. By the definition, $\succ \subseteq \rhd^+\succ^* + \sqsupset$, giving $\sqsupset\,\rhd \subseteq \rhd^+\succ^* + \sqsupset$, exactly as required. ☐

If $\vartriangleright$ is universal (the full Cartesian product), then $\succ$ does obey the subterm condition. If, in addition, following ($\gg$) is transitive, then $\succ$ is also. In that case, we have a considerably simpler definition, namely:

$$\succ \;:=\; \rhd\succeq + [\gg \cap \succ/\rhd]\ ,$$

where $\succeq$ is the reflexive closure of $\succ$.

Theorem 5. *The path ordering $\succ$ is transitive if $\gg$ is transitive and $\vartriangleright$ is universal.*[2]

PROOF. Let $\sqsupset$ be short for $\gg \cap \succ/\rhd$. We proceed by induction with respect to well-founded $\rhd$ in any of the three positions, s, t, or u in $s \succ t \succ u$. (i) If $s \rhd s' \succeq t$, then $s' \succ u$ by induction in the first position and $s \succ u$ by definition. (ii) If $s \sqsupset t \rhd t' \succeq u$, then $s \succ t' \succeq u$ on account of the division clause and $s \succ u$ by induction in the second position. (iii) Finally, if $s \sqsupset t \sqsupset u$, then we have $s \gg u$ and $s \succ t' \succ v$ for all $v \lhd u$. By induction in the third position, $s \succ v$ for all $v \lhd u$, from which it follows that $s \sqsupset u$ and, hence, $s \succ u$.

Being that $\succ$ is transitive and enjoys the subterm property ($\rhd \subseteq \succ$), the simpler definition (with $\rhd\succeq$) is equivalent to the original (with $\rhd^+\succ^*$). ☐

2.3. Well-Definedness

The following is an alternative mutually-recursive definition of $\succ$, which, together with its transitive closure $\succ^*$, can be implemented "bottom-up":

$$\succ \;:=\; (\vartriangleright \cap \rhd^+\succ^*) + \sqsupset \qquad (a')$$

$$\sqsupset \;:=\; \gg \cap\, (\rhd^+\succ^* + \sqsupset)/\rhd\ . \qquad (b')$$

We can have $\sqsupset$ on the right side of the second line instead of $\succ$ as appears in case (b) of the original definition of $\succ$, since case (a) of $\succ$ is subsumed by the first alternative, $\rhd^+\succ^*$. Note that (by induction on the computation) $\succ \subseteq (\rhd + \sqsupset)^+$.

As we will see, $\gg$ is often itself defined in terms of $\succ$, lexicographically on proper subterms, for instance. The path ordering is still well-defined, and may be viewed in the following stratified fashion, with the empty relation serving for the base cases:

$$\succ_n \;:=\; (\vartriangleright \cap \rhd^+\succ^*_{n-1}) + \sqsupset_n + \succ_{n-1} \qquad (a'')$$

$$\sqsupset_n \;:=\; \gg_n \cap\, (\rhd^+\succ^*_{n-1} + \sqsupset_{n-1})/\rhd + \sqsupset_{n-1} \qquad (b'')$$

$$\gg_n \;:=\; \succ^{\text{lex}}_{n-1} + \gg_{n-1}\ , \qquad (c'')$$

where $\succ^{\text{lex}}_{n-1}$ looks at certain $\succ_{n-1}$ relations between $\rhd$-neighbors of the points in question.

[2] I thank Alfons Geser for this result.

2.4. Well-Foundedness

Modularity of termination follows from jumping (Theorem 53 below with $\rhd$ for A and $\sqsupset$ for B). In other words, $\succ \subseteq (\rhd + \sqsupset)^+$ is well-founded if $\sqsupset$ is, since $\rhd$ is. Of course, $\sqsupset$ is well-founded if the "followed by" relation $\gg$ is. So:

Proposition 6. *A path ordering $\succ$ is well-founded whenever $\gg$ is.*

This works, as is, for simple interpretation-based termination orderings, as when $\gg$ compares terms on the basis of an arbitrary interpretation in the naturals. The problem is that, for path orderings, $\gg$ is normally defined recursively in terms of $\succ$ applied to subterms.

We will establish the following (by Lemma 4 and Corollaries 48 and 59):

Theorem 7. *A path ordering $\succ$ is well-founded if $\rhd$ lifts to $\sqsupset$.*

A minori ad maius this is so if it lifts to $\gg$.

This applies to the nested multiset ordering [12], where $\gg$ is an element ordering combined with its nested multiset extensions, since the first item of an infinite sequence of multisets must have an element that is immortal in the same ordering. So $\gg$ is well-founded if its elemental part is. The general case of such "lifted" definitions was first studied in [21] and pursued further in [15, 18].

