# COMPLETION WITHOUT FAILURE [1]

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**Abstract.** We present an "unfailing" extension of the standard Knuth-Bendix completion procedure that is guaranteed to produce a desired canonical system, provided certain conditions are met. We prove that this unfailing completion method is refutationally complete for theorem proving in equational theories. The method can also be applied to Horn clauses with equality, in which case it corresponds to positive unit resolution plus oriented paramodulation, with unrestricted simplification.

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# 1 Introduction

The design of efficient methods for dealing with the equality predicate is one of the major goals in automated theorem proving. Just adding equality axioms almost invariably leads to unacceptable inefficiencies. Instead, a number of special methods have been devised for reasoning about equality. Within resolution-based provers, demodulation, that is, using equations in only one direction to rewrite terms to a simpler form, is frequently employed (Wos, et al. 1967). Unfortunately, demodulation is an incomplete adhoc method. A complete method for handling equations is paramodulation (Robinson and Wos 1969) in which equational consequences are generated by using all equations in both directions. In general, paramodulation is difficult to control and may produce hosts of irrelevant or redundant formulas.

In this paper, we consider the purely equational case in which a theory is presented as a set of equations and one is interested in proving a given equation to be valid in that equational theory. Validity in equational theories is, of course, semi-decidable: an equation $s = t$ is true in all models of a (countable) set of equations $E$, if $s$ can be obtained from $t$ by using the axioms of $E$ to substitute equals for equals. In important special cases, validity can be decided using canonical (i.e., terminating Church-Rosser) rewrite systems that have the property that all equal terms (and only equal terms) simplify to an identical (canonical) form. Deciding validity in theories for which canonical systems are known (e.g., group theory) is thus easy and reasonably efficient.

Knuth and Bendix (1970) designed a procedure that attempts to construct a canonical rewrite systems from a given set of equational axioms. Hullot (1980) and Le Chenadec (1986) present a large number of systems derived in this manner. The Knuth-Bendix completion procedure suffers from two major problems, however. It must be supplied with a well-founded ordering which it uses to determine in which direction a generated equation is to be oriented into a one-way rule. Finding such an ordering is not always easy. Secondly, even when an appropriate ordering is chosen, the procedure may fail to find any canonical system, though one exists (Dershowitz, Marcus, and Tarlecki 1988). In this paper we address the latter problem by presenting an unfailing extension of the completion procedure.

Unfailing completion is guaranteed to produce the desired canonical system, provided certain conditions are met. It is also refutationally complete for equational theories and has the advantage over paramodulation in that terms are always kept in fully-simplified form and that fewer equational consequences need to be considered, since the ordering supplied to the procedure gives some measure of direction to the prover. The method works with all general purpose orderings that have been proposed for rewriting, e.g., polynomial interpretations and recursive path orderings. We demonstrate that unfailing completion can also be applied to Horn clauses with equality and prove that for such theories the inference rules of positive unit resolution and a strong restriction of paramodulation are refutationally complete, even in the presence of unrestricted simplification.

We follow the approach of Bachmair, Dershowitz, and Hsiang (1986) and Bachmair (1987), and formulate unfailing completion at an abstract level, as an equational inference system. Consequently, our results apply to a large class of procedures, not just a single version.

# 2 Definitions

We shall consider (first-order) *terms* over some set of operator symbols $\mathcal{F}$ and some set of variables $\mathcal{V}$. The symbols $s$, $t$, and $u$ denote terms; $f$ and $g$ denote operator symbols; and $x$, $y$, and $z$ denote variables. We assume that $\mathcal{F}$ contains at least one constant. Thus the set of *ground* terms, i.e., terms containing no variables, is non-empty. For example, if $+$ is a binary operator, $-$ is a unary operator, and 0 and 1 are constants, then $(-x+y)+0$ is non-ground and $1+0$ is ground.

A subterm of a term $t$ is called *proper* if it is distinct from $t$. The expression $t/p$ denotes the subterm of $t$ at position $p$ (positions may, for instance, be represented in Dewey decimal notation). We write $t[s]$ to indicate that the term $t$ contains $s$ as a subterm and (ambiguously) denote by $t[u]$ the result of replacing a particular occurrence of $s$ by $u$.

By $t\sigma$ we denote the result of applying the substitution $\sigma$ to the term $t$, and call $t\sigma$ an *instance* of $t$. An instance $s$ of $t$ is *proper* if $t$ is not an instance of $s$. Thus, $-x+0$ and $x+x$ are proper instances of $x+y$, whereas $x+z$ is a non-proper instance.

A binary relation $\rightarrow$ on terms is *monotonic with respect to the term structure* if $s \rightarrow t$ implies $u[s] \rightarrow u[t]$, for all terms $s$, $t$, and $u$. It is *monotonic with respect to instantiation* if $s \rightarrow t$ implies $s\sigma \rightarrow t\sigma$, for all terms $s$ and $t$, and substitutions $\sigma$. A relation that satisfies both properties is simply called *monotonic*. The symbols $\rightarrow^{+}$, $\rightarrow^{*}$ and $\leftrightarrow$ denote the transitive, transitive-reflexive, and symmetric closure of $\rightarrow$, respectively. The inverse of $\rightarrow$ is denoted by $\leftarrow$. An (strict partial) *ordering* is an irreflexive and transitive binary relation. An ordering $\succ$ is *well-founded* if there is no infinite sequence $t_1 \succ t_2 \succ t_3 \succ \cdots$. A *reduction ordering* is a well-founded monotonic ordering on terms.

An *equation* is a pair of terms, written $s = t$. Given a set of equations $E$, we denote by $\leftrightarrow_E$ the smallest symmetric and monotonic relation that contains $E$. That is, $u \leftrightarrow_E v$ if and only if $u$ is $w[s\sigma]$ and $v$ is $w[t\sigma]$, for some term $w$, substitution $\sigma$, and equation $s \doteq t$ in $E$ ($s \doteq t$ ambiguously denotes $s = t$ or $t = s$). The relation $\leftrightarrow_E^*$ is the smallest monotonic congruence that contains $E$ (a congruence is, by definition, monotonic with respect to the term structure). We shall refer to $\leftrightarrow_E^*$ as the *equational theory* defined by $E$.

Directed equations are called *rewrite rules* and are written $s \rightarrow t$. By a *rewrite system* we mean a set $R$ of rewrite rules. The corresponding *rewrite relation* $\rightarrow_R$ is the smallest monotonic relation that contains $R$. That is, $u \rightarrow_R v$ ($u$ *rewrites* to $v$) if and only if $u$ is $w[s\sigma]$ and $v$ is $w[t\sigma]$, for some term $w$, substitution $\sigma$, and rewrite rule $s \rightarrow t$ in $R$. We will have occasion to write $u \rightarrow_{R,s\rightarrow t} v$ to indicate that $u$ rewrites to $v$ by application of a rule $s \rightarrow t$ in $R$. A term that can not be rewritten is said to be *irreducible* (with respect to $R$). By $NF(R)$ we denote the set of all irreducible terms. A *normal form* of $t$ is any irreducible term $u$ for which $t \rightarrow_R^* u$.

A rewrite system $R$ is *Church-Rosser* if, for all terms $s$ and $t$ with $s \leftrightarrow_R^* t$, there exists a term $u$, such that $s \rightarrow_R^* u \leftarrow_R^* t$. In a Church-Rosser system the normal form of a term, if it exists, is unique. A rewrite system $R$ *terminates* if the ordering $\rightarrow_R^+$ is well-founded. In an untyped terminating system all variables appearing on the right-hand side of a rule must also appear on the corresponding left-hand side. Terminating Church-Rosser systems are called *complete*. They define a unique normal form for each term.

## 3 The Knuth-Bendix Completion Procedure

Knuth and Bendix (1970) designed a procedure that attempts to construct a complete system for a given set of equations. Bachmair, Dershowitz and Hsiang (1986) and Bachmair (1987) have reformulated the completion procedure as an equational inference system.

Let $\succ$ be a reduction ordering on terms. *Standard completion* consists of the following *inference rules*, where $E$ may be any set of equations and $R$ any rewrite system contained in $\succ$:

$$\textit{Orientation:} \quad \frac{(E \cup \{s \doteq t\}, R)}{(E, R \cup \{s \to t\})} \quad \text{if } s \succ t \tag{1}$$

$$\textit{Deduction:} \quad \frac{(E, R)}{(E \cup \{s = t\}, R)} \quad \text{if } s \leftarrow_R u \to_R t \tag{2}$$

$$\textit{Deletion:} \quad \frac{(E \cup \{s = s\}, R)}{(E, R)} \tag{3}$$

$$\textit{Simplification:} \quad \frac{(E \cup \{s \doteq t\}, R)}{(E \cup \{u \doteq t\}, R)} \quad \text{if } s \to_R u \tag{4}$$

$$\textit{Composition:} \quad \frac{(E, R \cup \{s \to t\})}{(E, R \cup \{s \to u\})} \quad \text{if } t \to_R u \tag{5}$$

$$\textit{Collapse:} \quad \frac{(E, R \cup \{s \to t\})}{(E \cup \{v = t\}, R)} \quad \text{if } s \to_{R, l \to r} v \text{ and } s \rhd l \tag{6}$$

The symbol $\rhd$ denotes the *specialization* ordering: $s \rhd l$ *if and only if some subterm of* $s$ *is an instance of* $l$*, but not vice versa.*

We write $(E, R) \vdash (E', R')$ to indicate that the pair $(E', R')$ can be obtained from $(E, R)$ by an application of an inference rule. A (possibly infinite) sequence $(E_0, R_0) \vdash (E_1, R_1) \vdash \cdots$ is called a *derivation* from $(E_0, R_0)$. The *limit* of a derivation is the pair $(E^\infty, R^\infty)$ of the set $\bigcup_i \bigcap_{j \geq i} E_j$ of all *persisting equations* and the set $\bigcup_i \bigcap_{j \geq i} R_j$ of all *persisting rules.*

A *completion procedure* is a program that accepts as input a set of equations $E_0$, a rewrite system $R_0$, and a reduction ordering $\succ$ containing $R_0$, and uses the above inference rules to generate a derivation from $(E_0, R_0)$. We say that a completion procedure *fails* for the given inputs, if $E^\infty \neq \emptyset$. A completion procedure is *correct*, if $R^\infty$ is complete whenever $E^\infty = \emptyset$.

