Non-deterministic Semantics for Intuitionistic Paraconsistent Logics

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1 Introduction

The survey [5] is devoted to paraconsistent extensions of positive classical logics (and especially to C-systems of this kind). It seems however that positive *intuitionistic* logic is a better starting point for investigating negation. The valid sentences of this fragment are all intuitively correct. Positive classical logic, in contrast, includes counterintuitive tautologies like $(A \supset B \vee C) \supset (A \supset B) \vee (A \supset C)$ or $((A \supset B) \supset A) \supset A$. Moreover: the classical natural deduction rules for the positive connectives ($\wedge, \vee$ and $\supset$) define the intuitionistic positive logic, not the classical one. It is only with the aid of the classical rules for the strong classical negation that one can prove the counterintuitive positive tautologies mentioned above!

Now it is well known that it is impossible to conservatively add to intuitionistic positive logic a negation which is both explosive (i.e.: $\neg A, A \vdash B$ for all A, B) and for which LEM (the Law of Excluded Middle: $\neg A \vee A$) is valid. With such an addition we get classical logic. The intuitionists reject indeed LEM, retaining the explosive nature of negation. Paraconsistent logics, in contrast, choose the other alternative.

In this paper we show that one can conservatively add to intuitionistic positive logic both types of negation. By this we get conservative extensions of intuitionistic logic which are LFIs (logics of Formal Inconsistency) in the sense of [5], as well as paraconsistent conservative extensions of intuitionistic positive logic which enjoy the relevance properties of the logic Pac from [2] (One of these logics is da Costa famous system C_ω ([6, 5])). The intuitionistic negation is added exactly as it is usually done in intuitionistic logic: by adding a bottom element $\bot$ (satisfying $\bot \vdash A$ for every A) and defining the strong negation of A to be $A \supset \bot$. This results, of course, with the full propositional intuitionistic logic. This logic and its positive fragment serve then as bases for several conservative extensions having a nonexplosive negation respecting LEM. The weakest of these system is obtained by adding only LEM to positive intuitionistic logic. The strongest - by adding to full intuitionistic logic almost all the properties (with only one exception) of the negation of the maximal paraconsistent logics Pac and J_3 ([1, 2, 7, 8, 5]). We provide Gentzen-type systems for all these systems and prove an appropriate version of the non-analytic cut elimination theorem for them. We also provide Kripke-style semantics for all the logics, and prove soundness and completeness for this semantics. The main idea in our semantics is to allow a certain amount of nondeterminism in the valuations we use. We note that in the case of C_ω this semantics is significantly simpler than the one given in [4].

2 The Gentzen-Type Systems

In what follows $\mathcal{F}$ denotes the set of formulas of the propositional language which is based on either $\{\vee, \wedge, \supset, \neg\}$ or $\{\vee, \wedge, \supset, \neg, \bot\}$ (depending on the context). p, q, r denote atomic formulas, $A, B, C, \psi, \varphi, \phi$ denote arbitrary formulas (of $\mathcal{F}$), and Γ, Δ denote finite sets of formulas of $\mathcal{F}$. A sequent of $\mathcal{F}$ has the form $\Gamma \Rightarrow \Delta$. Following tradition, we write Γ, φ and Γ, Δ for $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \Delta$ (respectively).

THE SYSTEM LJm^+

Axioms: $A \Rightarrow A$

Structural Rules: Cut, Weakening (and the standard rules)

Logical Rules:

$$(\supset\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \quad (\Rightarrow\supset)$$

$$(\wedge \Rightarrow) \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \quad (\Rightarrow \wedge)$$

$$(\vee \Rightarrow) \quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \quad (\Rightarrow \vee)$$

THE SYSTEM LJm: This is the system obtained from LJm^+ by adding to it the following axioms for $\bot$:

$$\bot \Rightarrow A$$

The system LJm is the propositional fragment of a well-known (see [11]) multiple-conclusion version of LJ, Gentzen's sequent calculus for Intuitionistic logic, while LJm^+ is its positive fragment. The systems are sound and complete for these logics and admit cut-elimination. Note that (Positive) Classical logic is obtained from LJm (LJm^+) simply by removing the restriction on $(\Rightarrow\supset)$ to single-conclusion sequents. An alternative method of obtaining classical logic from LJm or LJm^+ is to add $\neg$ to the language together with its two classical rules. It is easy to see (using cuts) that this extension is not conservative and indeed produces classical logic. What we do next is to conservatively add *paraconsistent* negation to these systems:

THE SYSTEM GPB ($GPB^{\bot}$): This is the system obtained from LJm^+ (LJm) by adding to it the following classical rule for negation:

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \quad (\Rightarrow \neg)$$

THE SYSTEM GPJ ($GPJ^{\perp}$): This is the system obtained from GPB ($GPB^{\perp}$) by adding to it the following rules for negation:

$$(\neg\neg \Rightarrow) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A} \qquad (\Rightarrow \neg\neg)$$

$$(\neg \supset\Rightarrow) \qquad \frac{\neg B, \Gamma \Rightarrow \Delta}{\neg(A \supset B), \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \supset B)} \qquad (\Rightarrow \neg \supset)$$

$$(\neg\vee \Rightarrow) \qquad \frac{\Gamma, \neg A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg A \qquad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)} \qquad (\Rightarrow \neg\vee)$$

$$(\neg\wedge \Rightarrow) \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \qquad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg A, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \qquad (\Rightarrow \neg\wedge)$$

Note 1 GPJ and $GPJ^{\perp}$ are respectively subsystems of the 3-valued systems Pac ([2]) and J_3 ([7]) [1]. Hence they are paraconsistent. $GPJ^{\perp}$ is also an LFI in the sense of [5], and even a C-system (see [5]) which is based on intuitionistic logic rather than on classical logic (this will be proved below — see Corollary 8).

Note 2 The $\{\vee, \wedge, \neg\}$-rules of GPJ and $GPJ^{\perp}$ are identical to those of Pac (and J_3), and the $\{\vee, \wedge, \neg, \perp\}$-rules of $GPJ^{\perp}$ are identical to those of J_3. The main difference is therefore the treatment of implication. In the Logic Pac the rule $(\neg \supset\Rightarrow)$ allows to infer $\neg(A \supset B), \Gamma \Rightarrow \Delta$ from $A, \neg B, \Gamma \Rightarrow \Delta$. This is equivalent to the $(\neg \supset\Rightarrow)$ rule of GPJ taken together with a rule which allows to infer $\neg(A \supset B), \Gamma \Rightarrow \Delta$ from $A, \Gamma \Rightarrow \Delta$. As we shall see, by adding the latter to GPJ we get the full power of positive classical logic. Hence this extension is not conservative.

Note 3 In the original formulation of Gentzen ([9]) the rules $(\wedge \Rightarrow)$ and $(\Rightarrow \vee)$ were split into two rules, each with only one side formula. It will be convenient to do the same here to $(\neg\vee \Rightarrow)$ and $(\Rightarrow \neg\wedge)$. So we assume henceforth that instead of these rules GPJ and $GPJ^{\perp}$ have the following equivalent four rules:

$$(\neg\vee \Rightarrow)_1 \quad \frac{\Gamma, \neg A \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad\qquad \frac{\Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \quad (\neg\vee \Rightarrow)_2$$

$$(\Rightarrow \neg\wedge)_1 \quad \frac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \quad (\Rightarrow \neg\wedge)_2$$

[1] J_3 is renamed $LFI1$ in [5]. That the language of J_3 can be taken as $\{\vee, \wedge, \supset, \neg, \perp\}$ easily follows from the functional completeness of this set for the set of classically closed 3-valued operations (proved in [3]).

Note 4 Several paraconsistent intermediate systems can be obtained from the basic systems GPB and $GPB^{\perp}$ by adding to them any subset of the set of the rules for negation of GPJ. In particular: by adding to GPB the rule $(\neg\neg \Rightarrow)$ we get Raggio's Gentzen-type formulation of Da Costa's C_{ω} ([10]).

GPB, $GPJ^{\perp}$, and all the systems in between have one defect: the cut-elimination theorem fails in all of them. It is easy to see in fact that the sequent $\Rightarrow p, q \supset \neg p$ cannot have a cut-free proof in $GPJ^{\perp}$, but it can be derived using a cut from $\Rightarrow \neg p, p$ and $\neg p \Rightarrow q \supset \neg p$, which are both easy theorems of GPB. Note however that the cut in this example is *analytic*: the cut formula is a subformula of the conclusion. The subformula property is indeed the most important consequence of cut-elimination. Hence elimination of non-analytic cuts is usually a quite satisfactory substitute for full cut-elimination. Now in the case of GPJ the logical rules themselves do not have the standard subformula property. We introduce therefore a version of this property which is adequate for GPB and leads to an appropriate, useful version of the cut-elimination theorem for the various systems we investigate here.

