HDX Condensers

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Abstract

More than twenty years ago, Capalbo, Reingold, Vadhan and Wigderson gave the first (and up to date only) explicit construction of a bipartite expander with almost full combinatorial expansion. The construction incorporates zig-zag ideas together with extractor technology, and is rather complicated. We give an alternative construction that builds upon recent constructions of hyper-regular, high-dimensional expanders. The new construction is, in our opinion, simple and elegant.

Beyond demonstrating a new, surprising, and intriguing, application of high-dimensional expanders, the construction employs totally new ideas which we hope may lead to progress on the still remaining open problems in the area.

1 Introduction

Expanders are regular graphs of low-degree and high expansion (or connectivity). There are several ways to measure the expansion of a graph, and, most notably:

- One way to measure the "expansion" of a graph is by its *set expansion*, i.e., the minimal fraction between the number of neighbors of S and the size of S itself, for not too large sets S. we call this measure *combinatorial expansion*.
- Another way to measure the "expansion" of a graph is by its *spectral expansion*. We look at the adjacency matrix of the graph and view it as a linear operator. The gap between the first and second largest eigenvalues is the *spectral gap* of the graph and is intimately connected with connectivity properties of the graph.

A D-regular graph is *Ramanujan* if its spectral gap is at least $1-2\frac{\sqrt{D-1}}{D}$. Alon and Boppana (see [Nil91]) proved a matching lower bound, showing that any family of D-regular graphs, with the number of vertices going to infinity, has a spectral gap that in the limit is at least as high as this bound. Friedman [Fri03] showed that random graphs are (w.h.p.) *almost* Ramanujan. Lubotzky, Philips and Sarnak [LPS88], and Margulis [Mar88] explicitly constructed truly Ramanujan graphs, building on deep mathematical results in representation theory. An alternative method based on lifts is given in [MSS13]. It is an amazing fact that the explicit constructions achieve spectral expansion that is *better* than what is guaranteed by the random construction.

We next turn to combinatorial expansion. Pinsker [Pin73] observed that random, constant-degree D graphs have (w.h.p.) combinatorial expansion close to D. Several works [AM85, Alo86, Kah95] showed intimate connections between combinatorial expansion and spectral expansion, and in particular, show that all Ramanujan graphs have combinatorial expansion at least $D/2$ [Kah95]. Kahale [Kah95] showed an example of a Ramanujan graph with combinatorial expansion of only $D/2$. Thus, with respect to combinatorial

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expansion, random constructions provably beat spectral techniques. It is an outstanding open problem to explicitly construct D-regular graphs with combinatorial expansion above $D/2$.

We remark that if G is a graph with combinatorial expansion $(1-\varepsilon)D$, then any small enough set must have at least $(1-2\varepsilon)$ unique neighbors. Alon and Capalbo [AC02], building on the zig-zag construction, showed an explicit construction of a graph in which every small enough set has many unique neighbors. This construction inspired our work. However, the combinatorial expansion of the Alon-Capalbo construction does not exceed $D/2$.

One can view expanders as hash functions. A D-regular graph $G = (V, E)$ represents a function $G : V \times [D] \to V$ where $G(v, i)$ is the i'th neighbor of v. As we often want to hash a large domain to a smaller domain, it is natural to ask for unbalanced structures $G : V \times [D] \to W$ where W is much smaller than V. In graph terminology, we are looking for sparse *bipartite* graphs $G = (V, W, E)$ where every "not too large" subset of V has many neighbors in W. These objects retain the original "expansion" property, while simultaneously mapping elements of V into a much smaller domain. Often this last feature is crucial, e.g., in the error correcting codes of [SS96]. Unbalanced expanders are often called *condensers*. Condensers that have combinatorial expansion close to D are called *lossless*.

Capalbo, Reingold, Vadhan, and Wigderson [CRVW02] give constructions of lossless condensers with optimal dependence in D and construction time that is doubly exponential in the shrinking factor, i.e., if $|V| = 2^n$ and $|W| = 2^m$ then the running time is doubly exponential in $n - m$. This, in particular, solves the balanced, and slightly unbalanced cases, and is sufficient for many important applications, including the error-correcting code construction of [SS96]. Specifically, the theorem states that:

Theorem 1.1. *[CRVW02, Thm 7.1] For every $N = 2^n$, $T = 2^t < N$, and $0 < \varepsilon \leq \frac{1}{2}$, there exists a $D = 2^d$ left-regular bipartite graph $B = (L = [N], R = [M = 2^{n-t}], E)$ such that for every $A \subseteq L$ of cardinality at most $K = 2^k$, the set of neighbors of A has cardinality $|\Gamma(A)| \geq (1-\varepsilon)D \cdot |A|$, where*

- *$k = n - t - d - \log \frac{1}{\varepsilon} - O(1)$, and,*
- *$d = O(\log(t+1) + \log(\frac{1}{\varepsilon}))$.*

Furthermore, given $x \in L$ and $i \in [D]$, the i'th neighbor of x can be computed in time $\mathrm{poly}(\frac{n}{\varepsilon}, 2^{2^t})$.