A complete system provides a decision procedure for the validity problem in the given equational theory: two terms are equivalent if and only if they reduce to identical normal forms. The unsolvability of the word problem for certain (even finitely-based) equational theories implies that the construction of a complete system is not always possible. For example, theories with commutativity can usually not be represented as terminating systems. Hence, completion fails for such theories. A (correct) completion procedure may (i) construct a (finite) complete system, (ii) fail, or (iii) not terminate and instead compute successive approximations $R_n$ of an infinite complete system $R^\infty$. In this paper we address the problem of failure by presenting an unfailing extension of standard completion.

## 4 Unfailing Completion

Even if there exists no complete rewrite system for a given equational theory, it may still be possible to construct a system of equations with a certain Church-Rosser property on *ground* terms, so that every ground term has a unique normal form. Such a ground Church-Rosser property is sufficient for most purposes, including theorem proving.

Let us first refine the notion of rewriting. We call $u\sigma \to v\sigma$ an *orientable instance* (with respect to a reduction ordering $\succ$) of the equation $u = v$ if and only if $u\sigma \succ v\sigma$. By $E_\succ$ we denote the rewrite system of all orientable instances of equations in $E$. Thus, $s \to_{E_\succ} t$ if $s \leftrightarrow_E t$ by applying some equation $u = v$ at a position $p$ with a substitution $\sigma$, for which $u\sigma \succ v\sigma$. Evidently, $s \to_{E_\succ} t$ implies $s \succ t$.

A set of equations $E$ is said to be *ground Church-Rosser with respect to* $\succ$ if $s \leftrightarrow_E^* t$ implies $s \to_{E_\succ}^* v \leftarrow_{E_\succ}^* t$, for all ground terms $s$ and $t$. A system which is ground Church-Rosser with respect to some reduction ordering defines unique ground normal forms. Normal forms can be computed, provided $E$ is finite and the given reduction ordering is decidable.

The following inference rule will be needed to construct pairs $(E, R)$, such that $E \cup R$ is ground Church-Rosser with respect to a given reduction ordering $\succ$ :

$$\textit{Deduction}_2\text{:} \qquad \frac{(E, R)}{(E \cup \{s = t\}, R)} \qquad \text{if } s \leftrightarrow_{E\cup R} u \leftrightarrow_{E\cup R} t, \quad s \not\succ u, \text{ and } t \not\succ u \tag{7}$$

The deduction rule (2) is a special case of (7), since $s \leftarrow_R u \to_R t$ implies $u \succ s$ and $u \succ t$. Since orientable instances of equations can be regarded as rewrite rules, they can also be used for simplification:

$$\textit{Simplification}_2\text{:} \qquad \frac{(E \cup \{s \doteq t\}, R)}{(E \cup \{u \doteq t\}, R)} \qquad \text{if } s \to_{E_\succ, l=r} u \text{ and } s \rhd l \tag{8}$$

$$\textit{Composition}_2\text{:} \qquad \frac{(E, R \cup \{s \to t\})}{(E, R \cup \{s \to u\})} \qquad \text{if } t \to_{E_\succ} u \tag{9}$$

$$\textit{Collapse}_2\text{:} \qquad \frac{(E, R \cup \{s \to t\})}{(E \cup \{v = t\}, R)} \qquad \text{if } s \to_{E_\succ, l=r} v \text{ and } s \rhd l \tag{10}$$

Note the difference between the simplification rules (4) and (8). While the latter is restricted to cases in which $s \rhd l$, no such restriction is imposed on the former.

An *unfailing completion procedure* is a program that takes as input a reduction ordering $\succ$, a set of equations $E_0$, and a rewrite system $R_0$ contained in $\succ$, and uses the above inference rules, plus the inference rules for standard completion, to generate a derivation from $(E_0, R_0)$. Unfailing completion is *sound*:

**Proposition 1 (Soundness)** *If $(E, R) \vdash (E', R')$ in unfailing completion, then the congruence relations $\leftrightarrow^*_{E \cup R}$ and $\leftrightarrow^*_{E' \cup R'}$ are the same.*

We shall adopt the approach of Bachmair, Dershowitz, and Hsiang (1986) and Bachmair (1987) of viewing completion as a process of *proof simplification*, to derive conditions under which unfailing completion is guaranteed to find a ground Church-Rosser system.

# 5 Proof Simplification

By a *proof* of $s = t$ in $E \cup R$ (or a proof $s \leftrightarrow^*_{E \cup R} t$ ) we mean a sequence of terms $(t_0, \ldots, t_n)$, such that $t_0$ is $s$, $t_n$ is $t$ and, for $1 \leq i \leq n$, one of $t_{i-1} \leftrightarrow_E t_i$, $t_{i-1} \rightarrow_R t_i$, or $t_{i-1} \leftarrow_R t_i$ holds. Every single proof step $(t_{i-1}, t_i)$ has to be justified by an equation $u_i = v_i$, a substitution $\sigma_i$, and a position $p_i$, such that $t_{i-1}/p_i$ is $u_i\sigma_i$, $t_i$ is $t_{i-1}[v_i\sigma_i]$ (where the replacement takes place at position $p_i$), and $u_i \doteq v_i$ is in $E \cup R$. The *justification* of a proof is the sequence of tuples $(t_{i-1}, t_i, u_i, v_i, \sigma_i, p_i)$, $1 \leq i \leq n$. We shall be concerned with *justified proofs*, but for simplicity usually leave the justification implicit or indicate it only partially by writing, for instance, $t_0 \leftrightarrow_E t_1 \rightarrow_R \cdots \leftarrow_R t_n$.

The symbols $P$ and $Q$ are used to denote (justified) proofs. Let $P$ be a proof $(t_0, \ldots, t_n)$. By $P^{-1}$ we mean the proof $(t_n, \ldots, t_0)$; by $P\sigma$, the proof $(t_0\sigma, \ldots, t_n\sigma)$; and by $u[P]$, the proof $(u[t_0], \ldots, u[t_n])$. A *subproof* of $P$ is any proof $(t_i, \ldots, t_j)$, where $0 \leq i \leq j \leq n$. We write $P[Q]$ to indicate that $P$ contains $Q$ as a subproof. A proof step $s \leftrightarrow_E t$ is called an *equality step*; a step $s \rightarrow_R t$ or $s \leftarrow_R t$, a *rewrite step*; a proof $s \leftarrow_R u \rightarrow_R t$, a *peak*. We usually abbreviate a proof of the form $t_0 \rightarrow_R \cdots \rightarrow_R t_n$ by $t_0 \rightarrow^*_R t_n$, and call a proof $t_0 \rightarrow^*_R t_k \leftarrow^*_R t_n$ a *rewrite proof*.

A binary relation $\Rightarrow$ on proofs is *monotonic with respect to the proof structure* if $Q \Rightarrow Q'$ implies $P[Q] \Rightarrow P[Q']$, for all proofs $P$, $Q$, and $Q'$. It is *monotonic with respect to the term structure* if $P \Rightarrow Q$ implies $u[P] \Rightarrow u[Q]$, for all proofs $P$ and $Q$ and terms $u$; and *monotonic with respect to instantiation* if $P \Rightarrow Q$ implies $P\sigma \Rightarrow Q\sigma$, for all proofs $P$ and $Q$ and substitutions $\sigma$. A relation satisfying all three properties is called *monotonic*. A *proof* *reduction ordering*, or simply *proof ordering*, is a well-founded monotonic ordering on proofs.