Definition 1

1. $nsf(\varphi)$ is inductively defined as follows:
   (a) If φ is atomic then $nsf(\varphi) = \{\varphi\}$
   (b) If $\varphi = \neg p$ (p atomic) then $nsf(\varphi) = \{p, \neg p\}$
   (c) If $\varphi \in \{\psi_1 \wedge \psi_2, \psi_1 \vee \psi_2, \psi_1 \supset \psi_2\}$ then $nsf(\varphi) = \{\varphi\} \cup nsf(\psi_1) \cup nsf(\psi_2)$
   (d) If $\varphi \in \{\neg(\psi_1 \wedge \psi_2), \neg(\psi_1 \vee \psi_2), \neg(\psi_1 \supset \psi_2)\}$ then $nsf(\varphi) = \{\varphi\} \cup nsf(\neg\psi_1) \cup nsf(\neg\psi_2)$
2. ψ is called an *n-subformula* of φ if $\psi \in nsf(\varphi)$
3. A cut in a proof of $\Gamma \Rightarrow \Delta$ is called *n-analytic* if it is done on some n-subformula of a formula in $\Gamma \cup \Delta$

We shall show that in all our systems cuts that are not n-analytic can be eliminated.

3 The Non-deterministic Kripke-style Semantics

Definition 2 A *non-deterministic frame (ndf)* for $\mathcal{F}$ is a triple $\langle V, \leq, v\rangle$ such that:

1. $\langle V, \leq\rangle$ is a nonempty, partially ordered set.
2. $v : V \times \mathcal{F} \rightarrow \{t, f\}$ is a valuation which satisfies the following condition:
   If $v(a, \varphi) = t$ and $a \leq b$ then $v(b, \varphi) = t$.
3. v satisfies the following conditions:
   (a) $v(a, \varphi \wedge \psi) = t \Leftrightarrow v(a, \varphi) = t$ and $v(a, \psi) = t$
   (b) $v(a, \varphi \vee \psi) = t \Leftrightarrow v(a, \varphi) = t$ or $v(a, \psi) = t$

(c) $v(a, \varphi \supset \psi) = t \Leftrightarrow \forall b \geq a(v(b, \varphi) = f$ or $v(b, \psi) = t)$

(d) $v(a, \perp) = f$ (in case $\perp \in \mathcal{F}$)

4. If $v(a, \varphi) = f$ then $v(a, \neg\varphi) = t$

Definition 3 Let R be a Gentzen-type rule. Suppose the premises of R are $\Gamma, A_i \Rightarrow \Delta$ $(1 \leq i \leq n)$ and $\Gamma \Rightarrow \Delta, B_j$ $(1 \leq j \leq k)$, and its conclusion is $\Gamma \Rightarrow \Delta, C$ or $\Gamma, C \Rightarrow \Delta$. Let $|C|_R$ be t if the conclusion is $\Gamma \Rightarrow \Delta, C$, and f if it is $\Gamma, C \Rightarrow \Delta$. The *local constraint* on frames which is induced by R is:

If $v(a, A_i) = f$ for $1 \leq i \leq n$ and $v(a, B_j) = t$ for $1 \leq j \leq k$ then $v(a, C) = |C|_R$.

Here are the local constraints that are induced by the negation rules of GPJ:

$(\Rightarrow \neg)$	If $v(a, \varphi) = f$ then $v(a, \neg\varphi) = t$
$(\Rightarrow \neg\neg)$	If $v(a, \varphi) = t$ then $v(a, \neg\neg\varphi) = t$
$(\neg\neg \Rightarrow)$	If $v(a, \varphi) = f$ then $v(a, \neg\neg\varphi) = f$
$(\Rightarrow \neg\wedge)_1$	If $v(a, \neg\varphi) = t$ then $v(a, \neg(\varphi \wedge \psi)) = t$
$(\Rightarrow \neg\wedge)_2$	If $v(a, \neg\psi) = t$ then $v(a, \neg(\varphi \wedge \psi)) = t$
$(\neg\wedge \Rightarrow)$	If $v(a, \neg\varphi) = f$ and $v(a, \neg\psi) = f$ then $v(a, \neg(\varphi \wedge \psi)) = f$
$(\Rightarrow \neg\vee)$	If $v(a, \neg\varphi) = t$ and $v(a, \neg\psi) = t$ then $v(a, \neg(\varphi \vee \psi)) = t$
$(\neg\vee \Rightarrow)_1$	If $v(a, \neg\varphi) = f$ then $v(a, \neg(\varphi \vee \psi)) = f$
$(\neg\vee \Rightarrow)_2$	If $v(a, \neg\psi) = f$ then $v(a, \neg(\varphi \vee \psi)) = f$
$(\Rightarrow \neg \supset)$	If $v(a, \varphi) = t$ and $v(a, \neg\psi) = t$ then $v(a, \neg(\varphi \supset \psi)) = t$
$(\neg \supset \Rightarrow)$	If $v(a, \neg\psi) = f$ then $v(a, \neg(\varphi \supset \psi)) = f$

Note. The constraint $(\Rightarrow \neg)$ was included already in our definition of an ndf, since the corresponding rule is included in our basic system GPB.