As can be seen from the theorem, highly unbalanced lossless condensers also exist, alas with non-constant degrees. This is (up to constant factors) also what is achieved by random constructions, see, e.g., Lemma 2.5 for concrete bounds. It also matches (again, up to constant factors) the lower bounds. The only drawback of the theorem is that the construction running time is double-exponential in $n-m$. Having such a construction with a running time polynomial in n would almost immediately solve the long-standing open problem of constructing an almost optimal explicit extractor with a small entropy loss (see, e.g., [TSUZ07]). We refer the reader to [TSUZ07, TSU12] for further work on the lossless condenser problem in the non-constant degree regime.

Recently, Asherov and Dinur [AD23] generalized the Alon-Capalbo approach and obtained explicit constructions of arbitrarily bounded-degree bipartite graphs $G = (V, W, E)$, where every "not too large" subset of V has unique neighbors in W, and V and W may be arbitrarily unbalanced. While weaker than the [CRVW02] result (that has almost lossless expansion) their construction is much simpler. Another recent work of Hsieh *et al.* [HMMP23] extends the Alon-Capalbo construction in a generic way and constructs highly unbalanced bipartite graphs where small enough sets (from either side) have unique neighbors. We elaborate on both of these works later on in the introduction.

1.1 High-dimensional expanders

In this paper we give an alternative construction of slightly unbalanced lossless condensers, building on high-dimensional expanders. High-dimensional expanders (HDXs) are hyper-graphs with downward closure. A graph X is a set of points $X(0)$ together with a set of edges $X(1)$, where an edge e is a subset of cardinality two of vertices. A dimension t hyper-graph X is a set of vertices $X(0)$ along with hyper-edges e that are subsets of vertices of cardinality at most $t+1$. A hyper-graph is a *complex* if it is downward-closed, i.e., whenever e is a hyper-edge then so does every subset of it, for example, if $e = (i, j)$ is an edge in a graph, then i and j are vertices in the graph.

The question of how to measure the expansion of a hyper-graph is fascinating. In a sense, the question already arises for graphs, and we have already mentioned two different natural measures: spectral expansion and combinatorial expansion. In this paper we need two properties from the complex:

- We need that if we restrict attention to the graph $G = (V = X(0), E = X(1))$ that contains only the vertices and edges of the complex, then G is a good spectral expander, and,
- That G is hyper-regular, which we explain next.

A hyper-graph is *hyper-regular* if any face $x \in X$ of cardinality i is contained in the same number of faces of X of cardinality j (where $i < j \leq t+1$). For example, a graph is regular if every vertex is contained in the same number of edges. A 2-dimensional complex is hyper-regular, if every vertex is contained in the same number of edges, and every edge is contained in the same number of triangles.

Random complexes are far from being good high-dimensional expanders because typically the neighbors of a vertex in a random bounded-degree graph are completely disconnected (i.e., the vertex v does not participate in any triangle). The first high-dimensional expanders of [LSV05] were not hyper-regular. A few years ago, Kaufman and Oppenhein gave an alternative construction based on coset geometries. Their construction gives, among other things, hyper-regular dimension-two high-dimensional expander. Other hyper-regular constructions appear in [CLP20, CLST22]. Shortly after, Friedgut and Iluz [FI20] showed a hyper-regular construction for any dimension, see, e.g., Theorem 2.20 for a precise statement of their result.

Unlike expanders, condensers, and other pseudo-random objects, HDXs are not pseudo-random as they possess strong combinatorial properties, that are not present in random objects, and their existence is a miracle. Thus, they perhaps should be more adequately termed "pseudo-diamond" objects. However, surprisingly, so far these marvelous objects have only found a few applications. Dinur and Kaufman [DK17] showed HDXs are good agreement testers. HDXs were used for list decoding direct-product and direct-sum codes [DHK+21], but it turned out this task can be done better using the Sum-Of-Squares (SOS) semi-definite programming hierarchy [AJQ+20] or weak regularity decomposition [JST21]. HDXs are also the inspiration behind the recent constructions of locally testable codes with a constant rate, distance, and locality ([DEL+21] and independently Panteleev and Kalachev [PK22] who also constructed quantum LDPC codes) but the actual construction does not use HDXs.

Our construction may be seen as a generalization of Alon-Capalbo using HDXs. Thus, our result joins a very short list of examples where HDXs serve as a building block for some other combinatorial or algorithmic application. Remarkably, we do not require the links to be good spectral expanders[1], and the most crucial property we require from the HDX is *hyper-regularity*, on top of the requirement that the underlying graph is a good spectral expander.

We conclude the introduction by explaining how HDXs help in the construction of constant-degree lossless condensers.

1.2 The bi-variate Alon-Capalbo construction

Our starting point is the zig-sag product of [RVW02] and the related replacement product. Let $G = (V, E)$ be a D-regular graph, and H a graph on D-vertices. In the replacement product $G' = (V', E')$, we put a *cloud* around each vertex, yielding vertex set $V' = V \times [D]$. On V' we put edges E' of two types:

- Inter-cloud edges E'_1: We first choose a two-side labeling of the edges, such that every vertex v labels the edges connected to it with distinct labels. We define a permutation $Rot : V \times [D] \to V \times [D]$ by letting $Rot(v, i) = (w, j)$ where w is the i'th neighbor of v, and v is the j'th neighbor of w. Notice that Rot is an involution, i.e., $Rot^2 = id$. We put an inter-cloud edge $((v, i), (w, j))$ whenever $Rot(v, i) = (w, j)$.
- Intra-cloud edges E'_2: For every vertex $v \in V$, we put a copy of H on the D vertices $\{(v, i) \mid i \in [D]\}$.