The inference rules for unfailing completion induce certain *proof transformations* that can be described by (conditional) rewrite rules on proofs. For example, the inference rules for orientation (1) and deletion (3) are reflected on the proof level by the following rewrite rules:

$$\begin{array}{lcll} s \leftrightarrow_E t & \Rightarrow & s \rightarrow_R t & \text{if } s \succ t \\ s \leftrightarrow_E t & \Rightarrow & s \leftarrow_R t & \text{if } t \succ s \\ s \leftrightarrow_E s & \Rightarrow & s & \end{array}$$

Simplification (4) is mirrored by

$$\begin{array}{lcll} s \leftrightarrow_{E,l=r} t & \Rightarrow & s \rightarrow_{R,l' \rightarrow r'} u \leftrightarrow_E t & \text{if } l \trianglerighteq l' \\ s \leftrightarrow_{E,l=r} t & \Rightarrow & s \leftrightarrow_E u \leftarrow_{R,r' \leftarrow l'} t & \text{if } l \trianglerighteq l' \end{array}$$

where $l \trianglerighteq l'$ indicates that either $l \triangleright l'$, or $l$ and $l'$ are instances of each other. For composition (5) and collapse (6) we have

$$\begin{array}{lcll} s \rightarrow_{R,l \rightarrow r} t & \Rightarrow & s \rightarrow_{R,l \rightarrow r'} u \leftarrow_R t & \\ s \leftarrow_{R,r \leftarrow l} t & \Rightarrow & s \rightarrow_R u \leftarrow_{R,r' \leftarrow l} t & \\ s \rightarrow_{R,l \rightarrow r} t & \Rightarrow & s \rightarrow_{R,l' \rightarrow r'} u \leftrightarrow_E t & \text{if } l \triangleright l' \\ s \leftarrow_{R,l \rightarrow r} t & \Rightarrow & s \leftrightarrow_E u \leftarrow_{R,r' \leftarrow l'} t & \text{if } l \triangleright l' \end{array}$$

The transformation rules for inference rules (8), (9), and (10) are similar to the rules for (4), (5), and (6). We list only one of the two symmetric cases for each:

$$\begin{array}{lcll} s \leftrightarrow_{E,l=r} t & \Rightarrow & s \rightarrow_{E_\succ,l'=r'} u \leftrightarrow_E t & \text{if } l \triangleright l' \\ s \rightarrow_{R,l \rightarrow r} t & \Rightarrow & s \rightarrow_{E_\succ,l=r'} u \leftarrow_R t & \\ s \rightarrow_{R,l \rightarrow r} t & \Rightarrow & s \rightarrow_{E_\succ,l'=r'} u \leftrightarrow_R t & \text{if } l \triangleright l' \end{array}$$

Among all the proof transformations induced by the deduction rule (7), we are only interested in the following:

$$s \leftarrow_{E_\succ \cup R} u \rightarrow_{E_\succ \cup R} t \quad \Rightarrow \quad s \leftrightarrow_E t.$$

In addition, we will need another rule

$$s \leftarrow_{E_\succ \cup R} u \rightarrow_{E_\succ \cup R} t \quad \Rightarrow \quad s \rightarrow^*_{E_\succ \cup R} v \leftarrow^*_{E_\succ \cup R} t$$

which specifies a transformation that can be performed without applying an inference rule (see the Critical Pair Lemma below). In all rules above, it is

Figure 1: Proof transformation rules

assumed that the rewrite system $R$ is contained in the given reduction ordering $\succ$. Representative proof transformation rules are depicted in Figure 1. By $\Rightarrow_{\mathcal{U}_\succ}$ (or simply $\Rightarrow_{\mathcal{U}}$) we denote the rewrite relation on proofs induced by the above rewrite rules. The connection between unfailing completion and the rewrite relation $\Rightarrow_{\mathcal{U}}$ can be formally expressed as follows.

**Lemma 1** *Whenever $(E, R) \vdash (E', R')$ in unfailing completion and $P$ is a proof in $E \cup R$, then there exists a proof $P'$ in $E' \cup R'$, such that $P \Rightarrow_{\mathcal{U}}^{*} P'$.*

In other words, completion can be interpreted as a process of proof transformation. Moreover, the ordering $\Rightarrow_{\mathcal{U}}^{+}$ can be shown to be well-founded. Hence, we may speak of *proof simplification*. The concept of *multiset orderings* is useful in this context.

A *multiset* is an unordered collection of elements in which elements may appear more than once. If $\succ$ is a partial ordering on a set $S$, then the corresponding multiset ordering $\succ_M$ on the set of all finite multisets of elements in $S$ is the smallest transitive relation such that

$$N \cup \{x\} \succ_M N \cup \{y_1, ..., y_n\}, \text{ whenever } n \geq 0 \text{ and } x \succ y_i, \text{ for } 1 \leq i \leq n.$$

According to this ordering an element of a multiset can be replaced by any finite number of elements that are smaller in $\succ$. Dershowitz and Manna (1979) have shown that *the multiset ordering $\succ_M$ is well-founded if and only if $\succ$ is well-founded.*

**Lemma 2** *The ordering $\Rightarrow_{\mathcal{U}}^{+}$ is a proof reduction ordering.*

*Proof.* We have to prove that $\Rightarrow_{\mathcal{U}}^{+}$ is well-founded. We first define a complexity measure $c(s,t)$ on single proof steps $s \leftrightarrow_{E \cup R} t$ as follows:

$$c(s,t) = \begin{cases} (\{s\}, l, \{t\}) & \text{if } s \rightarrow_{R, l \rightarrow r} t \\ (\{t\}, l, \{s\}) & \text{if } s \leftarrow_{R, r \leftarrow l} t \\ (\{s\}, l, \{t, max\}) & \text{if } s \rightarrow_{E_\succ, l=r} t \\ (\{t\}, l, \{s, max\}) & \text{if } s \leftarrow_{E_\succ, r=l} t \\ (\{s,t\}, -, -) & \text{if } s \leftrightarrow_E t \end{cases}$$

where $max$ is a new symbol with $max \succ u$, for all terms $u$. (Only the first component is relevant in the last clause.) Let the ordering $\succ^c$ be the lexicographic combination of the multiset extension $\succ_M$ of the reduction ordering $\succ$, the specialization ordering $\rhd$, and the multiset ordering $\succ_M$. (Two terms that are instances of each other are considered to be identical when compared in the specialization ordering.) We define: $(s_0, \ldots, s_m) \succ_{\mathcal{U}} (t_0, \ldots, t_n)$ *if and only if* $\{c(s_0, s_1), \ldots, c(s_{m-1}, s_m)\} \succ_M^c \{c(t_0, t_1), \ldots, c(t_{n-1}, t_n)\}$, where $\succ_M^c$ is the multiset extension of $\succ^c$. This ordering can easily be shown to be a proof reduction ordering. (Its well-foundedness is ultimately a consequence of the the well-foundedness of all components of $\succ^c$.) We next show that $\Rightarrow_{\mathcal{U}}$ is contained in $\succ_{\mathcal{U}}$.

i) $s \leftrightarrow_E t \succ_{\mathcal{U}} s \rightarrow_R t$, because $(\{s\}, l, \{t, max\}) \succ^c (\{s\}, l, \{t\})$.

ii) If $s \succ t$ and $l \unrhd l'$, then $s \leftrightarrow_{E, l=r} t \succ_{\mathcal{U}} s \rightarrow_{R, l' \rightarrow r'} u \leftrightarrow_E t$, because $s \succ u$ and $(\{s\}, l, \{t, max\}) \succ^c (\{s\}, l', \{u\})$.

iii) If $t \succ s$, then $s \leftrightarrow_{E, l=r} t \succ_{\mathcal{U}} s \rightarrow_{R, l' \rightarrow r'} u \leftrightarrow_{E, l''=r} t$, because $s \succ u$ and $\{(\{t\}, r, \{s, max\})\} \succ_M^c \{(\{s\}, l', \{u\}), (\{t\}, r, \{u, max\})\}$.

iv) If neither $s \succ t$ nor $t \succ s$, then $s \leftrightarrow_E t \succ_{\mathcal{U}} s \rightarrow_{E_\succ \cup R} u \leftrightarrow_E t$, because $s \succ u$ and $\{s, t\} \succ_M \{s\}$.

v) $s \leftrightarrow_E s \succ_{\mathcal{U}} s$, because $\{s, s\} \succ_M \emptyset$.

vi) $s \leftarrow_{E_\succ \cup R} u \rightarrow_{E_\succ \cup R} t \succ_{\mathcal{U}} s \leftrightarrow_E t$, because $u \succ s$ and $u \succ t$.

vii) $s \leftarrow_{E_\succ \cup R} u \rightarrow_{E_\succ \cup R} t \succ_{\mathcal{U}} s \rightarrow^*_{E_\succ \cup R} v \leftarrow^*_{E_\succ \cup R} t$, because $u$ is bigger than every term in $s \rightarrow^*_{E_\succ \cup R} v \leftarrow^*_{E_\succ \cup R} t$.

viii) $s \rightarrow_{R, l \rightarrow r} t \succ_{\mathcal{U}} s \rightarrow_{R, l \rightarrow r'} u \leftarrow_{R, r'' \leftarrow l''} t$, because $s \succ t \succ u$, and therefore $\{(\{s\}, l, \{t\})\} \succ^c_M \{(\{s\}, l, \{u\}), (\{t\}, l'', \{u\})\}$.

ix) If $l \rhd l'$, then $s \rightarrow_{R, l \rightarrow r} t \succ_{\mathcal{U}} s \rightarrow_{R, l' \rightarrow r'} u \leftrightarrow_E t$, because $s \succ t$, $s \succ u$, and $(\{s\}, l, \{t\}) \succ^c (\{s\}, l', \{u\})$.

x) If $s \succ t$ and $l \rhd l'$, then $s \leftrightarrow_{E, l=r} t \succ_{\mathcal{U}} s \rightarrow_{E_\succ, l'=r'} u \leftrightarrow_E t$, because $s \succ u$ and $(\{s\}, l, \{t, max\}) \succ^c (\{s\}, l', \{u, max\})$.

xi) If $t \succ s$, then $s \leftrightarrow_{E, l=r} t \succ_{\mathcal{U}} s \rightarrow_{E_\succ, l'=r'} u \leftrightarrow_{E, l''=r} t$, because $s \succ u$ and $\{(\{t\}, r, \{s, max\})\} \succ_{\mathcal{U}} \{(\{s\}, l', \{u, max\}), (\{t\}, r, \{u, max\})\}$.

xii) $s \rightarrow_{R, l \rightarrow r} t \succ_{\mathcal{U}} s \rightarrow_{R, l \rightarrow r'} u \leftarrow_{E_\succ} t$, because $s \succ t \succ u$ and $(\{s\}, l, \{t\}) \succ^c (\{s\}, l, \{u\})$.

xiii) If $l \rhd l'$, then $s \rightarrow_{R, l \rightarrow r} t \succ_{\mathcal{U}} s \rightarrow_{E_\succ, l'=r'} u \leftrightarrow_E t$, because $s \succ t$, $s \succ u$, and $(\{s\}, l, \{t\}) \succ^c (\{s\}, l', \{u, max\})$.