Using the more powerful Theorem 60 below, we arrive inescapably at this:

Theorem 8. *A path ordering $\succ$ is well-founded if $\rhd$ escapes from $\sqsupset$.*

For the multiset [8] path ordering, $\rhd$ is the transitive proper-subterm relation combined with extracting the head symbol; $\vartriangleright$ is always true; and $\gg$ is the recursive lifting of $\succ$ to first compare precedences and then the multiset of subterms. For the lexicographic [21] and (mixed) recursive path orderings [24, 9], $\gg$ looks at precedence-cum-tuples lexicographically, and at a mixture of tuples and multisets, respectively, from both of which subterm ($\rhd$) escapes.

2.5. A Non-transitive Example

As a simple example of a non-transitive "ordering", we show the well-foundedness of a relation $> \; = \; \rhd >^* + \gg$ over binary strings $\{a, b\}^*$, defined by the axioms $az \rhd z$, $baz \rhd z$, $bbz \rhd bz$, $az \gg \varepsilon$, and $bab \gg \varepsilon$, and the rules

$$\frac{x >^+ y}{ax \gg ay}\ (i) \qquad \frac{x >^+ y}{bax \gg ay}\ (ii) \qquad \frac{ax \geq y}{ax \gg by}\ (iii)\ ,$$

where x, y, and z are string variables and ε is the empty string. We have $\cdots > bbb > bb > b$ and $aa > \cdots > baa > \cdots > abba > \cdots > aba > \cdots > abab > a > \cdots > bba > ba > \cdots > abb > ab > \cdots > bbab > bab > \varepsilon$, which is not total ($a$ and b are unrelated), does not have the subterm property ($a > ba$), and is not monotonic ($bba > ba$). (Recall that a total well-ordering cannot enjoy both monotonicity and subterm at one and the same time [8, p. 286].)

Clearly $\rhd$ is well-founded. Furthermore, $\rhd$ jumps over $\gg$: (i) If $ax \gg ay \rhd y$ on account of $x >^+ y$, then $ax \rhd x >^+ y$. (ii) If $bax \gg ay \rhd y$ on account of

$x >^+ y$, then $bax \rhd x >^+ y$. (*iii*) For the third rule, y might begin with an a or a b. It cannot be ε, since there is nothing smaller under $\rhd$. If $ax \gg baz \rhd z$ on account of $ax \geq az$, then $ax \rhd x >^* z$, whether $ax > az$ because $ax \gg az$ (in which case, $x >^+ z$), or because $ax \rhd >^* az$ (in which case $ax \rhd x >^* az \rhd z$), or if $ax = az$ (in which case $ax \rhd x = z$); if $ax \gg bbz \rhd bz$ on account of $ax \geq bz$, then actually $ax > bz$, since equality is precluded.

Lastly, $\rhd$ escapes from $\gg$. For one thing, there cannot be any use of the ε rules for $\gg$ in an infinite descending chain $s_1 \gg s_2 \gg \cdots$. There also cannot be consecutive words beginning with b, since there is no rule for that situation. Furthermore, were every s_i to begin with an a, then escape would be immediate: $s_1 \rhd x_1 >^+ x_2 >^+ \cdots$, where $s_i = ax_i$. So, the sequence is necessarily of the form $\cdots \gg ax_1 \gg bay_1 \gg az_2 \gg \cdots \gg ax_i \gg bay_i \gg az_{i+1} \gg \cdots$, with the $\gg$ pairs justified by $z_i >^* x_i$, $ax_i \geq ay_i$, and $y_i >^+ z_{i+1}$. Now, if any $ax_i \geq ay_i$ on account of $ax_i \rhd >^* ay_i$, then that constitutes an escape, since that continues $ay_i \gg az_{i+1} \gg \cdots$. This leaves only one possibility, viz. $ax_i \underline{\gg} ay_i$, implying $x_i >^* y_i$, for all i. But, again, there is an escape route: $ax_1 \rhd x_1 >^* y_1 >^+ z_2 >^* x_2 > \cdots$.

3. Mortality and Separation

As already mentioned, we denote the transitive and reflexive-transitive closures of a relation R by R^+ and R^*, respectively. For the reflexive closure, we will use R^ε. So, when we say that a sequence of steps has the "form" $A^*B^+A^\varepsilon$, for example, we mean that it comprises some number of A-steps, followed by one or more B-steps, followed by at most one A-step. Let R^n be the n-fold composition of R, $\varnothing$, the empty relation, and $\mathbf{1}$, the identity relation. For any R, then, $R^0 = \mathbf{1}$, $R^1 = R$, $R^{i+j} = R^i R^j$, $R\varnothing = \varnothing R = \varnothing$, and $R\mathbf{1} = \mathbf{1}R = R$.

3.1. Mortality

Define

$$R^\infty = \{\langle u, v\rangle \colon u, v \in V,\ u \text{ is immortal for } R\}\ ,$$

relating immortal elements to *all* vertices (as is commonly done in relational program semantics). Thus,

$$R^\infty S \subseteq R^\infty\ , \tag{1}$$

for any relation S. Relation R is well-founded, or *terminating*, if $R^\infty = \varnothing$. Clearly, $R^* R^\infty = R^\infty$.