Reingold, Vadhan, and Wigderson [RVW02] choose H to be a graph with a good spectral expansion and analyze the spectral properties of the replacement product. Alon and Capalbo [AC02] used a variant of the replacement product, where the goal is obtaining a graph with a good combinatorial expansion where the graph G is a good spectral expander and the graph H is a good combinatorial expander. Specifically, Alon and Capalbo construct a graph $G'' = (V'', E'')$ where:

[1] See Section 2 for the definition of a link.

- V'' is the set of inter-cloud edges E'_1. We may identify V'' with E.
- Two inter-cloud edges $e_1, e_2 \in E$ share an edge in E'', if they share a vertex $v \in V$, e_1 is the i'th edge leaving v, e_2 is the j'th edge leaving v, and $(i,j) \in E(H)$. In words, the vertices of G'' are the edges of G, and two vertices in G'' share an edge, if as edges of G they share a vertex v in G and they are connected by H in the cloud of v.

Alon and Capalbo use very specific graphs H. The work of Hsieh *et al.* [HMMP23] mentioned before extends the Alon-Capalbo construction to generic highly unbalanced bipartite graphs where small enough sets (from either side) have many unique neighbors.

Intuitively, the idea behind the Alon-Capalbo construction is very clear. Let $S_1 \subseteq V'' = E$ be a small (linear-size) subset of selected G-edges. Let S_0 be the set of G-vertices that sit on a selected edge (S_0 is the downward closure of S_1). Now, look at the graph $G = (V, E)$ and $S_0 \subseteq V$. As G is a good spectral expander, and S_0 is small, it must be the case that there are few edges in $E(S_0, S_0)$ (this is the Alon-Chung lemma, proved by the expander mixing lemma, see, e.g., Lemma 2.3 below). Thus, informally, for most G-edges $e = (v_1, v_2)$ in S_1, at most one of the two vertices in e has many S_0 neighbors. Let us call a vertex with "many" S_0 vertices "heavy", and "light" otherwise. Informally, light vertices expand well in H, and so we should expect at least roughly $D/2$ expansion. The challenge is surpassing that, or at least, achieving many unique neighbors (without beating the $D/2$ expansion barrier).

The work of [AD23] studies a bipartite version of Alon-Capalbo, that is easier to handle. In this version, the left-hand side of the bipartite graph is $V'' = E$. For every $v \in V$ there are D incoming edges, and we label them by $\{(v,i) \mid i \in [D]\}$. We call this set the "cloud" of v. We choose a fixed bipartite graph $H = ([D], W'')$ that is a good bounded-degree, combinatorial expander. We put a copy of H on every cloud. More precisely, the right-hand side of the bi-partite graph that we build is $V \times W''$, and we connect $e = (v_1, v_2) \in E$ on the left-hand size to all the H-neighbors of v_1 and v_2 in their respective clouds (for a precise description of the construction see Section 2.3). It is relatively straightforward to prove that this construction has about $D/2$ expansion, following the argument given above. It is also not complicated to come up with G and H and small sets that have no more than $D/2$ expansion.

1.3 Hyper-regular HDX to the rescue

In this work, we extend the argument to higher-dimension complexes. We first describe the construction. Assume for simplicity X is a two-dimensional complex with vertices, edges, and triangles. We construct a bipartite graph where on the left-hand side we have all the triangles $X(2)$ of the complex. Further, assume X is hyper-regular and every vertex $v \in X(0)$ as exactly $D_{0,2}$ triangles on it. Label the triangles on v with distinct $D_{0,2}$ labels. Let $\{(v,i) \mid i \in [D_{0,2}]\}$ be the "cloud" of v. Let H be a bounded-degree, bipartite graph with $D_{0,2}$ edges on the left-hand side. As before, connect a triangle $t = (v_1, v_2, v_3) \in X(2)$ on the left-hand side to all the H-neighbors of v_1, v_2 and v_3 in their respective clouds (for a precise description of the construction see Section 2.3).

Intuitively, we expect that if $S_2 \subseteq V'' = X(2)$ is small enough, then almost all triangles in S_2 have at most one heavy vertex, where a vertex v is heavy if there are many selected triangles containing v. Indeed, this is the case, although the actual argument is a bit more involved. The most fundamental observation is that every bad triangle $t = \{v_1, v_2, v_3\}$ has two heavy vertices, and therefore is responsible for one edge in $E(H_0, H_0)$, where H_0 is the set of heavy vertices. One should be careful because every edge in $E(H_0, H_0)$ may be contributed by $D_{1,2}$ triangles, where $D_{1,2}$ is the number of triangles sitting on an edge. The combinatorial argument also uses the fact that in the inclusion graph $(X(2), X(0))$ there are many edges leaving H_0 (because every vertex in H_0 is heavy) and they go to almost completely distinct vertices. The argument also naturally extends to dimension t HDXs for higher values of t. We give the complete argument in Section 3.

In a sense, the construction has two separate stages. In the first stage, we map a triangle (or a t-dimension face) to a cloud of a vertex with $D_{0,t}$ vertices, where $D_{0,t}$ is the number of t-faces sitting on a vertex. This mapping has the property that whatever $S_t \in X(t)$ is (as long as it is small enough), most clouds are light (or more precisely, most edges leaving S_t fall in a light cloud), and then the second stage solves the hashing problem (doing both contraction of the universe and expansion of the light set) on the cloud. This second stage is done on a much smaller universe and may be found, e.g., by brute force. We take this approach when presenting the correctness proof in Section 3.