The assertion now follows from the monotonicity of $\Rightarrow_{\mathcal{U}}$ and $\succ_{\mathcal{U}}$.

## 6 Correctness of Unfailing Completion

Let $\succ$ be a reduction ordering and $R$ be a rewrite system contained in $\succ$. The set $E \cup R$ is ground Church-Rosser with respect to $\succ$ if there is a ground rewrite proof for every valid equation between ground terms. A ground rewrite proof with respect to $\succ$, on the other hand, is a proof containing no equality step $s \leftrightarrow_E t$, wherein $s$ and $t$ are incomparable with respect to $\succ$, and no peak $s \leftarrow_{E_\succ \cup R} u \rightarrow_{E_\succ \cup R} t$. An unfailing completion procedure will produce a ground Church-Rosser system if it applies inference rules in such a way that all undesirable subproofs are simplified.

If the ordering $\succ$ is total on equivalent ground terms, that is, if $u \leftrightarrow^*_E v$ implies $u \succ v$ or $v \succ u$, for all distinct ground terms $u$ and $v$, then certainly any two distinct equivalent terms are comparable. Thus, in that case, there is no problem with equality steps. We say that a reduction ordering is *complete for $E$* if any two distinct ground terms that are equivalent in $E$, are comparable in $\succ$. If a reduction ordering is total on the set of all ground terms then it is simply called *complete*.

For elimination of peaks it suffices to compute certain equational consequences called critical pairs. Let $s = t$ and $l = r$ be two equations with no variables in common (the variables of one equation are renamed if necessary) and suppose that, for some position $p$, $s/p$ is not a variable and is unifiable with $l$, $\sigma$ being the most general unifier. The proof $t\sigma \leftrightarrow_E s\sigma \leftrightarrow_E s\sigma[r\sigma]$, where the replacement in $s\sigma$ takes place at position $p$, is called a *critical overlap* of $l = r$ on $s = t$. Furthermore, if $t\sigma \not\succ s\sigma$ and $r\sigma \not\succ l\sigma$, then the equation $t\sigma = s\sigma[r\sigma]$ is called an *extended critical pair*. By $EP_\succ(E)$ we denote the set of all extended critical pairs between equations in $E$. The ordering $\succ$ restricts the number of critical overlaps defining extended pairs. Thus, if the ordering $\succ$ is contained in another ordering $>$, then $EP_>(E)$ is a subset of $EP_\succ(E)$.

For example, the two equations $(xy)(zw) = (xz)(yw)$ and $(xy)x = x$ overlap in $((uv)u)(v'v) \leftrightarrow_E ((uv)v')(uv) \leftrightarrow_E uv$ to define an extended critical pair $((uv)u)(v'v) = uv$ with respect to the subterm ordering. The usual definition of critical pairs is a special case of extended pairs. Some subtle points may arise with extended pairs. For instance, overlapping the equation $f(a) = g(x,a)$ on itself at the top results in a non-trivial extended pair $g(x,a) = g(y,a)$.

**Critical Pair Lemma.** *Let $\succ$ be a complete reduction ordering for $E$. For all ground terms $s$, $t$, and $u$ with $s \leftarrow_{E_\succ} u \rightarrow_{E_\succ} t$, there is a term $v$, such that either $s \rightarrow^*_{E_\succ} v \leftarrow^*_{E_\succ} t$, or else $s$ is $v[l\sigma]$ and $t$ is $v[r\sigma]$, for some extended critical pair $l = r$ in $EP_\succ(E)$.*

*Proof.* The lemma is a straightforward adaption of the Critical Pair Lemma in Knuth and Bendix (1970). We sketch the basic ideas. Let $P$ be a peak $s \leftarrow_{E_\succ} u \rightarrow_{E_\succ} t$. Since the assertion holds trivially if $s$ and $t$ are identical, we assume that they are distinct. We distinguish three types of peaks.

If the two proof steps in $P$ apply at disjoint positions, i.e., do *not overlap*, then they commute. In other words, the peak $s \leftarrow_{E_\succ} u \rightarrow_{E_\succ} t$ can be replaced by $s \rightarrow_{E_\succ} v \leftarrow_{E_\succ} t$, for some appropriate term $v$, as indicated in Figure 2.

If one proof step applies in the variable part of the other, we speak of a *variable overlap*. More precisely, a variable overlap is characterized by the existence of a substitution $\sigma$, equations $l = r$ and $l' = r'$, and positions $p$ and $q$, such that $u/p$ is $l\sigma$, $l\sigma/q$ is $l'\sigma$, and either $q$ is not a position in $l$ at all or else $l/q$ is a variable. The peak

$$u[r\sigma[l'\sigma, \ldots, l'\sigma]] \leftarrow_{E_\succ} u[l\sigma[l'\sigma, \ldots, l'\sigma]] \rightarrow_{E_\succ} u[l\sigma[r'\sigma, l'\sigma, \ldots, l'\sigma]]$$

can be replaced by a proof

$$\begin{array}{llll} u[r\sigma[l'\sigma, \ldots, l'\sigma]] & \rightarrow^*_{E_\succ} & u[r\sigma[r'\sigma, \ldots, r'\sigma]] & \\ & \leftrightarrow_E & u[l\sigma[r'\sigma, \ldots, r'\sigma]] & \leftarrow^*_{E_\succ} \quad u[l\sigma[r'\sigma, l'\sigma, \ldots, l'\sigma]], \end{array}$$

Figure 2: *No overlap*

Figure 3: *Variable overlap*

as depicted in Figure 3. Since the reduction ordering $\succ$ is complete for $E$, any such proof has to be a rewrite proof with respect to $E_\succ$.

If one proof step applies below the other, but not in the variable part, then we speak of a *proper overlap*. (Critical overlaps, in particular, are proper overlaps.) In that case, the equation $s = t$ is of the form $v[s'\sigma] = v[t'\sigma]$, where $s' = t'$ is an extended critical pair in $EP_\succ(E)$.

The lemma indicates that computation of extended critical pairs suffices to eliminate peaks between equations in $E_\succ \cup R$. More formally, we say that a derivation $(E_0, R_0) \vdash (E_1, R_1) \vdash \cdots$ in unfailing completion is *fair* if $EP_\succ(E^\infty \cup R^\infty)$ is a subset of $\bigcup_k E_k$. An unfailing completion procedure is fair if it generates only fair derivations.

With these definitions we have:

**Theorem 1 (Correctness)** *Let $C$ be a fair unfailing completion procedure, $E$ be a set of equations, $R$ be a rewrite system, and $\succ$ be a reduction ordering that contains $R$ and can be extended to a complete reduction ordering $>$ for $E$. Then, for inputs $E_0 = E$, $R_0 = R$, and $\succ$, $C$ will generate a derivation such that $E^\infty \cup R^\infty$ is ground Church-Rosser with respect to $>$.*

*Proof.* Let $(E_0, R_0) \vdash (E_1, R_1) \vdash \cdots$ be the derivation produced by $C$ and let $>$ be a complete reduction ordering for $E$ that contains $\succ$. We prove, by induction on $>_{\mathcal{U}}$, that whenever there is a proof $P$ of $s = t$ in $E_i \cup R_i$, for some $i$, then there is a ground rewrite proof with respect to $>$ of $s = t$ in $E^\infty \cup R^\infty$. Soundness then implies that $E^\infty \cup R^\infty$ is ground Church-Rosser.

Let $P$ be a proof of $s = t$ in $E_i \cup R_i$. The assertion holds trivially if $P$ is a persisting ground rewrite proof. If $P$ uses a non-persisting equation or rule, we can use Lemma 1 to conclude that there is a proof $Q$ in $E_j \cup R_j$, for some $j$, such that $P \Rightarrow^+_{\mathcal{U}} Q$. If $P$ is a persisting proof containing a peak $s \leftarrow_{E_> \cup R} u \rightarrow_{E_> \cup R} t$, then, by the Critical Pair Lemma, the peak can either be replaced by a rewrite proof $s \rightarrow^*_{E^\infty_> \cup R^\infty} v \leftarrow^*_{E^\infty_> \cup R^\infty} t$, or else $s = t$ can be written as $v[l\sigma] = v[r\sigma]$, where $l = r$ is an extended pair in $EP_>(E^\infty \cup R^\infty)$. In the first case, there is a proof $Q$ with $P >_{\mathcal{U}} Q$. In the second case, using fairness and the fact that $EP_>(E^\infty \cup R^\infty)$ is a subset of $EP_\succ(E^\infty \cup R^\infty)$, we may conclude that there is a proof in $E_j \cup R_j$, for some $j$, such that $s \leftrightarrow_{E_j \cup R_j} t$. Using Lemma 1, we may infer that there is a proof $Q$ with $P \Rightarrow^+_{\mathcal{U}} Q$.