Definition 4 Let GL be some Gentzen-type system which is obtained from GPB or $GPB^{\perp}$ by adding to it some of the negation rules of GPJ. An nd-frame is called a *GL-frame* if it satisfies the local constraints which are induced by the rules of GL.

Examples:

1. A GPB-frame is simply an ndf.

2. A GPJ-frame (or $GPJ^{\perp}$-frame) is an ndf in which all the constraints in the list above are satisfied.

3. Let GC_ω be Raggio's system for da Costa's C_ω (see [10]). GC_ω is obtained from GPB by adding to it the rule $(\neg\neg \Rightarrow)$. Hence a GC_ω-frame is an ndf $\langle V, \leq, v\rangle$ such that $v(a, \neg\neg\varphi) = f$ wherever $v(a, \varphi) = f$.

The crucial property of GL-frames is given in the following

Theorem 5 *Let $\langle V, \leq\rangle$ be a nonempty partially ordered set, and let $\mathcal{F}'$ be a subset of $\mathcal{F}$ which is closed under n-subformulas. Assume that $v' : V \times \mathcal{F}' \rightarrow \{t, f\}$ satisfies with respect to $\mathcal{F}'$ the various conditions of Definition 2 as well as the various constraints induced by the rules of GL (for example, if $(\Rightarrow \neg\wedge)$ is a rule of GL then v' satisfies constraint $(\Rightarrow \neg\wedge)$ in the sense that if $\neg(\varphi \wedge \psi) \in \mathcal{F}'$ and $v'(a, \neg\varphi) = v'(a, \neg\psi) = f$ then $v'(a, \neg(\varphi\wedge\psi)) = f$). Then there exists a valuation $v : V \times \mathcal{F} \rightarrow \{t, f\}$ such that $\langle V, \leq v\rangle$ is a GL-frame, and v extends v' (i.e. $v/(V \times \mathcal{F}') = v'$).*

Proof. We do the case of GPJ (the proofs for the other systems are practically identical).

Define v recursively, letting $v(a, \varphi) = t$ except for the following ten cases (in which $v(a, \varphi) = f$):

1. φ is atomic, $\varphi \in \mathcal{F}'$, and $v'(a, \varphi) = f$
2. $\varphi = \neg\psi$, $\varphi \in \mathcal{F}'$, and $v'(a, \varphi) = f$
3. $\varphi = \psi_1 \vee \psi_2$, and $v(a, \psi_1) = v(a, \psi_2) = f$
4. $\varphi = \psi_1 \wedge \psi_2$, and $v(a, \psi_1) = f$ or $v(a, \psi_2) = f$
5. $\varphi = \psi_1 \supset \psi_2$ and there exists $b \geq a$ such that $v(b, \psi_1) = t$ and $v(b, \psi_2) = f$
6. $\varphi = \neg\neg\psi_1$ and $v(a, \psi_1) = f$
7. $\varphi = \neg(\psi_1 \vee \psi_2)$ and $v(a, \neg\psi_1) = f$
8. $\varphi = \neg(\psi_1 \vee \psi_2)$ and $v(a, \neg\psi_2) = f$
9. $\varphi = \neg(\psi_1 \wedge \psi_2)$ and $v(a, \neg\psi_1) = v(a, \neg\psi_2) = f$
10. $\varphi = \neg(\psi_1 \supset \psi_2)$ and $v(a, \neg\psi_2) = f$

We show now that v has the required properties.

(I) We show that if $\varphi \in \mathcal{F}'$ then $v(a, \varphi) = v'(a, \varphi)$. We use induction on the complexity of φ. The case where φ is atomic is immediate from the definition of v. If $\varphi = \neg\psi$ and $v'(a, \varphi) = f$ then again $v(a, \varphi) = v'(a, \varphi) = f$ by definition of v. If $\varphi = \neg\psi$ and $v'(a, \varphi) = t$ then by I.H. (the Induction Hypothesis) and the fact that v' respects the various constraints, none of the conditions that force $v(a, \varphi)$ to be f is satisfied, and so $v(a, \varphi) = t = v'(a, \varphi)$. For example: if $\varphi = \neg(\psi_1 \wedge \psi_2)$, $\varphi \in \mathcal{F}'$ and