Another issue that deserves attention is *explicitness*. Previous HDXs constructions have not explicitly addressed the explicitness issue. For that reason, we need to go back and check that previous constructions are as explicit as we wish.

The paper is organized as follows: In Section 2 we give some preliminaries on expanders, condensers, high-dimensional expanders and the routed product from [AD23]. In Section 3 we give the combinatorial proof that the construction gives a lossless condenser. In Section 4 we show the construction is fully-explicit. We give the definitions and top-level arguments in the main text, and in the Appendix we show previous constructions ([KO23] and [FI20]) are explicit. In Section 5 we put all the claims together, choose parameters and give our new lossless condenser construction.

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2 Preliminaries

2.1 Expanders

Definition 2.1. *Let $G = (V, E)$ be a graph[2] with N vertices, regular degree D, and normalized eigenvalues $-1 \leq \lambda_{(N)} \leq \ldots \leq \lambda_{(2)} \leq \lambda_{(1)} = 1$.*

- *We say G is a (N, D, λ) one-sided expander if $\lambda_{(2)} \leq \lambda$.*
- *We say G is a (N, D, λ) two-sided expander if $max(\lambda_{(2)}, |\lambda_{(N)}|) \leq \lambda$.*

Alon-Boppana gave a (tight) bound on the spectral gap:

Lemma 2.2 (Alon-Boppana)**.** *[Nil91] Let G be an (N, D) graph. Then $\lambda_{(2)}(G) \geq \frac{2\sqrt{D-1}}{D} - o_N(1)$.*

A simple consequence of the expander mixing lemma, as proved by [AC88] for the two-sided case and generalized to the one-sided case by [DEL+21] is the following lemma:

Lemma 2.3 (Alon-Chung lemma)**.** *[AC88] Let $G = (V, E)$ be a (N, D, λ) one-sided expander. For every set $S \subseteq V$ define $E(S, S) = \{(u, v) \in E \mid u, v \in S\}$. Then:*

$$\frac{E(S,S)}{D \cdot |S|} \leq \frac{|S|}{N} + \lambda.$$

Definition 2.4 (Bipartite expander)**.** *Let $B = (L_B, R_B, E_B)$ be a bipartite graph. We say B is D* left-regular *(resp.* right-regular*) if every vertex in L_B (resp. R_B) has degree D. We say B is (D_L, D_R) regular, if it is D_L left-regular and D_R right-regular. A D_L left-regular bipartite graph B is (K, ε)* expanding*, if for every $A \subseteq L_B$ such that $|A| \leq K$ it holds that:*

$$|\Gamma(A)| \geq (1 - \varepsilon) D_L \cdot |A|,$$

Where $\Gamma(A) \subseteq R_B$ is the set of neighbours of A in B. If B is D_L left-regular and (K, ε) expanding, we say B is a (D_L, K, ε) bipartite expander. Similarly, if B is (D_L, D_R) regular and (K, ε) expanding, we say B is a $(D_L, D_R, K, \varepsilon)$ bipartite expander.

Non-explicitly, using the probabilistic method one can show that good left-regular bipartite (K, ε) expanders exist. Specifically,

Lemma 2.5 (Non-explicit construction)**.** *[CRVW02] For every $N \geq M \geq 0$ and $\varepsilon > 0$, there exists a (D, K, ε) bipartite graph $B = (L = [N], R = [M], E)$ where*

- *$K = \Theta(\varepsilon \frac{M}{D})$, and,*

[2] When we say a *graph* we mean an *undirected* graph.

- $D = \Theta(\frac{\ln \frac{2N}{M}}{\varepsilon})$.

Finally, we define what makes a construction explicit. We begin by defining polynomially dense sets:

Definition 2.6. *We say a set $S \subseteq \mathbb{N}$ is* polynomially dense*, if there exists a polynomial p and a polynomial-time algorithm such that on input $N \in \mathbb{N}$ outputs $N' \in S$ such that $N \leq N' \leq p(N)$.*

For the rest of this section, let S be a polynomially dense set.

Definition 2.7 (Explicit bipartite graph)**.** *Let $\{G_N\}_{N \in S}$ be a family such that for every $N \in S$, $G_N = ([N], [M(N)], E_N)$ is a D_L left-regular bipartite graph. We say $\{G_N\}$ is* fully explicit *if S is polynomially-dense and there exists an algorithm that on inputs $N \in S$, $x \in [N]$ and $j \in [D_L]$ outputs the j'th neighbor of x in G_N in polynomial time in the input length, i.e., in time* $\text{polylog}(N + D_L)$.