In summary, unless $P$ is a persisting ground rewrite proof, there is a proof $Q$ of $s = t$ in $E_j \cup R_j$, for some $j$, such that $P >_{\mathcal{U}} Q$. By the induction hypothesis, there is a ground rewrite proof of $s = t$ in $E^\infty \cup R^\infty$.

Theorem 1 applies to all reduction orderings that are commonly used in practice. Any ordering based on polynomial interpretations (Lankford 1975, 1979), for instance, can be extended to a complete ordering by combining it with a well-founded ordering to distinguish ground terms having the same interpretations. Furthermore, any partial ordering on the set of operator symbols (a *precedence* ordering) can be extended to a complete ordering by way of a *recursive* or *lexicographic path ordering* (see the survey in Dershowitz 1987). (If the precedence ordering is total, then the corresponding lexicographic path ordering is total on ground terms.)

Theorem 1 also applies to a large class of completion procedures, not just a single version. We only require that a procedure compute all extended critical pairs necessary to satisfy fairness. Correctness does not depend on the strategy used to simplify equations or compose and collapse rules. We can incorporate further inference rules in unfailing completion, such as the following generalization of subsumption:

$$\frac{(E \cup \{s \doteq t, u[s\sigma] \doteq u[t\sigma]\}, R)}{(E \cup \{s \doteq t\}, R)}$$

Correctness of completion procedures that use this inference rule can easily be established in the proof ordering framework. Other refinements of standard completion, based on critical pair criteria (Bachmair and Dershowitz 1987), can also be carried over to unfailing completion.

## 7 Construction of Canonical Rewrite Systems

Standard completion fails for every equational theory that can not be represented as a complete system. More annoyingly, it may fail even when a complete system does exist. For example, suppose that $E_0$ consists of equations $a = b$, $a = c$, $fb = b$, and $fa = d$, and let $\succ_{rpo}$ be the recursive path ordering (Dershowitz 1982) corresponding to a precedence $\succ$ in which $a \succ b \succ d$ and $a \succ c \succ d$. In standard completion the derivation

$$\begin{array}{rcl}
(E_0, \emptyset) & \vdash & (\{b = c, fb = b, fb = d\}, \{a \rightarrow b\}) \\
& \vdash & (\{b = c, b = d\}, \{a \rightarrow b, fb \rightarrow b\}) \\
& \vdash & (\{d = c, fd = d\}, \{a \rightarrow d, b \rightarrow d\}) \\
& \vdash & (\emptyset, \{a \rightarrow d, b \rightarrow d, c \rightarrow d, fd \rightarrow d\})
\end{array}$$

results in a canonical system, whereas the derivation

$$\begin{array}{rcl}
(E_0, \emptyset) & \vdash & (\{c = b, fb = b, fc = d\}, \{a \rightarrow c\}) \\
& \vdash & (\{c = b, a = c\}, \{fb \rightarrow b, fc \rightarrow d\}) \\
& \vdash & (\{c = b\}, \{a \rightarrow c, fb \rightarrow b, fc \rightarrow d\})
\end{array}$$

fails (Dershowitz, Marcus, and Tarlecki 1988). In other words, success or failure of standard completion may depend on the order in which inference rules are applied. The kind of failure illustrated above can be avoided by a strategy that systematically enumerates all possible derivations, e.g., via backtracking. In some cases, however, standard completion is bound to fail even with backtracking.

For example, suppose we use the recursive path ordering corresponding to a precedence in which $f \succ g \succ h \succ c$. Standard completion fails for the equations $f(g(h(x),x)) = c$, $f(g(x,h(x))) = g(x,h(x))$, $g(h(x),x) = g(y,h(y))$, wheras with unfailing completion we obtain a complete system of three rules, $g(h(x),x) \to c$, $g(x,h(x)) \to c$, and $fc \to c$. For most reduction orderings, unfailing completion also obviates the need for backtracking.

A rewrite system $R$ is called *reduced* if, for every rewrite rule $l \to r$ in $R$, the term $r$ is irreducible in $R$ and $l$ is irreducible in $R \setminus \{l \to r\}$. In a reduced system, proper subterms of a left-hand side $l$ are irreducible, as are all terms of which $l$ is a proper instance. Also note that if a rewrite system $R$ is complete, and $\succ$ is a reduction ordering containing $R$, then a term is irreducible in $R$ if and only if it is minimal in its congruence class with respect to $\succ$. Reduced, complete systems are called *canonical*.

Two complete system $R$ and $R'$ are said to be *equivalent* if they define the same equational theory (i.e., $\leftrightarrow_R^*$ and $\leftrightarrow_{R'}^*$ are the same) and the same set of normal forms (i.e., $NF(R) = NF(R')$). Every complete system can be transformed into an equivalent reduced system, and any two equivalent canonical systems are identical up to renaming of variables (e.g., Metivier 1983).

The systems constructed by completion need not be canonical, however, as the final system $R^\infty$ may contain redundant rules. We call a completion procedure *simplifying* if, for all inputs $E_0$, $R_0$, and $\succ$, the system $R^\infty$ is reduced and whenever $u = v$ is contained in $E^\infty$, then $u$ and $v$ are incomparable with respect to $\succ$ and irreducible in $R^\infty$. Most implementations of standard completion (e.g., Huet 1981) are simplifying.

**Theorem 2 (Completeness)** *Let $R$ be a canonical system for $E$, $\succ$ be a reduction ordering containing $R$, and $C$ be a fair and simplifying unfailing completion procedure. If $\succ$ is contained in some complete reduction ordering for $E$, then $C$ will generate a derivation for inputs $E_0 = E$, $R_0 = \emptyset$, and $\succ$, such that $E^\infty = \emptyset$ and $R^\infty$ and $R$ are identical up to renaming of variables.*

*Proof.* Let $\mathcal{F}$ be the set of function symbols of the given equational theory, $\mathcal{V}$ be the set of variables. (We assume that $\mathcal{F}$ contains at least one constant.) By $\mathcal{T}(\mathcal{F},\mathcal{V})$ we denote the set of terms built from symbols in $\mathcal{F}$ and $\mathcal{V}$; by $\mathcal{T}(\mathcal{F})$, the set of ground terms over $\mathcal{F}$. Suppose that $R$ is a canonical system for $E$ and that $>$ is a complete reduction ordering for $E$, which is defined on the set $\mathcal{T}(\mathcal{F},\mathcal{V})$ and contains $\succ$. We introduce a new constant $\perp$ plus Skolem constants for all variables occurring in $R$. By $\mathcal{K}$ we denote the set of these constants; by $\hat{t}$, the result of replacing in $t$ all variables by their corresponding Skolem constants. Evidently, a term $t$ is irreducible in $R$ if and only if $\hat{t}$ is.

Let $f$ be a constant in $\mathcal{F}$. By $D_f$ we denote the mapping from $\mathcal{T}(\mathcal{F}\cup\mathcal{K},\mathcal{V})$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$, for which $D_f(t)$ is the result of replacing in $t$ every Skolem constant by $f$. We extend the ordering $>$ to the set $\mathcal{T}(\mathcal{F}\cup\mathcal{K},\mathcal{V})$ by defining: *$s > t$ if and only if either $D_f(s) > D_f(t)$, or $D_f(s) = D_f(t)$ and $s \gg_{lpo} t$,* where $f$ is the smallest constant (with respect to $>$) in $\mathcal{F}$, $\gg$ is a total well-founded ordering on $\mathcal{F} \cup \mathcal{K}$ wherein all symbols in $\mathcal{F}$ are bigger than symbols in $\mathcal{K}$ and $\perp$ is minimal, and $\gg_{lpo}$ is the corresponding lexicographic path ordering (see Kamin and Levy 1980). Note that a term $t$ is minimal with respect to $>$ if and only if $\hat{t}$ is.

Now suppose that $(E_0, R_0) \vdash (E_1, R_1) \vdash \cdots$ is the derivation generated by $C$. If $l \to r$ is a rule in $R$, then $\hat{l} = \hat{r}$ is valid in $E^\infty \cup R^\infty$. By Theorem 1, there is a ground rewrite proof (with respect to $>$) of $\hat{l} = \hat{r}$. Let $P$ be one such proof that is minimal with respect to $>_{\mathcal{U}}$. This proof must be of the form $(\hat{l}, t_1, \ldots, t_{n-1}, \hat{r})$, where $\hat{l} > \cdots > t_k < \cdots < \hat{r}$, for some $k$. Since $r$ is minimal in its congruence class, so is $\hat{r}$. Hence, $\hat{l} > \cdots > \hat{r}$. Moreover, all proper subterms of $l$, and thus all proper subterms of $\hat{l}$, are minimal in their respective equivalence classes. Therefore the first proof step in $P$ must be by application of an equation $u = v$ in $E^\infty \cup R^\infty$ at the *top*. In other words, $\hat{l} = u\sigma$, for some substitution $\sigma$. Thus, $l$ is an instance of $u$. If it were a proper instance, then $u$ would be irreducible, which would imply $v \succ u$ and contradict $u\sigma > v\sigma$. Therefore, $l$ and $u$ have to be identical up to renaming of variables, and we may assume, without loss of generality, that $E^\infty \cup R^\infty$ contains an equation $l = v$ with $v \not\succ l$. The substitution $\sigma$ replaces each variable $x$ in $l$ by its corresponding Skolem constant and, since the proof $P$ is assumed to be minimal, replaces each variable occurring in $v$, but not in $l$, by the minimal constant $\perp$.