$v'(a,\varphi) = t$, then $\neg\psi_1 \in \mathcal{F}'$, $\neg\psi_2 \in \mathcal{F}'$, and it cannot be the case that $v'(a,\neg\psi_1) = v'(a,\neg\psi_2) = f$, because otherwise $v'(a,\varphi) = f$ by constraint $(\neg\wedge \Rightarrow)$. Hence $v'(a,\neg\psi_1) = t$ or $v'(a,\neg\psi_2) = t$, and so, by I.H., $v(a,\neg\psi_1) = t$ or $v(a,\neg\psi_2) = t$. There is accordingly no condition that forces $v(a,\varphi)$ to be f, and so $v(a,\varphi) = t = v'(a,\varphi)$. If $\varphi = \psi_1 \supset \psi_2$ then $v(a,\varphi)$ is f iff there exists $b \geq a$ such that $v(b,\psi_1) = t$ and $v(b,\psi_2) = f$. By I.H. this is the case iff there exists $b \geq a$ such that $v'(b,\psi_1) = t$ and $v'(b,\psi_2) = f$ (since $\psi_1, \psi \in \mathcal{F}'$ in this case), which happens iff $v'(a,\varphi) = f$. It follows that $v(a,\varphi) = v'(a,\varphi)$ in this case too. The proofs in the cases $\varphi = \psi \wedge \psi_2$ and $\varphi = \psi_1 \vee \psi_2$ are similar.

(II) We show, again by induction on the complexity of φ, that if $v(a,\varphi) = t$ and $a \leq b$ then $v(b,\varphi) = t$. This is obvious in case $\varphi \in \mathcal{F}'$ by (I) and our assumption on v'. Assume therefore that $\varphi \notin \mathcal{F}'$. We need to show that none of the conditions that might force $v(b,\varphi)$ to be f obtains. This is trivial for the first two conditions (since $\varphi \notin \mathcal{F}'$) and is proved exactly as in the intuitionistic case if φ is not of the form $\neg\psi$. It remains to treat the cases where $\varphi = \neg\neg\psi_1$, $\varphi = \neg(\psi_1 \wedge \psi_2)$, $\varphi = \neg(\psi_1 \vee \psi_2)$, $\varphi = \neg(\psi_1 \supset \psi_2)$. We do the last one as an example (the others are similar). So assume (for contradiction) that $v(b,\varphi) = f$ where $\varphi = \neg(\psi_1 \supset \psi_2)$. The only possible reason for this is that the premise of condition (10) is satisfied, i.e., that $v(b,\neg\psi_2) = f$. By I.H. this entails that $v(a,\neg\psi_2) = f$, and so $v(a,\varphi) = f$ (by condition (10) again), contradicting our assumption about φ.[2]

(III) We show by induction that if $v(a,\varphi) = f$ then $v(a,\neg\varphi) = t$. By (I) and our assumption on v', this is obvious in case $\neg\varphi \in \mathcal{F}'$. Assume that $\neg\varphi \notin \mathcal{F}'$. Since t is the default value, it suffices to show that no constraint forces $v(a,\neg\varphi)$ to be f. We have four cases:

(i) $\varphi = \neg\psi$. Since $v(a,\varphi) = f$, $v(a,\psi) \neq f$ (otherwise $v(a,\varphi) = v(a,\neg\psi) = t$ by I.H. for ψ). Hence the only condition on v which might be relevant here, condition (6), is not applicable. Hence $v(a,\neg\varphi) = v(a,\neg\neg\psi) = t$.

(ii) $\varphi = \psi_1 \vee \psi_2$. Since $v(a,\varphi) = f$, necessarily $v(a,\psi_1) = v(a,\psi_2) = f$. It follows by I.H. that $v(a,\neg\psi_1) = v(a,\neg\psi_2) = t$. Hence neither condition (7) not condition (8) is applicable here, and so there is no condition that forces $v(a,\neg\varphi)$ to be f. It follows that $v(a,\neg\varphi) = t$.

(iii) $\varphi = \psi_1 \wedge \psi_2$. Similar.

(iv) $\varphi = \psi_1 \supset \psi_2$. Since $v(a,\varphi) = f$, there exists $b \geq a$ such that $v(b,\psi_1) = t$ and $v(b,\psi_2) = f$. The latter implies (by I) that $v(a,\psi_2) = f$. Hence $v(a,\neg\psi_2) = t$ by I.H. It follows that condition (10) (the only one which is relevant in this case) is not applicable, and so $v(a,\neg\varphi) = t$. [3]

(IV) The intuitionistic conditions concerning $\vee, \wedge$, and $\supset$ have been built into the definition of v.

[2] Note that this point of the proof would break down if we add the classical $(\neg \Rightarrow)$ rule or if we add the other $(\neg \supset\Rightarrow)$ rule of Pac [2], which allows the inference of $\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta$ from $\Gamma, \varphi \Rightarrow \Delta$.

[3] Again this point of the proof would break down if we add the rule which allows the inference of $\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta$ from $\Gamma, \varphi \Rightarrow \Delta$!