2.2 High dimensional expanders

Definition 2.8 (Hypergraph and simplicial complex)**.** *A hypergraph is a pair (V, E) with V a set of vertices and E a set of subsets of V. $G = (V, E)$ is called a* simplicial complex *if E is downwards closed with respect to containment, i.e., for every $s \in E$ and $r \subseteq s$ it also holds that $r \in E$.*

Definition 2.9 (Dimension)**.** *Let X be a simplicial complex and $s \in X$. The dimension of s is $|s| - 1$, i.e., one less the cardinality of s. We partition a simplicial complex X to*

$$X = X(-1) \cup X(0) \cup X(1) \cup \ldots \cup X(t),$$

where $X(i)$ is the set of all faces of size $i+1$ (and dimension i). Notice that $X(-1) = \{\emptyset\}$ and $X(0) = V$. If $s \in X(i)$ we say s is an i-face. We say X is t-dimensional if $\max_{s \in X} |s| \leq t+1$. A t-dimensional simplicial complex X is pure *if every $s \in X$ is contained in some $s \in X(t)$.*

Definition 2.10 (Link)**.** *Let X be a D-dimensional simplicial complex and $s \in X(i)$. The link of s is a $(D - |s|)$-dimensional simplicial complex defined by*

$$X_s = \{r \setminus s : s \subseteq r \in X\}.$$

Definition 2.11 (The graph of a link)**.** *Let X be a simplicial complex, and $s \in X$. The* graph of the link *s is the graph $G_s = (V_s = X_s(0), E_s = X_s(1))$. The graph of $s = \emptyset$ is called the 1-skeleton of X and is the graph whose vertices are $X(0)$ and whose edges are $X(1)$.*

Definition 2.12. *(Clique complex) Let G be a graph. The clique complex of the graph G, denoted by $X(G)$ is the simplicial complex formed by the sets of vertices in all the cliques of G.*

2.2.1 Hyper-regularity

Definition 2.13. *Let X be a t-dimensional simplicial complex and let $0 \leq i < j \leq t$. The j-degree of $s \in X(i)$, denoted by $D_j(s)$ is the number of j-faces in X containing s.*

Definition 2.14. *(Faces regularity) Let X be a t-dimensional simplicial complex. Let $0 \leq i < j \leq t$. We say X is (i, j)-regular with regularity $D_{i,j}$ if the j-degree of every i-face in X is exactly $D_{i,j}$*

For example $D_{0,1}$ denotes the edge regularity of the 1-skeleton, and $D_{0,2}$ denotes the number of triangles connected to each vertex.

Definition 2.15 (Hyper-regular simplicial complex)**.** *Let X be a t-dimensional simplicial complex. We say X is $(D_{-1}, D_0, \ldots, D_{t-1})$* hyper-regular *if for every $-1 \leq i \leq t-2$ and every $s \in X(i)$, the graph of the link X_s has D_i vertices and is D_{i+1} regular.*

A family $\mathcal{X} = \{X_N\}_{N \in I \subseteq \mathbb{N}}$ of t-dimensional simplicial complexes is called $(D_0, \ldots, D_{t-1})$ hyper-regular if X_N is $(D_{-1,N}, D_0, \ldots, D_{t-1})$ hyper-regular for some values $D_{-1,N}$ for all $N \in I$

Note that by this definition $D_{-1} = |X(0)|$. We sometimes omit D_{-1}.

Claim 2.16. *Let X be a t-dimensional $(D_{-1},\dots,D_{t-1})$ hyper-regular simplicial complex. Then, for every $0 \le i < j \le t$ X is (i,j)-regular with regularity*

$$D_{i,j} = \prod_{m=i}^{j-1} \frac{D_m}{j-m} = \frac{1}{(j-i)!} \prod_{m=i}^{j-1} D_m. \tag{2.1}$$

Proof. Fix $0 \le i < t$ and $s \in X(i)$. We prove that for every $i < j \le t$, X is (i,j)-regular, by induction on $j-i$.

Base case, $j = i+1$: By assumption $G_s = (V_s, E_s)$, the graph of the link X_s, has D_i vertices and is D_{i+1} regular. Notice that V_s is the set of $(i+1)$-faces containing s. Thus, the number of $(i+1)$-faces containing s is D_i, independent of s, and therefore $D_{i,i+1} = D_i$.

The induction step : Assume the existence of $D_{i,j}$. Fix $s \in X(i)$. By the induction hypothesis, there are $D_{i,j}$ j-faces containing s, and each such face has $D_{j,j+1}$ $(j+1)$-faces containing it. Furthermore, each $(j+1)$-face is counted exactly $j+1-i$ times, corresponding to the last item added to the face. We see that the number of $(j+1)$-faces containing s is independent of s and

$$D_{i,j+1} = \frac{D_{i,j} \cdot D_{j,j+1}}{j+1-i}.$$

Note that this relation holds trivially for $j = i$ under the convention $D_{i,i} = 1$.

We now prove Equation (2.1) by induction on $j-i$. The base case is $j = i+1$, and then $D_{i,i+1} = D_i = \prod_{m=i}^{j-1} \frac{D_m}{j-m}$. Then, by induction, and using $D_{j,j+1} = D_j$:

$$D_{i,j+1} = \frac{D_{i,j} \cdot D_{j,j+1}}{j+1-i} = \prod_{m=i}^{j-1} \frac{D_m}{j-m} \cdot \frac{D_j}{j+1-i} = \frac{\prod_{m=i}^{j} D_m}{\prod_{m=i}^{j} j+1-m}.$$

□

A simple corollary is:

Corollary 2.17. *Suppose X is a t-dimensional $(D_{-1}, D_0, ..., D_{t-1})$ hyper-regular simplicial complex. Then the bipartite inclusion graph $G = (V = X(t), W = X(0), E)$ where $(t,v) \in E$ iff $v \in t$ is $(t+1, D_{0,t})$ regular.*