If $l \not\succ v$, then $v \neq r$. Consequently, $P$ must contain at least two proof steps. Since the first proof step applies at the top, the first two steps must form a variable or a proper overlap. A variable overlap is impossible, since $x\sigma$ is irreducible, for every variable $x$ in $l = v$. On the other hand, if there is a proper overlap, then there is also a critical overlap $l\tau \leftrightarrow_{E^\infty \cup R^\infty} v\tau[s\tau] \leftrightarrow_{E^\infty \cup R^\infty} v\tau[t\tau]$ of some equation $s = t$ on $l = v$. That is, there is a substitution $\sigma'$, such that $x\sigma = (x\tau)\sigma'$, for all variables $x$ in $l = v$ and $s = t$. We may assume, without loss of generality, that $\tau$ does not change any variable in $l$, and only renames those variables that occur in $v$, but not in $l$, in such a way that $x\tau$ does not occur in $l$. Consequently, if all variables in $v$ also appear in $l$, then $l\tau$ is $l$ and $v\tau$ is $v$; hence $l\tau \not\succ v\tau$. If $v$ contains a variable not appearing in $l$, then $v\tau$ contains a variable not appearing in $l\tau$; hence, Again, we may infer that $l\tau \not\succ v\tau$. In addition, $s\sigma > t\sigma$ implies

$t\tau \not> s\tau$. Therefore, $l\tau = v\tau[t\tau]$ is an extended critical pair in $EP_{>}(E^\infty \cup R^\infty)$ as well as in $EP_{\succ}(E^\infty \cup R^\infty)$. Using the fairness assumption and Lemma 1, we may conclude that $P$ can be simplified; hence, is not minimal. In other words, the assumption $l \not\succ v$ leads to a contradiction.

We may therefore conclude that $R^\infty$ contains a rule $l \to v$, for every left-hand side $l$ in $R^\infty$. This means that every term that is reducible with respect to $R$ is also reducible with respect to $R^\infty$. Since $C$ is simplifying, we may infer that $E^\infty = \emptyset$ (equivalent terms have identical normal forms) and $R^\infty$ and $R$ are identical up to renaming of variables (each right-hand side of a rule in $R^\infty$ must be irreducible). We emphasize that only the hypothetical complete ordering $>$, but not the reduction ordering $\succ$ used by the completion procedure, has to be complete (for $E$).

Reduction orderings induced by canonical reduced rewrite systems can not always be extended to complete reduction orderings. For example, the rewrite system $R$, consisting of rules $f(h(x)) \to f(i(x))$, $g(i(x)) \to g(h(x))$, $h(a) \to c$, and $i(a) \to c$, is canonical. Any complete reduction ordering for $R$ must contain $h(a) > i(a)$ or $i(a) > h(a)$. If $h(a) > i(a)$, then, by monotonicity, $g(h(a)) > g(i(a))$; but from the second rule in $R$ we infer, by monotonicity, $g(i(a)) > g(h(a))$. A similar contradiction can be derived from the assumption $i(a) > h(a)$.

We next characterize a class of rewrite systems for which the ordering $\to_R^+$ can be extended to a complete reduction ordering. A *reduction sequence* (of length $n$) is any sequence $t_0 \to_R t_1 \to_R \cdots \to_R t_n$. If $R$ is finite and terminating, then (by König's Lemma) there are only finitely many reduction sequences from any given term $t$. A reduction sequence is called *innermost* if, for $1 \leq i \leq n$, the reduction step $t_{i-1} \to_R t_i$ applies at a position $p_i$, such that each proper subterm of $t_{i-1}/p_i$ is irreducible in $R$. We denote by $I(t)$ the length of the shortest innermost reduction sequence from $t$ to a normal form $t'$, and define the ordering $\succ_R^i$ by: *$s \succ_R^i t$ if and only if $s \leftrightarrow_R^* t$ and either $I(s) > I(t)$ or $I(s) = I(t)$ and $s \gg t$*, where $\gg$ is some complete reduction ordering for $R$. (Note that $\gg$ need not contain $R$.)

**Lemma 3** *If $R$ is a finite canonical system, then the ordering $\succ_R^i$ is monotonic with respect to the term structure and contains $R$.*

*Proof.* If $R$ is reduced, then $I(l) = 1$ and $I(r) = 0$, for every rule $l \to r$ in $R$. Hence, $\succ_R^i$ contains $R$. Now suppose that $s \succ_R^i t$, and let $s'$ be the (unique) normal form of $s$ and $t$ in $R$. Any shortest innermost reduction sequence of $u[s]$ can be rearranged so that $s$ is reduced to $s'$ before any other rewrite steps are applied. In other words, there is a shortest innermost sequence of the form $u[s] \to_R^* u[s'] \to_R^* u'$. Since the corresponding innermost sequence $u[t] \to_R^* u[s'] \to_R^* u'$ is shorter, we have $u[s] \succ_R^i u[t]$.

The ordering $\succ_R^i$ need not be monotonic with respect to instantiation. For example, if $R$ is $\{f(x) \to g(x,x,x), a \to b\}$, then $f(x) \succ_R^i g(x,x,x)$, but $f(a) \not\succ_R^i g(a,a,a)$. The restriction of $\succ_R^i$ to ground terms is monotonic, of course.

We say that a term $s$ *overlaps* another term $t$ if it can be unified with some non-variable subterm of $t$.

**Proposition 2** *If $R$ is a canonical system wherein no variable appears more often in a right-hand side than in the corresponding left-hand side and no left-hand side overlaps any right-hand side, then $\to_R^+$ can be extended to a complete reduction ordering.*

*Proof.* Let $\succ$ be the transitive closure of the union of the reduction ordering $\to_R^+$ and the restriction of $\succ_R^i$ to ground terms. This ordering is monotonic and well-founded. To establish well-foundedness, we prove that $l\sigma \succ_R^i r\sigma$, for every ground instance $l\sigma \to r\sigma$ of a rule in $R$. Let us denote by $\sigma'$ the substitution that maps each variable $x$ to the normal form of $x\sigma$. Since no variable appears more often in a right-hand side of a rule than in the corresponding left-hand side, no shortest innermost reduction sequence $r\sigma \to_R \cdots \to_R r\sigma'$ can be longer than a shortest innermost sequence $l\sigma \to_R \cdots \to_R l\sigma'$. Since no left-hand side of $r$ overlaps any right-hand side, the term $r\sigma'$ is irreducible in $R$, whereas $l\sigma'$ is reducible. Thus $I(l\sigma) > I(r\sigma)$, which implies $l\sigma \succ_R^i r\sigma$.

The above proposition implies that every ground canonical rewrite system is contained in a complete reduction ordering, a result that has also been proved by Dershowitz and Marcus (1985). The following result applies to arbitrary canonical systems.

**Proposition 3** *Let $R$ be a canonical system for $E$ and $\succ$ be any reduction ordering contained in both $\to_R^+$ and $\succ_R^i$. Then any fair unfailing completion procedure will generate a derivation for inputs $E_0 = E$, $R_0 = \emptyset$, and $\succ$, such that $R$ is contained in $E^\infty \cup R^\infty$.*

*Proof.* We proceed along similar lines as in the proof of Theorem 2. Let $>$ be the union of $\succ$ and the restriction of $\succ_R^i$ to ground terms. Without loss of generality, we may assume that $>$ is defined on terms containing Skolem constants. It is a complete reduction ordering for $E$. A term $t$ is irreducible in $R$ if and only if it is minimal with respect to $>$.

Let $l \to r$ be a rule in $R$. Since the Skolemized equation $\hat{l} = \hat{r}$ is valid in $E^\infty \cup R^\infty$, there is a ground rewrite proof with respect to $>$. Let $P$ be one such proof that is minimal with respect to $>_{\mathcal{U}}$. Since $\hat{r}$ is irreducible, $P$ must be of the form $(\hat{l}, t_1, \ldots, t_{n-1}, \hat{r})$, where $\hat{l} > \cdots > \hat{r}$. Since all proper subterms of $\hat{l}$ are irreducible, the first proof step in $P$ must be by application of an equation $u = v$ at the top. Thus $\hat{l} = u\sigma$, for some substitution $\sigma$, which implies that $l$ is an instance of $u$.

Now suppose that $I(u) \geq I(v)$. If $I(u) = I(v) = 0$, then both $u$ and $v$ would be irreducible in $R$, which is impossible. (Since $P$ is minimal, $u$ and $v$ must be distinct.) Thus $I(u) > 0$, which implies that $u$ is reducible in $R$. If $u$ is reducible, so is $l$. Since $R$ is reduced, $u$ and $l$ must be the same up to renaming of variables. We may assume that they are identical. If $l \not\succ v$, then $P$ must contain two proof steps, hence can be simplified, which contradicts the assumption that it is minimal. Thus $l \succ v$, which implies $l \to_R^+ v$. Hence, $v$ and $r$ are identical. In other words, $E^\infty \cup R^\infty$ contains (a variant of) the equation $l = r$.