(V) We show that v satisfies the various constraints which correspond to the rules of GPJ. For a constraint which forces some formula to get the value f this follows from (I) and our assumption on v' (in case that formula is in $\mathcal{F}'$) and the definition of v (in case it is not). The validity of $(\Rightarrow \neg)$ was shown in (III). The other cases follow easily from the choice of t as the default value. We do the case of $(\Rightarrow \neg \supset)$ as an example. So assume that $v(a, \varphi) = t$ and $v(a, \neg\psi) = t$. If $\neg(\varphi \supset \psi) \in \mathcal{F}'$ then $v'(a, \neg(\varphi \supset \psi)) = t$ by (I) and our assumption on v'. By (I) again, this entails that $v(a, \neg(\varphi \supset \psi)) = t$. If $\neg(\varphi \supset \psi) \notin \mathcal{F}$ then $v(a, \neg(\varphi \supset \psi))$ can be f only in case $v(a, \neg\psi) = f$. Here, however, $v(a, \neg\psi) = t$, and so this case is not applicable. It follows that $v(a, \neg(\varphi \supset \psi)) = t$, as condition $(\Rightarrow \neg \supset)$ requires. □

We turn now to the use of GL-frames for providing semantics for GL.

Definition 6 A sequent $\Gamma \Rightarrow \Delta$ is valid in an ndf $\langle V, \leq, v\rangle$ if for every $a \in V$, either $v(a, \varphi) = f$ for some $\varphi \in \Gamma$, or $v(a, \psi) = t$ for some $\psi \in \Delta$.

Theorem 7 *Let GL be as in Definition 4. If $\vdash_{GL} \Gamma \Rightarrow \Delta$ then $\Gamma \Rightarrow \Delta$ is valid in any GL-frame.*

Proof. By induction on the complexity of the proof of $\Gamma \Rightarrow \Delta$. The proof is like in the intuitionistic case for the positive rules, while the various constraints take care of the negation rules.

Corollary 8 *GPJ ($GPJ^{\perp}$) is a conservative extension of GMI^{+} (GMI).*

Proof. It is well known that GMI^{+} (GMI) is sound and complete for ordinary (intuitionistic) Kripke frames (see e.g. [11]). Now with respect to the language of GMI^{+} (GMI) there is no difference between such frames and our ndfs.

We next simultaneously prove our two main results: the converse of Theorem 7, and and the possibility to eliminate cuts which are not n-analytic.

Theorem 9 *Let GL be like in Definition 4. If $\Gamma \Rightarrow \Delta$ is valid in any GL-frame then $\Gamma \Rightarrow \Delta$ has a proof in GL in which every cut is n-analytic.*

Proof. Let $\Gamma \Rightarrow \Delta$ be a sequent, and let $\mathcal{F}'$ be the set of n-subformulas of formulas in $\Gamma \cup \Delta$. Call a proof in GL "good" if every cut in it is on a formula in $\mathcal{F}'$. We show that if $\Gamma \Rightarrow \Delta$ does not have a good proof then it has a finite countermodel. We start with a simple lemma:

Lemma. *Suppose that $\Gamma' \cup \Delta' \subseteq \mathcal{F}'$ and $\Gamma' \Rightarrow \Delta'$ does not have a good proof. Then there exist $\Gamma^* \supseteq \Gamma'$, $\Delta^* \supseteq \Delta'$, such that $\Gamma^* \cup \Delta^* = \mathcal{F}'$ and $\Gamma^* \Rightarrow \Delta^*$ does not have a good proof.*

The Lemma follows easily from the fact that if $\varphi \in \mathcal{F}'$ and $\varphi \notin \Gamma' \cup \Delta'$ then at least one of $\Gamma' \Rightarrow \Delta', \varphi$ and $\varphi, \Gamma' \Rightarrow \Delta'$ does not have a good proof (otherwise a cut on φ of these two sequents would give a good proof of $\Gamma' \Rightarrow \Delta'$). Hence $\Gamma' \Rightarrow \Delta'$ can gradually be extended to a sequent $\Gamma^* \Rightarrow \Delta^*$ as required.

Let now V be the set of all sequents $\Gamma' \Rightarrow \Delta'$ such that $\Gamma' \cup \Delta' = \mathcal{F}'$ and $\Gamma' \Rightarrow \Delta'$ does not have a good proof. Define $\leq$ on V by:

$$(\Gamma_1 \Rightarrow \Delta_1) \leq (\Gamma_2 \Rightarrow \Delta_2) \quad \text{iff} \quad \Gamma_1 \subseteq \Gamma_2 \quad (\text{iff } \Delta_1 \supseteq \Delta_2) \ .$$

Obviously $\langle V, \leq \rangle$ is a partially ordered set. Define next a partial valuation $v' : V \times \mathcal{F}' \to \{t, f\}$ by:

$$v'(\Gamma' \Rightarrow \Delta', \varphi) = \begin{cases} t & \varphi \in \Gamma' \\ f & \psi \in \Delta' \ . \end{cases}$$

Note that v' is well defined because $\Gamma' \Rightarrow \Delta'$ does not have a good proof (and so, in particular, $\Gamma' \cap \Delta' = \emptyset$). We show now that it satisfies with respect to $\mathcal{F}'$ all the necessary conditions and constraints.