We will make repeated use of *well-founded* (a.k.a. *noetherian*) induction in the form of the following:

Proposition 9. *If, for relations A, B, and C, one has $B \subseteq AB + C$, then $B \subseteq A^*C + A^\infty$.*

Proof. Suppose $s\ B\ t$, for points s and t. Either s is immortal in A, or else we can reason by induction with respect to A. We are told that either $s\ C\ t$ or $s\ A\ s'\ B\ t$. In the latter case, and for mortal s, we may assume that the induction hypothesis holds for s' (which must also be mortal in A), namely, that $s' \; A^*C \; t$, obtaining $s \; A^+C \; t$. Hence, $B \subseteq A^\infty + C + A^+C = A^*C + A^\infty$. □

This holds also when B may be nonterminating. In particular, if $B^\infty \subseteq AB^\infty + A^\infty$, then $B^\infty \subseteq A^\infty$, meaning that B is well-founded if A is.

3.2. Finite and Infinite Separations

Two main properties will command our attention.

Definition 10 (Finite Separation). Two relations A and B are *finitely separable* if

$$(A+B)^* = A^*B^* \; .$$

Definition 11 (Infinite Separation). Two relations A and B are *infinitely separable* if

$$(A+B)^\infty = (A+B)^*(A^\infty + B^\infty) \; .$$

The point is that infinite separability ensures that the union $E = A + B$ is well-founded if each of A and B is, even without finite separability.

Example 12. (a) An easy example of finite, but not infinite, separability is $s \; A \; t \; B \; s$. (b) A simple example of infinite, but not finite, separability is $s \; A \; s \; BA \; t$.

Nota bene. Finite separation is not a symmetric property: we want the A's before the B's. Infinite separation, on the other hand, is symmetric.

It is easy to see [30, Ex. 1.3.5(ii)] that finite separability (called "postponement" in [30]) is equivalent to the (global) "commutation" property $B^*A^* \subseteq A^*B^*$. Another trivial induction also shows that this is equivalent to an even simpler property:

Proposition 13 ([29])**.** *Two relations A and B are finitely separable if, and only if, $B^+A \subseteq A^*B^*$.*

This condition cannot be weakened further to the purely local property, $BA \subseteq A^*B^*$, which holds in the inseparable example, $s \; B \; t \; B \; u$ and $t \; A \; u \; A \; v$. (See, for example, [30, Ex. 1.3.2].)

A prettier shape of infinite separation is the following:

Definition 14 (Full Separation). Two relations A and B are *fully separable* if

$$(A+B)^\infty \quad = \quad A^*B^*A^\infty + A^*B^\infty \quad [= \; A^*B^*(A^\infty + B^\infty)] \; .$$

With finite separability, infinite separability is equivalent to full separability.

Clearly, if A and B are fully separable, and A is well-founded, then $E^\infty = A^*B^\infty$.

Definition 15 (Partial Separation). Two relations A and B are *partially separable* if

$$(A+B)^* \quad \subseteq \quad A^*B^* + A^*B^*A^\infty + A^*B^\infty \quad [= \quad A^*B^*(\mathbf{1} + A^\infty + B^\infty)] \; .$$

The following is straightforward:

Proposition 16. *Infinite separability plus partial separability give full separability.*

3.3. Infinite Chains

Obviously, infinite E-chains either have finitely many A's, with some B's in between, or else infinitely many A's.

Proposition 17. *For all relations A and B,*

$$(A+B)^\infty \quad = \quad (B^*A)^*B^\infty + (B^*A)^\infty \;\; = \;\; (A^*B)^*A^\infty + (A^*B)^\infty \; .$$

If A is well-founded, then

$$(A+B)^\infty \quad = \quad (A^*B)^\infty \; .$$

Cf. [2, Lemma 1] and [6, Lemma 12B]. It follows that if infinitely many interspersed B-steps are precluded (a property called *relative termination* in [23, Example 1.7 (11)]), then $A+B$ is well-founded if, and only if, A is.

3.4. Constriction

The point of "constriction" is to consider the impact on immortality of a color preference for steps. This is very useful in arguments for well-foundedness, as we will see.

Definition 18 (Constriction [27]). Let A and B be two (not necessarily disjoint) relations and let $E = A + B$. An infinite sequence $s_0 \; E \; s_1 \; E \; \cdots$ of A- and B-steps is *constricting in B* if wherever there is a B-step $s_i \; B \; s_{i+1}$ in the sequence, it is the case that all neighbors t, such that $s_i \; A \; t$, are mortal in E.

In other words, B-steps are "forced", in the sense that every possible A-step from the source point of a B-step (tail end of the arrow) in the sequence results in mortality. The use of a version of constriction was first suggested by Plaisted [27, proof of Lemma 2] for rewriting (A is subterm) and by Sørensen [28] for the lambda calculus.