On the other hand, if $I(v) > I(u)$, then $v$ is reducible in $R$. Therefore $P$ must contain at least two proof steps, of which the first two must overlap. Since a variable overlap could be simplified, we must have a proper overlap $u\sigma \leftrightarrow_{E^\infty \cup R^\infty} v\sigma[s\sigma] \leftrightarrow_{E^\infty \cup R^\infty} v\sigma[t\sigma]$. By the Critical Pair Lemma, there is a critical overlap $u\tau \leftrightarrow_{E^\infty \cup R^\infty} v\tau[s\tau] \leftrightarrow_{E^\infty \cup R^\infty} v\tau[t\tau]$, which, in fact, defines an extended critical pair (the assumption $u\tau \succ v\tau$ would imply $l \to_R (v\tau)\sigma'$ and force $(v\tau)\sigma'$ and $r$ to be identical, which would contradict $I(v) > 0$). Using Lemma 1, we may conclude that $P$ can be simplified. Hence, $I(v) > I(u)$ leads to a contradiction, which concludes the proof.

# 8 Refutational Theorem Proving

Completion procedures have been primarily used for constructing canonical rewrite systems. Huet (1981) suggested the use of standard completion for theorem proving in equational theories. The possibility of failure is particularly bothersome here, as the validity problem in equational theories is semi-decidable. Unfailing completion, on the other hand, is refutationally complete for equational theories.

We introduce new constants *true* and *false*, denoting truth and falsehood, respectively, and a new binary operator *eq*, denoting equality, so as to be able to use unfailing completion as a refutational theorem proving procedure.

**Theorem 3 (Refutation Completeness)** *Let $C$ be a fair unfailing completion procedure, $E$ be a set of equations, and $\succ$ be a reduction ordering that can be extended to a complete reduction ordering for $E$. If $s = t$ is valid in $E$, then $C$ will generate a derivation for inputs $E_0 = E \cup \{eq(x,x) = true, eq(\hat{s},\hat{t}) = false\}$, $R_0 = \emptyset$, and $\succ$, such that some set $E_i \cup R_i$ contains a contradictory statement $true = false$.*

*Proof.* Let $E$, $s = t$, and $\succ$ be as described. First note that the equation $s = t$ is valid in $E$ if and only if $true = false$ is valid in $E_0$.

Now, let $\mathcal{F}$ be the given set of function symbols, $\mathcal{V}$ the set of variables, and $\mathcal{K}$ be the set of all Skolem constants occurring in $\hat{s}$ and $\hat{t}$. Let $>$ be a complete reduction ordering for $E$ on the set $\mathcal{T}(\mathcal{F})$. We may assume without loss of generality (vide Theorem 2) that $>$ is defined on the set $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$. We extend $>$ to a complete reduction ordering $\gg$ for $E_0$ by defining, for all terms $s$, $t$, $u$, and $v$ in $\mathcal{T}(\mathcal{F} \cup \mathcal{K}, \mathcal{V})$: (i) $eq(s,t) \gg u \gg true \gg false$ and (ii) $eq(s,t) \gg eq(u,v)$ *if and only if* $\{s,t\} >_M \{u,v\}$, where $>_M$ is the extension of $>$ to multisets.

Since $true = false$ is valid in $E_0$, there is, by Theorem 1, a ground rewrite proof with respect to $\gg$ of $true = false$ in $E_i \cup R_i$, for some $i$. Since no term is smaller than $true$ or $false$, this proof can only be of the form $true \leftrightarrow_{E_i \cup R_i} false$, which implies that the equation $true = false$ is contained in $E_i \cup R_i$.

Consider, for example, an *entropic groupoid* defined by the two axioms

$$\begin{aligned} (xy)(zw) &= (xz)(yw) \\ (xy)x &= x. \end{aligned}$$

The first equation is *permutative* and cannot be oriented in any reduction ordering. Thus standard completion will fail for this problem. With unfailing completion, we can obtain a system (Hsiang and Rusinowitch 1987)

$$\begin{aligned} (xy)z &= (xw)z \\ (xy)x &\to x \\ x(yz) &\to xz \\ ((xy)z)w &\to xw \end{aligned}$$

which is ground Church-Rosser with respect to any complete reduction ordering satisfying the following subterm property: $t[s] \succeq s$, *for all terms* $s$ *and* $t$. The system $E$ therefore provides a decision procedure for the word problem in the above theory. Pedersen (1984) has given another system with a similar completeness property.

By the above theorem, unfailing completion is guaranteed to find a proof of an (valid) equation, provided the given reduction ordering can be extended to a complete ordering, which is the case for all general purpose orderings. The efficiency of the completion process crucially depends on the ordering that is employed, as the main strengths of completion derive from simplification and the restriction of equational consequences to (extended) critical pairs. In short, systematic rewriting may considerably reduce the search space of a proof procedure, without destroying refutation completeness. Unfailing completion can be used with the "empty" ordering, in which case it degenerates to ordinary paramodulation (Robinson and Wos 1969).

The idea of extending completion by computing equational consequences of unorientable equations can be traced back to the work of Brown (1975) and Lankford (1975) on integrating resolution and simplification by rewriting. Peterson (1983) proved the refutation completeness of an inference system combining resolution, paramodulation, and simplification with respect to orderings isomorphic to $\omega$ on ground terms. Fribourg (1985) proved the completeness of a restricted version of paramodulation with locking resolution. The refutation completeness of a specific version of unfailing completion has been proved independently by Hsiang and Rusinowitch (1987). Implementations of completion without failure have been reported by Mzali (1986) and Ohsuga and Sakai (1986).

# 9 Horn Clauses with Equality

Paul (1986) has studied the application of *standard* completion to sets of Horn clauses (with equality). We adapt his techniques to unfailing completion. In addition to terms built from function symbols in $\mathcal{F}$ and variables in $\mathcal{V}$, we shall consider in this section *atoms*, that is, expressions $p(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms and $p$ is an element of a given set $\mathcal{P}$ of *predicate symbols*. The constants $true$ and $false$ are considered to be atoms as well. We assume that $\mathcal{P}$ contains the symbol $eq$ which denotes equality.

A *literal* is either an atom (a *positive* literal) or the negation of an atom (a *negative* literal). A *Horn clause* is a disjunction of literals with at most one positive literal. A *unit* clause is a clause containing only one literal. If this literal is positive, we speak of a *positive unit* clause. Each clause is represented as a rewrite rule $L_1 \vee \ldots \vee L_n \rightarrow true$, where $L_1 \vee \ldots \vee L_n$ is an abbreviation for $(\cdots(L_1 \vee L_2) \vee \ldots) \vee L_n)$.

We deal with the equality predicate *eq* by extending unfailing completion by the inference rule

$$\frac{(E, R \cup \{eq(s,t) \to true\})}{(E \cup \{s = t\}, R)} \tag{11}$$

In other words, we convert a positive unit literal $eq(s,t)$ into an equation $s = t$.

When reasoning with the rewrite representation of Horn clauses, we need the following rules of Boolean algebra, denoted collectively by $BA$:

$$\begin{array}{cc}
\neg true \to false & \neg false \to true \\
x \vee true \to true & true \vee x \to true \\
x \vee false \to x & false \vee x \to x
\end{array}$$

Two results will be instrumental in proving that unfailing completion is refutationally complete for Horn clauses with equality: the correctness of unfailing completion and the refutation completeness of hyper-resolution for Horn clauses without equality.

Let $C$ be a clause $\neg A_1 \vee \ldots \vee \neg A_n \vee B$. Furthermore, let $B_1, \ldots, B_n$ be positive unit clauses and $\sigma$ be a most general substitution for which $A_i\sigma$ and $B_i\sigma$ are identical, for $1 \leq i \leq n$. Then $B\sigma$ is called a *hyper-resolvent* of $C$ with $B_1, \ldots, B_n$, whereas the constant *false* is a hyper-resolvent of $\neg A_1 \vee \ldots \vee \neg A_n$ with $B_1, \ldots, B_n$.

Given a set of Horn clauses $S$, we denote by $\mathcal{H}_0(S)$ the set of all ground instances $A\sigma$ of positive unit clauses $A$ in $S$. The set $\mathcal{H}_{n+1}(S)$ is defined recursively as the union of $\mathcal{H}_n(S)$ and the set of all ground-instances of hyper-resolvents of a clause $C$ in $S$ with unit clauses in $\mathcal{H}_n(S)$. By $\mathcal{H}(S)$ we denote the set $\bigcup_i \mathcal{H}_i(S)$.

A set of clauses $S$ is called *equality unsatisfiable* if the set $S \cup EQ$ is unsatisfiable, where $EQ$ consists of the following equality axioms, formulated as Horn clauses:

$$\begin{gathered}
eq(x,x), \\
\neg eq(x,y) \vee eq(x,y), \\
\neg eq(x,y) \vee \neg eq(y,z) \vee eq(x,z), \\
\neg eq(x,y) \vee eq(f(x_1,\ldots,x,\ldots,x_n), f(x_1,\ldots,y,\ldots,x_n)), \\
\neg eq(x,y) \vee \neg p(x_1,\ldots,x,\ldots,x_n) \vee p(x_1,\ldots,y,\ldots,x_n),
\end{gathered}$$

with $f$ ranging over all function symbols and $p$ over all predicate symbols. The following proposition follows from the completeness of hyper-resolution (Robinson 1965).

**Proposition 4** *Let $S$ be a set of Horn clauses. (i) If $S$ is unsatisfiable, then $\mathcal{H}(\mathcal{S})$ contains the propositional constant false, and (ii) if $S$ is equality unsatisfiable, then $\mathcal{H}(S \cup EQ)$ contains false.*

The equality axioms, except for the reflexivity law, are not needed for unfailing completion. We have to refine the notion of fairness, though. When applying unfailing completion to Horn clauses, we may use any reduction ordering $\succ$ on terms that can be extended to a complete reduction ordering. We slightly generalize such an ordering by asserting that $A \succ t$, $t \succ \mathit{false}$, and $\mathit{false} \succ \mathit{true}$, for all terms $t$ and atoms $A$.