- Suppose that $v'(\Gamma_1 \Rightarrow \Delta_1, \varphi) = t$ and $\Gamma_1 \Rightarrow \Delta_1 \leq \Gamma_2 \Rightarrow \Delta_2$. Then $\varphi \in \Gamma_1$, and $\Gamma_1 \subseteq \Gamma_2$. Hence $\varphi \in \Gamma_2$ and so $v'(\Gamma_2 \Rightarrow \Delta_2, \varphi) = t$.

- Suppose that $v'(\Gamma' \Rightarrow \Delta', \varphi \wedge \psi) = t$. Then $\varphi \wedge \psi \in \Gamma'$, and so $\varphi \wedge \psi, \varphi$ and ψ are in $\mathcal{F}'$. Since $\varphi \wedge \psi \Rightarrow \varphi$ has a cut-free proof (in GMI^+), $\varphi \notin \Delta'$. Hence $\varphi \in \Gamma'$ and so $v'(\Gamma' \Rightarrow \Delta', \varphi) = t$. The proof of the other conditions concerning $\wedge$ and $\vee$ similarly follows from the cut-free provability in GMI^+ of $\varphi \wedge \psi \Rightarrow \psi$; $\varphi, \psi \Rightarrow \varphi \wedge \psi$; $\varphi \vee \psi \Rightarrow \varphi, \psi$; $\varphi \Rightarrow \varphi \vee \psi$; $\psi \Rightarrow \varphi \vee \psi$.

- Assume $v'(\Gamma' \Rightarrow \Delta', \varphi \supset \psi) = t$. Then $\varphi \supset \psi \in \Gamma'$. Assume now that $\Gamma'' \Rightarrow \Delta'' \geq \Gamma' \Rightarrow \Delta'$ (i.e., that $\Gamma'' \supseteq \Gamma'$). We have to show that either $v'(\Gamma'' \Rightarrow \Delta'', \varphi) = f$ or $v'(\Gamma'' \Rightarrow \Delta'', \psi) = t$. Assume $v'(\Gamma'' \Rightarrow \Delta'', \varphi) \neq f$. Then $\varphi \in \Gamma''$. Since $\varphi \supset \psi \in \Gamma'$, $\varphi \supset \psi \in \Gamma''$ too. But $\varphi, \varphi \supset \psi \Rightarrow \psi$ has a cut-free proof in GMI$^+$. Hence ψ cannot be in Δ''. It follows also that $\psi \in \Gamma''$, and so $v'(\Gamma'' \Rightarrow \Delta'', \psi) = t$.

- Assume that $v'(\Gamma' \Rightarrow \Delta', \varphi \supset \psi) = f$. Then $\varphi \supset \psi \in \Delta'$. Hence $\Gamma' \Rightarrow \varphi \supset \psi$ does not have a good proof. Because of the ($\Rightarrow\supset$) rule this implies that $\Gamma', \varphi \Rightarrow \psi$ does not have a good proof. It follows by the lemma that there exists $\Gamma'' \supseteq \Gamma' \cup \{\varphi\}$, $\Delta'' \supseteq \{\psi\}$ such that $\Gamma'' \Rightarrow \Delta'' \in V$. Now $\Gamma'' \Rightarrow \Delta'' \geq \Gamma' \Rightarrow \Delta'$ (since $\Gamma'' \supseteq \Gamma'$), and we have $v'(\Gamma'' \Rightarrow \Delta'', \varphi) = t$, $v'(\Gamma'' \Rightarrow \Delta'', \psi) = f$.

- Assume that $\perp \in \mathcal{F}'$, and let $\Gamma' \Rightarrow \Delta' \in V$. Then $\Gamma' \Rightarrow \Delta'$ does not have a good proof, and so $\perp \notin \Gamma'$. Hence $\perp \in \Delta'$, and so $v'(\Gamma' \Rightarrow \Delta', \perp) = f$.

- Assume $v'(\Gamma' \Rightarrow \Delta', \varphi) = f$ and $\neg\varphi \in \mathcal{F}'$. Then $\varphi \in \Delta'$. Since $\Rightarrow \neg\varphi, \varphi$ has a cut-free proof in GPB, $\neg\varphi \notin \Delta'$. Hence $\neg\varphi \in \Gamma'$ (because $\neg\varphi \in \mathcal{F}'$), and so $v'(\Gamma' \Rightarrow \Delta', \neg\varphi) = t$.