One can always build a constricting sequence from an immortal element. Simply ignore B-steps whenever an immortalizing A-step is available.

Proposition 19 ([27]). *For relations A and B, if a point s is immortal in $A+B$, then there is an infinite B-constricting sequence in $A+B$ originating with s.*

Proof. Just take an A-step whenever possible, that is, whenever there is one that still leads to immortality. Only take B if there is no choice. □

We define a B-*sharp* step as a B-step from an immortal point from which no A-step leads to an immortal point:

$$B_\sharp \;=\; B \setminus AE^\infty \;.$$

Sharp, constricting steps are the only kind of B-step in a constricting sequence. So,

$$E^\infty \;=\; (A + B_\sharp)^\infty \;=\; (A^* B_\sharp)^* A^\infty + (A^* B_\sharp)^\infty \;. \qquad (2)$$

Escaping (Definition 2) can be characterized as the absence of infinite sequences of B-sharp steps.

Proposition 20. *If relation A escapes from relation B, then $B_\sharp^\infty = \varnothing$, where $B_\sharp$ are the sharp steps of a B-constricted sequence.*

4. Commutation

We begin with relatively simple cases of separability. We are looking for *local* conditions on double-steps BA that help establish separability, finite or infinite. The point is that these local conditions suggest how to eliminate BA patterns, which are exactly what are prohibited in separated sequences.

4.1. Finite Separation

Note that $CB^\varepsilon = C(B + \mathbf{1}) = CB + C$, and recall the following:

Proposition 21 ([19])**.** *Relations A and B are finitely separable if*

$$BA \subseteq A^* B^\varepsilon \;.$$

By symmetry (of A and B and left and right):

Proposition 22. *Relations A and B are finitely separable if*

$$BA \subseteq A^\varepsilon B^* \;,$$

When A is well-founded, the following property suffices for separation:

Definition 23 (Quasi-commutation [2]). Relation A *quasi-commutes over* relation B when

$$BA \subseteq A(A + B)^* \;.$$

The right-hand side, AE^*, is equal to $(AB^*)^+$, in that $(A + B)^* = B^*(AB^*)^*$.

Obviously (by induction):

Lemma 24. *Relation A quasi-commutes over relation B if, and only if, $B^* A \subseteq A(A + B)^*$.*

Theorem 25. *If relation A quasi-commutes over relation B, then*

$$(A+B)^+ \quad \subseteq \quad A^+B^* + B^+ + A^\infty \ .$$

This is a special case of partial separation (Definition 15).

Proof. Consider a nonempty E-sequence. If it is not already of the separated form $C = A^+B^*+B^+$, then it must contain an occurrence of B^+A, that is, $E^+ = C+A^*B^+AE^*$. By the above lemma, we have $E^+ \subseteq C+A^*AE^*E^* = AE^++C$. Applying Proposition 9, we get $E^+ \subseteq A^*C + A^\infty = A^+B^* + B^+ + A^\infty$. □

Corollary 26. *Quasi-commutation of a well-founded relation A over a relation B implies their finite separability.*

Quasi-commutation applies to combinatory logic and to orthogonal (left-linear non-overlapping) term-rewriting systems, where A is a leftmost (outermost) step and B is anything but. This means that leftmost steps may precede all non-leftmost ones. Combined with the fact—in these cases—that $A^-B \subseteq B^*A^-$, where A^- is the inverse (converse) relation, this gives *standardization* (leftmost rewriting suffices for computing normal forms). See [22].

4.2. Infinite Separation

Quasi-commutation is a simple local condition guaranteeing (a particularly simple form of) full separability:

Theorem 27 ([2, Thm. 1])**.** *If relation A quasi-commutes over relation B, then A and B are infinitely separable. More precisely, $(A+B)^\infty \ = \ A^\infty + A^*B^\infty$.*

Proof. Consider any infinite E-chain. It has an A or else is all B: $E^\infty = B^*AE^\infty + B^\infty$. By quasi-commutation and Lemma 24, we have: $E^\infty = AE^\infty + B^\infty$, and, by Proposition 9, we end up with $E^\infty = A^\infty + A^*B^\infty$. □

This form of separability was used in [7] to show that “forward closure” termination suffices for orthogonal term rewriting, where B are “residual” steps (at redexes already appearing *below the top* in the initial term) and A-steps are at “created” redexes (generated by earlier rewrites).