Also note that by Claim 2.16

$$D_{0,t} = \frac{D_{0,1} D_{1,t}}{t}. \tag{2.2}$$

2.2.2 Hyper-expansion

Definition 2.18 (i-level expansion)**.** *Let X be a t-dimensional $(D_{-1},\dots,D_{t-1})$ hyper-regular simplicial complex. Let $-1 \le i \le t-2$. We say X has i-level one sided (resp. two sided) expansion λ_i, if for all $s \in X(i)$ the graph of its link, G_s, is a $(D_i, D_{i+1}, \lambda_i)$ one sided (resp. two sided) expander.*

Lemma 2.19 (Trickle-Down Theorem)**.** *([Opp18], as presented, e.g., in [FI20, Lemma 1.6]) Let X be a t-dimensional pure simplicial complex such that the graph of every link is connected and X has $(t-2)$ level expansion λ_{t-2}. Then, X has (-1)-level expansion at least*

$$\lambda_{-1} \le \frac{\lambda_{t-2}}{1-(t-1)\lambda_{t-2}}.$$

Hyper-regular, hyper-expanding graphs exist:

Theorem 2.20. *[FI20, Theorem 1.1] For every $t \in \mathbb{N}$ and $\lambda < 1$ there exist constants $D_{t-1} < D_{t-2} \dots < D_0$ and an infinite sequence $\{X_i\}_{i \in \mathbb{N}}$ such that*

- X_i *is a* t*-dimensional* $(D_{-1} = M_i, D_0, \ldots, D_{t-1})$ *hyper-regular pure simplicial complex,*
- $M_i \to \infty$ *and*
- X_i *has* $(t-2)$*-level one-sided expansion* $\lambda_{t-2} \leq \lambda$.

We need one slight modifications of the theorem: We want λ to be associated with λ_{-1} rather then λ_{t-2}.

Lemma 2.21. *For every* $t \in \mathbb{N}$ *and for every* $\lambda < 1$ *there exist constants* $D_{t-1} < D_{t-2} \ldots < D_0$ *and an infinite sequence* $\{X_i\}_{i\in\mathbb{N}}$ *such that*

- X_i *is a* t*-dimensional* $(D_{-1} = M_i, D_0, \ldots, D_{t-1})$ *hyper-regular pure simplicial complex,*
- $M_i \to \infty$*,*
- X_i *has* (-1)*-level one-sided expansion* $\lambda_{-1} \leq \lambda$

Proof. Given λ set $\lambda' = \beta\lambda$ for $\beta = \frac{1}{\sqrt{bt}+(t-1)\lambda}$. By Theorem 2.20 there exist constants $D_{t-1} < D_{t-2} \ldots < D_0$ and an infinite sequence $\{X_i\}_{i\in\mathbb{N}}$ such that X_i is a t-dimensional $(D_{-1} = M_i, D_0, \ldots, D_{t-1})$ hyper-regular pure simplicial complex with $(t-2)$-level expansion $\lambda_{t-2} \leq \lambda'$.

By Lemma 2.19, we get that

$$\lambda_{-1} \leq \frac{\lambda'}{1-(t-1)\lambda'} = \frac{\beta\lambda}{1-(t-1)\beta\lambda} = \frac{\lambda}{\sqrt{bt}}. \tag{2.3}$$

In particular, $\lambda_{-1} \leq \lambda$.

□

2.3 The high-dimension routed product

Definition 2.22 (Routed product)**.** *[AD23] Let:*

- $G = (V, W, E)$ *be a* (D_L, D_R)*-regular graph, and,*
- $H = (V' = [D_R], W', E')$ *be a* D'_L *left-regular graph.*

Let $\pi : E \to V'$ *be a labelling of the edges in* E *such that for every* $w \in W$*, the* D_R *edges in* E *containing* w *have distinct labels (and recall that* $V' = [D_R]$*). Then, the routed product* $G \circ H$ *is a bipartite graph*

$$(V'' = V, W'' = W \times W', E'')$$

where $(v, (w, j)) \in E''$ *iff* $(v, w) \in E$ *and* $(\pi(v, w), j) \in E'$.

We illustrate the definition in Figure 1.

Claim 2.23. $G \circ H$ *is* $D_L \cdot D'_L$ *left-regular. If, further,* H *is* (D'_L, D'_R) *regular then* $G \circ H$ *is* $(D_L \cdot D'_L, D'_R)$ *regular.*

Proof. For the first assertion, every vertex $v \in V'' = V$ has D_L edges going out of it in G, and each such edge is captured by a unique $(w, j) \in W \times [D'_R]$, which, in turn, has D'_L neighbors in W'.

For the second assertion assume H is (D'_L, D'_R) regular. Every vertex $w' \in W''$ has D'_R neighbors $(w, j) \in W \times [D'_R]$, which, in turn, has a unique neighbor in $V'' = V$. See Figure 1. Indeed one can check that

$$|V| \cdot D_L \cdot D'_L = |W| \cdot D_R \cdot D'_L = |W| \cdot |V'| \cdot D'_L = |W| \cdot |W'| \cdot D'_R = |W''| \cdot D'_R.$$

□

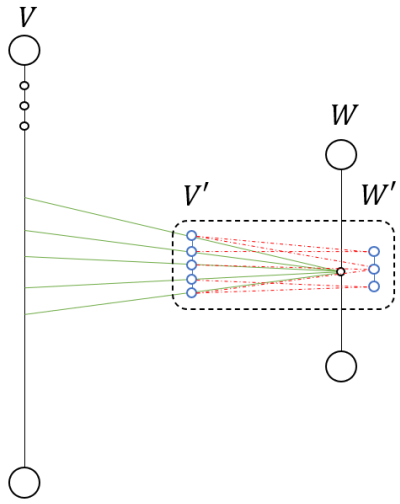

Figure 1: Illustration of the routed product $G \circ H = (V'', W'', E'')$. A node $w \in W$ is replaced with a copy of H.