We say that an unfailing completion procedure is *fair for Horn clauses* if (a) $EP_{\succ}(E^{\infty} \cup R^{\infty})$ is a subset of $\bigcup_k E_k$, (b) $E^{\infty}$ contains only equations $s = t$ between terms, and (c) $R^{\infty}$ contains $eq(x,x) \rightarrow \mathit{true}$, but no other rule of the form $eq(s,t) \rightarrow \mathit{true}$.

**Proposition 5** *Let $S$ be a set of Horn clauses and $R$ be their representation as a rewrite system. Let $\succ$ be any reduction ordering on terms as described above. If $(E,R)$ is the ground canonical system generated by any fair unfailing completion procedure for inputs $E_0 = \emptyset$, $R_0 = R \cup BA \cup \{eq(x,x) \rightarrow \mathit{true}\}$, and $\succ$, then (i) $A \leftrightarrow^*_{E \cup R} \mathit{true}$, for all atoms $A$ in $\mathcal{H}(S \cup EQ)$, and (ii) $s \leftrightarrow^*_{E \cup R} t$, for all atoms $eq(s,t)$ in $\mathcal{H}(S \cup EQ)$.*

*Proof.* Let $(E,R)$ be the ground canonical system constructed by a fair unfailing completion procedure for inputs $E_0 = \emptyset$, $R_0 = R \cup BA \cup \{eq(x,x) \rightarrow \mathit{true}\}$, and $\succ$. First note that property (ii) is a consequence of (i), because of fairness. We prove, by induction on $n$, that property (i) holds for all atoms in $\mathcal{H}_n(S \cup EQ)$, for all $n \geq 0$.

If $A$ is in $\mathcal{H}_0(S \cup EQ)$, then it must be an instance of some positive unit clause in $S$ or must be of the form $eq(s,s)$. In the first case, we may infer, by the soundness of completion, that $A \leftrightarrow^*_{E \cup R} \mathit{true}$. The same is true in the second case, as fairness implies that $eq(x,x) \rightarrow \mathit{true}$ is a rule in $R$.

Suppose properties (i) and (ii) are true for all atoms in $\mathcal{H}_n(S \cup EQ)$, and let $A$ be in $\mathcal{H}_{n+1}(S \cup EQ)$. We may assume that $A$ is a ground instance of a hyper-resolvent of some clause $C$ in $S \cup EQ$ with atoms in $\mathcal{H}_n(S \cup EQ)$.

1) Suppose $C$ is a clause $\neg A_1 \vee \ldots \vee \neg A_n \vee B$ in $S$. Let $B_1, \ldots, B_n$ be (ground) atoms in $\mathcal{H}_n(S \cup EQ)$ and $\sigma$ be a substitution, such that $B_i$ is identical with $A_i\sigma$, for $1 \leq i \leq n$. Let $\tau$ be a substitution, such that $A$ is $(B\sigma)\tau$. By soundness of completion, $\neg(A_1\sigma)\tau \vee \ldots \vee \neg(A_n\sigma)\tau \vee (B\sigma)\tau \leftrightarrow^*_{E \cup R} \mathit{true}$. By the induction hypothesis, $B_i \leftrightarrow^*_{E \cup R} \mathit{true}$, for $1 \leq i \leq n$. Since $B_i$

and $(A_i\sigma)\tau$ are identical, we have $\neg(A_1\sigma)\tau \vee \ldots \vee \neg(A_n\sigma)\tau \vee (B\sigma)\tau \leftrightarrow^*_{E\cup R}$ $\neg true \vee \ldots \vee \neg true \vee (B\sigma)\tau \leftrightarrow^*_{E\cup R} (B\sigma)\tau$. In summary, $A \leftrightarrow^*_{E\cup R} true$.

If $C$ is $\neg A_1 \vee \ldots \vee \neg A_n$, then $A$ is the constant *false*. Since $\neg A_1\sigma \vee \ldots \vee$ $\neg A_n\sigma \leftrightarrow^*_{E\cup R} false$, we do get $false \leftrightarrow^*_{E\cup R} true$, as required.

2) Let $C$ be the symmetry axiom $\neg eq(x,y) \vee eq(y,x)$ and $B_1$ be $eq(s,t)$. Then $A$ is simply $eq(t,s)$. By the induction hypothesis, $B_1 \leftrightarrow^*_{E\cup R} true$, which implies $\neg eq(s,t) \vee eq(t,s) \leftrightarrow^*_{E\cup R} eq(t,s)$. On the other hand, we can also infer that $s \leftrightarrow^*_{E\cup R} t$. Hence, $\neg eq(s,t) \vee eq(t,s) \leftrightarrow^*_{E\cup R} \neg eq(t,t) \vee eq(t,t)$. The latter disjunction can be simplified to $true$, using $eq(x,x) \rightarrow true$ and the rules in $BA$. Summarizing, we obtain $eq(t,s) \leftrightarrow^*_{E\cup R} true$.

3) Assume that $C$ is a substitutivity axiom

$$\neg eq(x,y) \vee eq(f(\ldots,x,\ldots), f(\ldots,y,\ldots))$$

and $B_1$ is $eq(s,t)$. Then $A$ is (a ground instance of)

$$eq(f(\ldots,s,\ldots), f(\ldots,t,\ldots)).$$

We can use the induction hypothesis to show that $(C\sigma)\tau \leftrightarrow^*_{E\cup R} A$. Since $s \leftrightarrow^*_{E\cup R} t$, we may conclude that

$$(C\sigma)\tau \leftrightarrow^*_{E\cup R} \neg eq(t,t) \vee eq(f(\ldots,t,\ldots), f(\ldots,t,\ldots)).$$

As above, the latter disjunction can be simplified to $true$, which implies $A \leftrightarrow^*_{E\cup R} true$.

The remaining cases can be proved similarly.

The proposition shows that any ground atom deducible from $S \cup EQ$ by hyper-resolution can be reduced to the normal form $true$ with respect to $E \cup R$. This immediately yields

**Theorem 4** *Unfailing completion is refutationally complete for theories of Horn clauses with equality.*

*Proof.* Let $S$ be an equality unsatisfiable set of Horn clauses and $R$ be their representation as a rewrite system. Let $(E,R)$ be the system constructed (or approximated) by a fair unfailing completion procedure for inputs $E_0 = \emptyset$, $R_0 = S \cup BA \cup \{eq(x,x) \rightarrow true\}$, and $\succ$ (with $\succ$ being a reduction ordering as described above). By Proposition 4, *false* is contained in $\mathcal{H}(S \cup EQ)$. Hence, by Proposition 5, $false \leftrightarrow^*_{E\cup R} true$. Since $E \cup R$ is ground canonical with respect to $\succ$, there must be a rewrite proof of $false = true$. This is only possible if the contradictory equation *false* $=$ *true* itself is generated by the completion procedure.

Let us take a closer look at the (extended) critical pairs that have to be computed by unfailing completion in the case of Horn clauses.

From two rules $C_1 \to true$ and $C_2 \to true$, both representing Horn clauses, we can only obtain a non-trivial critical pair if one of the two clauses is a positive unit clause. More precisely, if $B'$ unifies with $A_i$, then the superposition of $B' \to true$ on $\neg A_1 \vee \ldots \vee \neg A_n \vee B \to true$ results in a critical pair $\neg A_1\sigma \vee \ldots \vee \neg true \vee \ldots \vee \neg A_n\sigma \vee B\sigma = true$, which can be simplified and oriented into a rule $\neg A_1\sigma \vee \ldots \vee \neg A_{i-1}\sigma \vee \neg A_{i+1}\sigma \ldots \vee \neg A_n\sigma \vee B\sigma \to true$. Thus, superposition plus simplification in this case corresponds to positive unit resolution. Since rewrite rules $eq(s,t) \to true$ do not persist, by fairness, they need not be considered for critical pair computations. The only remaining critical overlaps are those involving an equation $s = t$ or a rule $s \to t$. In these cases superposition corresponds to a restricted version of paramodulation, called *oriented paramodulation*.

We can thus rephrase Theorem 4 as follows:

**Corollary 1** *Positive unit resolution plus oriented paramodulation is refutationally complete for Horn clauses with equality, even if unrestricted simplification by rewriting is permitted.*

This result improves the completeness theorem of Henschen and Wos (1974) in several respects: (a) the functional reflexivity axioms are not needed, (b) factoring is not needed, (c) paramodulation into variables is not needed, (d) oriented paramodulation generates fewer paramodulants than ordinary paramodulation, and (e) unrestricted simplification is permitted.

We can refine the unfailing completion method by including, for instance, in the initial system $R_0$ all rewrite rules of the form

$$x_1 \vee \ldots \vee x \vee \ldots \vee \neg x \vee \ldots \vee x_n \to true$$

or

$$x_1 \vee \ldots \vee \neg x \vee \ldots \vee x \vee \ldots \vee x_n \to true$$

to allow for elimination of tautologies. Factoring can be accomplished by including all rules of the form

$$x_1 \vee \ldots \vee x \vee \ldots \vee x \vee \ldots \vee x_n \to true.$$

The inclusion of these rules does not affect the above completeness result.

Unfailing completion can also be applied to proofs by consistency of inductive theorems, in a similar way as described by Paul (1986). For recent work on the word problem in Horn clause theories see Kounalis and Rusinowitch (1987).

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