More generically:

Lemma 28. *If relations A and B are infinitely separable and if $B^\infty \subseteq A^+B^\infty + (A+B)^*A^\infty$, then $(A+B)^\infty = (A+B)^*A^\infty$.*

Proof. The right-to-left inclusion is trivial. Applying Proposition 9 to the premise, with A^+ for A, B^∞ for B and E^*A^∞ for C, we get $B^\infty \subseteq E^*A^\infty$. Then, by infinite separation, $E^\infty = E^*(A^\infty + B^\infty) \subseteq E^*(A^\infty + E^*A^\infty) = E^*A^\infty$. □

4.3. Production and Promotion

With finite separability, a non-empty cycling sequence ($t\ BA\ t$, as in Example 12a) can be "reordered" to give an empty, separated one. To preclude an empty reordering, one can insist on the following:

Definition 29 (Productive Separation). Relations A and B are *productively separable* if

$$(A+B)^+ \quad = \quad A^+B^* + B^+ \ \ [= \ A^+B^* + A^*B^+] \, .$$

Given finite separation, productive separation is equivalent to requiring every "mixed" cycle to be productively separable: $E^+ \cap \mathbf{1} \subseteq A^+B^* + B^+$.

We will see (Theorem 44) that productive separation implies full separation. We have already seen (Theorem 25) that quasi-commutation provides partial separation, in general, and productive separation in the situation where A is well-founded.

Definition 30 (Promotion). Relation B *promotes* relation A if

$$BA \ \subseteq \ AB^* + B^+ \ \ [= \ (A+B)B^*] \, .$$

Lemma 31. *Relation B promotes relation A if, and only if, $B^+A^n \subseteq (A^1 + A^2 + \cdots + A^n)B^* + B^+$ for all n. In particular, $B^+A \subseteq AB^* + B^+$.*

Proof. For the "if" direction, just take $n = 1$. For the other, assume promotion and first note that $B^*A \subseteq EB^*$, by an easy induction (on the quantity of Bs), since $BB^*A \subseteq BEB^* \subseteq BAB^* + B^+ \subseteq EB^* + B^+ = EB^*$. Now, assuming the claim for n (for 0 it's trivial), and letting $C_n = A^1 + \cdots + A^n$, we have $(B^+A^n)A \subseteq C_nB^*A + B^+A \subseteq C_nEB^* + EB^* = C_nAB^* + C_nB^+ + AB^* + B^+ = C_{n+1}B^* + B^+$, as required. □

Theorem 32. *If relation B promotes relation A, then A and B are productively separable.*

The local condition $BA \subseteq A^+B^* + B^+$, or even $BA \subseteq AA + BB$, is insufficient for separability, as the first example in Note 55 below shows.

Proof. By Proposition 22, promotion gives finite separability, $E^* = A^*B^*$. By the previous lemma, $B^*A \subseteq AB^* + B^+$, whence the theorem follows: $E^+ = E^*E = A^*B^*A + A^*B^*B = A^*(AB^* + B^+) + A^*B^+ = A^+B^* + B^+$. □

Note 33. One can have productivity without promotion: $s\ A\ t\ A\ u$ with $s\ B\ t$. Obviously, promotion cannot be weakened to allow the erasure of both the A and B, as can easily be seen from $t\ BA\ t$ (Example 12a), which is finitely, but not productively, separable.

In symmetry with promotion (exchanging A with B and left with right), we also have the following:

Corollary 34. *If, for relations A and B, $BA \subseteq A^*B + A^+$, then A and B are productively separable.*

Here is a somewhat analogous way of obtaining a "productive" version of partial separability (Definition 15), similar in flavor to promotion as just used:

Theorem 35. *If, for relations A and B,*

$$B^+A \quad \subseteq \quad A(A+B)^* + B^+ + A^*B^*A^\infty \ ,$$

then

$$(A+B)^+ \quad \subseteq \quad A^+B^* + B^+ + A^*B^*A^\infty \ .$$

Proof. Let $D = AE^* + A^*B^*A^\infty$. By induction on n, we obtain $B^+AE^n \subseteq D + B^+$, since $DE \subseteq E$, $B^+A \subseteq B^+ + D$ by hypothesis, and $B^+B \subseteq B^+$ by definition. Considering that E^+ must look like $A^*(A+B^+ + B^+AE^*)$, it follows that $E^+ \subseteq A^*(A+B^+ + D) = AE^+ + A + B^+ + A^*B^*A^\infty$, and, by Proposition 9, $E^+ \subseteq A^*(A + B^+ + B^*A^\infty) + A^\infty = A^+B^* + B^+ + A^*B^*A^\infty$. □

5. Selection and Jumping

For the abstract path ordering, we need to reason about reorderings of sequences of two kinds of steps ($\rhd$ and $\sqsupset$). That's where commutation comes into play, but we need something better than quasi-commutation.

5.1. Selection

First, a non-local commutation property, weaker than promotion, for which constriction yields separation.

Definition 36 (Selection). Relation B *selects* relation A if

$$BA^+ \subseteq A(A+B)^* + B^+ \ .$$

When A is transitive, as it is in many of the cases of relevance to path orderings (where A is some sort of special subterm relation), selection amounts to the pleasant local condition, $BA \subseteq AE^* + B^+$.

Definition 37 (Partial Selection). Relation B *partially selects* relation A if

$$BA^+ \subseteq A(A+B)^* + B^+ + A^\infty + A^*B^\infty \ .$$

Clearly, partial selection is weaker than selection.