3 Lossless Bipartite expanders

3.1 Scattering the input

Definition 3.1. *Let $G = (L, R, E)$ be a (D_L, D_R)-regular bipartite graph. Given a set $S \subseteq L$ we say a vertex $r \in R$ is S-*heavy*, if $|E(S, r)| \geq K'$. Let $H \subseteq R$ be the set of all S-heavy vertices. We say G is (K, K', ε)-scattering, if for every $S \subseteq L$ of size at most K,*

$$|E(S, H)| \leq \varepsilon |S| D_L.$$

Theorem 3.2 (HDXs are scattering)**.** *Suppose X is a t-dimensional $(D_{-1}, D_0, ..., D_{t-1})$ hyper-regular simplicial complex with (-1)-level expansion λ_{-1}. Let $G = (L = X(t), R = X(0), E)$ be the bipartite inclusion graph, and note that by Corollary 2.17 G is $(D_L = t+1, D_R = D_{0,t})$ regular. Then, for every $\varepsilon > 0$, G is $(K, K', \frac{1}{t+1} + \varepsilon)$-scattering for:*

- $K = \frac{\lambda_{-1}}{D_{0,t}} |L|$
- $K' = 2t(t + \frac{1}{\varepsilon})\lambda_{-1} D_{0,t}$

Proof. Let $S_t \subseteq X(t)$ be a set of t-faces of cardinality at most K. Define:

Definition 3.3 (Heavy vertices and bad t-faces)**.**

- *An element $x \in X$ is* selected *if $x \in s$ for some $s \in S_t$. We let S_i be the set of selected elements of dimension i and we call S_1 the set of selected edges and S_0 the set of selected vertices.*
- *A vertex $v \in S_0$ is* heavy *if*

$$|\{t \in S_t \mid v \in t\}| \geq K',$$

  Let $H_0 \subseteq S_0$ be the set of heavy vertices.

- *A t-face $r \in S_t \subseteq X(t)$ is* bad *if $|r \cap H_0| \geq 2$, i.e., if at least 2 out of its $t+1$ vertices are heavy. Let $B_t \subseteq S_t$ denote the set of bad (and selected) t-faces.*

We illustrate the definition in Figure 2.

The most crucial observation in the proof is that bad t-faces contribute edges going inside H_0, in the 1-skeleton of the graph. This, in turn, implies by the Alon-Chung lemma that B_t is small. Specifically,

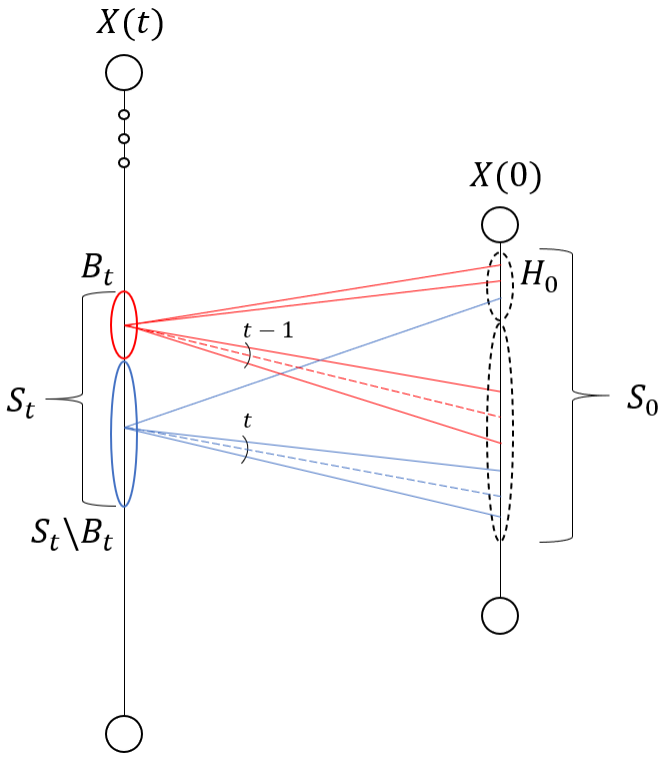

Figure 2: Illustration of the bipartite inclusion graph G. The left side vertices are the t-faces of X and the right vertices are the vertices of X. $H_0 \subseteq S_0$ is the set of heavy vertices, and $B_t \subseteq S_t$ is the set of bad t-face. A t-face is bad if at least two out of its $t+1$ vertices are heavy.

Lemma 3.4. $|B_t| \leq 2t\lambda_{-1} \cdot D_{0,t} \cdot |H_0|$.