- Let $(\Rightarrow \neg\neg)$ be one of the rules of GL. Suppose $v'(\Gamma' \Rightarrow \Delta', \varphi) = t$ and $\neg\neg\varphi \in \mathcal{F}'$. Then $\varphi \in \Gamma'$. Since $\varphi \Rightarrow \neg\neg\varphi$ has a cut-free proof in GL (using the $(\Rightarrow \neg\neg)$ rule), $\neg\neg\varphi \notin \Delta'$. Hence $\neg\neg\varphi \in \Gamma'$ and $v'(\Gamma \Rightarrow \Delta', \neg\neg\varphi) = t$.

- The proofs of the other constraints corresponding to the negation rules of GL (whatever they are) are similar (and in each case only the relevant rule is used).

It follows that $\langle V, \leq v'\rangle$ satisfies the assumptions of Theorem 5. Hence it can be extended to a GL-frame $\langle V, \leq, v\rangle$. Now since the original $\Gamma \Rightarrow \Delta$ does not have a good proof it can (by the lemma) be extended to a sequent $\Gamma^* \Rightarrow \Delta^*$ in V. Now if $\varphi \in \Gamma$ then $\varphi \in \mathcal{F}'$ and $\varphi \in \Gamma^*$ and so $v(\Gamma^* \Rightarrow \Delta^*, \varphi) = v'(\Gamma^* \Rightarrow \Delta^*, \varphi) = t$. Similarly, if $\psi \in \Delta$ then $v(\Gamma^* \Rightarrow \Delta^*, \psi) = f$. Hence $\Gamma \Rightarrow \Delta$ is refuted in the element $\Gamma^* \Rightarrow \Delta^*$ of V. $\square$

Corollary 10 *Let GL be any of the systems considered above. If $\vdash_{GL} \Gamma \Rightarrow \Delta$ then it has a proof in GL in which all the cuts are n-analytic (and so all formulas used in it are n-subformulas of $\Gamma \Rightarrow \Delta$).*

Corollary 11 *Let GL be any of the systems considered above. Then GL is a conservative extension of any of its fragments. In particular: the $\{\neg, \vee, \wedge\}$-fragment of GL is identical to that of Pac (and J_3).*

Corollary 12 *Let GL be any of the systems considered above. Then GL is decidable.*

First Proof. By Corollary 10, given a sequent $\Gamma \Rightarrow \Delta$ it suffices to search for a proof of $\Gamma \Rightarrow \Delta$ in which all cuts are n-analytic. Such a search requires only a finite number of steps (that can be determined in advance).

Second Proof. By Theorems 9 (and its proof), 5, and 7 it suffices to check whether a given sequent $\Gamma \Rightarrow \Delta$ is valid in all partial frames of the form $\langle V, \leq, v'\rangle$ where $v' : V \times \mathcal{F}' \to \{t, f\}$, $\mathcal{F}'$ is the set of n-subformulas of $\Gamma \Rightarrow \Delta$, and the number of elements of V is at most 2^n, where n is the number of elements of $\mathcal{F}'$.

Note 5 We have seen above that the proof of Theorem 5 (and so of Theorem 9 and of Corollary 10) fails in the presence of the following rule (which is valid in Pac): [4]

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$$

The addition of this rule to the basic system GPB is indeed *not* conservative, over GMI^+, since the sequent $\Rightarrow \varphi, \varphi \supset \psi$, which is not intuitionistically valid, is provable in the resulting system. Here is the proof:

$$\dfrac{\dfrac{\varphi \supset \psi \Rightarrow \varphi \supset \psi}{\Rightarrow \varphi \supset \psi, \neg(\varphi \supset \psi)} \qquad \dfrac{\varphi \Rightarrow \varphi}{\neg(\varphi \supset \psi) \Rightarrow \varphi}}{\Rightarrow \varphi \supset \psi, \varphi} \; .$$

Now the sequent $\Rightarrow \varphi, \varphi \supset \psi$ is equivalent to the axiom $\varphi \vee (\varphi \supset \psi)$. It is well known that the addition of this axiom to (positive) intuitionistic logic suffice for getting (positive) classical logic. It follows that by adding this rule to GPJ ($GPJ^{\perp}$) we get the maximal paraconsistent classical logic Pac (J_3).

[4] The reason is that the condition that is induced by this rule (i.e.: If $v(a, \varphi) = f$ then $v(a, \neg(\varphi \supset \psi)) = f$) may force $v(a, \neg(\varphi \supset \psi))$ to be f even in case $v(a, \varphi \supset \psi) = f$ (the latter only means that there is $b \geq a$ in which $v(b, \varphi) = t$ and $v(b, \psi) = f$. This still allows the possibility that $v(a, \varphi) = f$).

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