A weaker form of partial selection suffices for a weak ("partial") form of productive separation:

Lemma 38. *If $BA^+ \subseteq A(A+B)^* + B^+ + (A+B)^\infty$, then $(A+B)^+ \subseteq A^+B^* + B^+ + (A+B)^\infty$.*

Proof. We reason by well-founded induction on E. A finite nonempty E-sequence is either of the separated form A^+ or is of the form A^*BE^*. Assuming the initial point is mortal, induction yields $A^*BE^* \subseteq A^*BA^*B^*$, and the premise gives $A^*(BA^+ + B)B^* \subseteq A^*(AE^* + B^+ + B)B^* = A^*(AE^* + B^+)$. Another application of induction results in $A^*(AA^*B^* + B^+)$. All together, $E^+ \subseteq A^+ + E^\infty + A^+B^* + A^*B^+ = A^+B^* + B^+ + E^\infty$. □

Restricting attention to constricting sequences gives full separation:

Theorem 39. *If relation B partially selects relation A, then A and B are partially separable and fully separable.*

Proof. By the above lemma, $E^* \subseteq A^*B^* + E^\infty$, so it remains only to show full separation of E^∞. We know (reordering Eq. 2) that $E^\infty \subseteq A^*(B_\sharp A^*)^*A^\infty + A^*(B_\sharp A^*)^\infty$. By assumption, $B_\sharp A^* \subseteq AE^* + B^+ + A^\infty + A^*B^\infty$, but in such an infinite constricting sequence, $B_\sharp A^*$ cannot be replaced by anything beginning with A, so only B^+ and B^∞ are possible. We have, therefore, $E^\infty \subseteq A^*(B^+ + B^\infty)^*A^\infty + A^*(B^+ + B^\infty)^\infty = A^*B^\infty + A^*B^*A^\infty$. □

Note 40. Weaker versions of selection do not yield partial separability. To wit, $s\ BA\ t\ A\ t$ is not partially separable, though $BA^+ \subseteq B^*A^\infty$. And $s\ AB\ s\ BA\ t$ is not, despite the fact that $BA^+ \subseteq AE^* + B^*$.

Lemma 41. *If $BA^+ \subseteq A(A+B)^* + B^+ + (A+B)^*(A^\infty + B^\infty)$, for relations A and B, then they are infinitely separable.*

This weaker condition clearly does not suffice for full separation.

Proof. By the premise and constriction, any steps $B_\sharp A^*$ in an infinite constricting E-chain may be replaced by $B^+ + E^*(A^\infty + B^\infty)$, since AE^* is precluded. For infinitely many such steps, we have $(B_\sharp A^*)^\infty \subseteq (B^+ + E^*(A^\infty + B^\infty))^\infty \subseteq E^*(A^\infty + B^\infty)$. Looking at an infinite constricted sequence, we see that $E^\infty \subseteq A^*(B_\sharp A^*)^*A^\infty + A^*(B_\sharp A^*)^\infty \subseteq E^*A^\infty + E^*B^\infty$. □

With well-foundedness, Lemma 38 and Theorem 39 combine to provide productive separation:

Theorem 42. *If relation B selects relation A and both are well-founded, then A and B are productively separable and their union $A + B$ is also well-founded.*

Note 43. Without well-foundedness of A, one does not have separation, as may be seen from the following selecting, but inseparable, example: $s\ A\ s\ BA\ t$. Well-foundedness of B is also needed. To wit: $t\ A\ s\ B\ s\ B\ t\ BA\ u$.

Actually:

Theorem 44. *Whenever relations A and B are productively separable, they are also fully separable.*

Proof. Productive separability ($E^+ \subseteq A^+B^* + B^+$) implies that B^+ selects A^+ (i.e. $B^+A^+ \subseteq A^+E^* + B^+$), which means, thanks to Theorem 39, that A^+ and B^+ are fully separable, which is the same as A and B themselves being fully separable, since, for any R, $(R^+)^* = R^*$ and $(R^+)^\infty = R^\infty$. □

In particular, this is true for promotion (Definition 30).

Corollary 45. *If relation B promotes relation A, then A and B are fully separable.*

Theorem 32 provided productive separability in this case.

Promotion cannot be weakened to allow the erasure of both the A and B, as can be seen from $t\ BA\ t$, which only cycles in the union. Compare Note 33.

5.2. Jumping

A notion of "absorption" is convenient:

Definition 46. Relation X *absorbs* relation A if $XA \subseteq X$.

For example, A^*, E^*, B^∞, and $\varnothing$ all absorb A. (Recall Eq. 1.) Obviously, if X absorbs A, then it absorbs any number of A's, and CX absorbs A whenever X does.

Proposition 47. *If, for relations A, B, and X, $BA \subseteq B + X$ and X absorbs A, then $BA^* \subseteq B + X$.*

Proof. By induction on n, $BA^n \subseteq B + X$, since, trivially, $B \subseteq B + X$, while, by the inductive hypothesis and absorption, $BA^nA \subseteq (B+X)A = BA + XA \subseteq B + X$. □

Recall that jumping (Definition 1) means $BA \subseteq AE^* + B$. The following is a corollary of the above proposition, with $X = AE^*$:

Corollary 48 ([14, Eq. 4.5])**.** *If relation A jumps over relation B, then $BA^* \subseteq A(A + B)^* + B$.*

In short, jumping is stronger than selection (Definition 36).

The essence of the method espoused here is captured by the following observation.

Proposition 49. *If relation A jumps over relation B, then $(B_\sharp A^*)^\infty = B_\sharp^\infty$. where $B_\sharp$ are the sharp steps of a B-constricted sequence.*

Proof. Jumping, per the above corollary, says that $B_\sharp A^* \subseteq AE^* + B_\sharp$, but constriction says that an A-step is impossible at any such point in the infinite sequence. So, it must be that $B_\sharp A^* = B_\sharp$. □

Note 50. Jumping is noticeably weaker than quasi-commutation (Definition 23), which was known from [2] to separate well-foundedness of the union into well-foundedness of each. Theorem 39 is stronger. As pointed out in [13], jumping is also much weaker than transitivity of the union, which—by a direct invocation of a very simple case of the infinite version of Ramsey's Theorem—gives infinite separation. (See [16, 5] for some of the history of this idea.) Note that replacing BA with $AABB$ is a process that can continue unabated: BBA, $BAABB$, $AABBABB$, $AABAABBBB$, ….

A weaker version of jumping, which allows for infinite exceptions is the following (cf. the analysis in [26]):

Definition 51 (Partial Jumping). Relation A *partially jumps over* relation B if

$$BA \quad \subseteq \quad A(A+B)^* + B + A^*B^*(A^\infty + B^\infty) \; .$$

Theorem 52. *If relation A partially jumps over relation B, then A and B are fully separable.*

Proof. Applying Proposition 47 (E^*, A^∞, and B^∞ each absorb A), partial jumping implies $BA^* \subseteq AE^* + B + A^*B^*(A^\infty + B^\infty)$. By Lemma 41, this yields infinite separation. For full separation, consider a constricting sequence, in which only $B_\sharp A^* \subseteq B_\sharp + B^*(A^\infty + B^\infty)$ is possible. There are either finitely many sharp steps or infinitely many: $E^\infty \subseteq A^*(B_\sharp A^*)^*A^\infty + A^*(B_\sharp A^*)^\infty \subseteq A^*(B+B^*(A^\infty+B^\infty))^*A^\infty+A^*(B+B^*(A^\infty+B^\infty))^\infty = A^*B^*(A^\infty+B^\infty)$. □

Partial jumping gives only infinite separability, not finite or partial separability (Note 40). Jumping, itself, does, however. Stringing together the fact that jumping implies selection (Corollary 48) and that (partial) selection suffices for both partial and full separation (Theorem 39), we arrive at the following:

Theorem 53. *If relation A jumps over relation B, then A and B are partially and fully separable.*

The fact that jumping implies infinite separability is the main result of [13]. In other words, if A jumps over B, then E is well-founded whenever A and B are.

The example $s\ B\ t\ A\ t$ allows jumping, since $BA \subseteq B$. So, the relations are fully separable, but there must be a B-step before an A-step for immortality of s. A similar example, with a simple infinite chain, rather than a cycle, is $s\ B\ t_i$ and $t_i\ A\ t_{i+1}$, for all i. Jumping is clearly essential: for instance, $s\ BA\ s$ is not infinitely separable. The simple case $BA \subseteq AA + BB$ does not provide full separation. Just wrap the counterexample of Note 55 around itself: $s\ B\ t\ B\ u\ A\ s$ and $t\ A\ u$. So, in particular, jumping cannot be weakened to $BA \subseteq AE^* + B + BB$.

As a corollary, if $BA \subseteq A^*B$, then A and B are fully separable. This was used to show that "forward closure" termination suffices for right-linear term rewriting, where B-steps are "created" and As are "residual" ones [7]. When B

is the subterm relation, this condition is implied by quasi-monotonicity (weak monotonicity) of A (cf. [8, 1]). This condition, equivalent to $BA \subseteq A^+B + B$, is much more demanding than jumping. With jumping, an occurrence of BA need not *always* leave an A (as in quasi-commutation), nor *always* a B (as here), but might *sometimes* leave one and other times the other.

With well-foundedness, partial jumping turns into jumping. So, as a consequence of Theorem 42, we get productive separability (Definition 29):

Theorem 54. *If relation A jumps over relation B, and both are well-founded, then A and B are productively separable.*

Well-foundedness of A and of B are necessary, even in the presence of jumping. Refer to the examples in Note 43.

Note 55. Jumping cannot be weakened to $BA \subseteq AE^* + B^+$. Even $BA \subseteq AA + BB$ causes trouble, as can be seen from the (finitely) inseparable example $s\ B\ t\ B\ u\ A\ v$, and $t\ A\ u$. Jumping also cannot be weakened to $BA \subseteq AE^* + B^\varepsilon$, as was seen earlier (Note 40) from the inseparable example: $s\ AB\ s\ BA\ t$.

6. Lifting and Escaping

For various applications of commutation arguments, the well-foundedness of one relation is dependent on that of the other.

6.1. Lifting

Definition 56 (Lifting). Relation A *lifts to* relation B if

$$B^\infty \quad \subseteq \quad A(A+B)^\infty \ .$$

Equivalently, $B^\infty \setminus AE^\infty = B_\sharp B^\infty = \varnothing$. Regarding lifting, see [21, 15, 18]. In particular, A lifts to B if mortality of all A-neighbors of a point implies its own mortality.

Lifting means that, if there is an infinite E-chain, then there is one with infinitely many interspersed A-steps. Somewhat more generally:[3]

Lemma 57. *If $B^\infty \subseteq B^*A(A+B)^\infty$, for relations A and B, then $(A+B)^\infty = (B^*A)^\infty$.*

This is a stronger statement than the ones in Proposition 17.

Proof. We are given that $B^\infty \subseteq B^*AE^\infty = (B^*A)^+B^\infty + (B^*A)^\infty$. By Proposition 9, $B^\infty \subseteq ((B^*A)^+)^\infty + (B^*A)^\infty = (B^*A)^\infty$. Therefore, $E^\infty = (B^*A)^*B^\infty + (B^*A)^\infty = (B^*A)^\infty$. □

Theorem 58. *If relation A lifts to B and they separate nicely as $(A+B)^\infty = (A+B)^*A^\infty + A^*B^\infty$, then $(A+B)^\infty = (A+B)^*A^\infty$.*

[3] Personal communication of Alfons Geser.

In other words, $A + B$ is well-founded if A is.

Proof. We have that $E^\infty \subseteq A^* B^\infty + E^* A^\infty \subseteq A E^\infty + E^* A^\infty$. By Proposition 9, $E^\infty = A^\infty + A^* E^* A^\infty = E^* A^\infty$. ☐

Corollary 59. *If relation A lifts to relation B and B selects A, then $(A+B)^\infty = (A+B)^* A^\infty$.*

Proof. Use Theorems 39 and 58, bearing in mind that full separation is more than what is required by the theorem. ☐

6.2. Escaping

It turns out, however, that oftentimes (as in path orderings) we need a weaker alternative to lifting, in which the A-step need only take place *eventually*. This was captured by the crucial notion of *escaping* (Definition 2).

The following is our main result:

Theorem 60. *If relation A escapes from relation B and A jumps over B, then $(A + B)^\infty \subseteq (A + B)^* A^\infty$.*

Under these conditions, too, $A + B$ is well-founded as long as A is. Selection, however, is insufficient, without the stronger lifting, as in the previous corollary. To wit: $s\ A\ t$ and $s\ B\ t\ BB\ s$.

Proof. Consider any infinite constricting sequence. It is either of the form $E^* A^\infty$, in which case we are done, or else looks like $A^*(B_\sharp A^*)^\infty$. By jumping (Proposition 49), $A^*(B_\sharp A^*)^\infty \subseteq A^* B_\sharp^\infty$. But, by escape (Proposition 20), that option is impossible. ☐

6.3. Dependency Pairs

To capture the dependency-pair method of [1], let A be the well-founded immediate-subterm relation, I be inner rewriting, and D be dependency pairs, including outer rewrites (and including pairs with terms on the right that are headed by constructors). By the nature of rewriting, $IA \subseteq A + A(I + D)$; by the addition of dependencies, one has $DA \subseteq A + D$. Hence, A jumps over $B = I + D$, and $A + B$ is well-founded if B is. For termination of B, D-steps are made decreasing in some well-founded order (with constructor-headed terms minimal), while I-steps are non-increasing. So, any infinite B-chain must have a tail I^∞ of inner-only rewriting. But A (subterm) escapes I (inner rewriting); hence, escapes B. So, B (which contains the rewrite relation) must be terminating.

Since we only require A to jump over D, one can omit from D any pair that is contained in $A(A + B)^*$, including any whose right side is a reduct (by original rules or dependency pairs) of a proper subterm of the left (and not just subterms contained in A, as suggested in [10, n. 8]).

7. Conclusion

We are optimistic that the methods herein will help for advanced path orderings, like the general path ordering [11, 17] and higher-order recursive-path-ordering [20, 4], without recourse to reducibility/computability predicates. As pointed out in [10], there is an analogy between the use of reducibility predicates and the use in proofs of well-foundedness of constricting sequences.

Another avenue perhaps worth pursuing is to demonstrate modular termination by "completing" the two systems so that one jumps over the other. Combinations of transformations and orderings are used for termination proofs in [2, 3], and may benefit from such a perspective.

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