Proof. We have

$$|H_0| \leq |S_0| \leq (t+1)|S_t| \leq (t+1)K = (t+1)\frac{\lambda_{-1}}{D_{0,t}}|X(t)| = \lambda_{-1}|X(0)|,$$

where we have used the facts that $K = \frac{\lambda_{-1}}{D_{0,t}}|L| = \frac{\lambda_{-1}}{D_{0,t}}|X(t)|$, $(t+1)|X(t)| = D_{0,t}|X(0)|$. Using Lemma 2.3 on the 1-skeleton $G = (V = X(0), E = X(1))$:

$$\frac{E(H_0, H_0)}{D_{0,1} \cdot |H_0|} \leq \lambda_{-1} + \frac{|H_0|}{|X(0)|} \leq 2\lambda_{-1}.$$

Now, every bad t-face $t = \{v_1, ..., v_{t+1}\} \in B_t$ contains at least two vertices from H_0, and therefore at least one edge in $E(H_0, H_0)$. Also, every edge in $E(H_0, H_0)$ may be contributed by at most $D_{1,t}$ t-faces in B_t. Therefore,

$$E(H_0, H_0) \geq \frac{|B_t|}{D_{1,t}}. \tag{3.1}$$

It therefore follows that:

$$|B_t| \leq D_{1,t}E(H_0, H_0) \leq 2\lambda_{-1}D_{0,1}D_{1,t}|H_0| = 2t\lambda_{-1}\frac{D_{0,1}D_{1,t}}{t}|H_0| = 2t\lambda_{-1}D_{0,t}|H_0|.$$

□

We also note that:

Lemma 3.5. $|S_t \setminus B_t| \geq |E(S_t \setminus B_t, H_0)| \geq K'|H_0| - (t+1)|B_t|$.

Proof. For the first inequality, notice that in the subgraph $E(S_t \setminus B_t, H_0)$ the degree of vertices in $S_t \setminus B_t$ is at most one. This is true because otherwise there is some $s \in S_t \setminus B_t$ and two distinct vertices $v_1, v_2 \in H_0$ such that $v_1, v_2 \in s$, but then the t-face s contains at least two heavy vertices and therefore $s \in B_t$ which is a contradiction. Hence,

$$|E(S_t \setminus B_t, H_0)| \leq |S_t \setminus B_t|. \tag{3.2}$$

For the second inequality notice that since every vertex in H_0 is heavy, every vertex in H_0 participates in at least K' t-faces in S_t, and therefore

$$|E(S_t, H_0)| \geq K'|H_0|. \tag{3.3}$$

Also, since every t-face touches at most $t+1$ heavy vertices we have

$$|E(B_t, H_0)| \leq (t+1)|B_t| \tag{3.4}$$

Equations (3.3) and (3.4) together give the second inequality. □

We conclude that:

$$\begin{aligned}\frac{|S_t|}{|B_t|} &= \frac{|S_t \setminus B_t|}{|B_t|} + 1 \\ &\geq \frac{K'|H_0|}{|B_t|} - (t+1) + 1 \\ &\geq \frac{K'}{2t\lambda_{-1}D_{0,t}} - t \;\geq\; t + \frac{1}{\varepsilon} - t = \frac{1}{\varepsilon},\end{aligned}$$

where the first inequality is by Lemma 3.5, the second inequality is by Lemma 3.4 and the last inequality is by plugging $K' = 2t(t + \frac{1}{\varepsilon})\lambda_{-1} \cdot D_{0,t}$. Thus,

$$|B_t| \leq \varepsilon|S_t|. \tag{3.5}$$

Concluding the proof for Theorem 3.2, for every $S_t \subseteq L$ of size at most K,

$$\begin{aligned}|E(S_t, H_0)| &= |E(S_t \setminus B_t, H_0)| + |E(B_t, H_0)| \leq |S_t \setminus B_t| + |B_t| \cdot (t+1) \\ &\leq |S_t|(t+1) \cdot (\frac{1}{t+1} + \varepsilon),\end{aligned} \tag{3.6}$$

using Equations (3.2) and (3.5).

□

3.2 The composition theorem

Theorem 3.6. *If $G = (L_G, R_G, E_G)$ is a $(D_{L,G}, D_{R,G})$-regular bipartite graph that is $(K_{L,G}, K_{R,G}, \varepsilon_G)$-scattering, and $H = (L_H = [D_{R,G}], R_H = [\mu_H D_{R,G}], E_H)$ is a $(D_{L,H}, K_H, \varepsilon_H)$ bipartite expander, with $K_{R,G} \leq K_H$ then $G \circ H = (L, R, E)$ is such that:*

- $|L| = |L_G|$
- $|R| = \mu_H D_{L,G}|L|$,
- *$G \circ H$ is $D_{L,G}D_{L,H}$ left-regular, and,*
- *$G \circ H$ is $(K_{L,G}, (1-\varepsilon_G)(1-\varepsilon_H))$ expanding.*

If, further, H is $(D_{L,H}, D_{R,H})$ regular, then $G \circ H$ is $(D_{L,G}D_{L,H}, D_{R,H})$ regular.

Proof. To see the first claim notice that $L = L_G$, $R = R_G \times R_H$ and therefore

$$|R| = |R_G| \cdot |R_H| = |R_G| \cdot \mu_H D_{R,G}.$$

However, $|R_G| D_{R,G} = |E_G| = |L_G| \cdot D_{L,G}$ and therefore $|R| = \mu_H D_{L,G} |L_G|$. For the second claim, we recall that G is $(D_{L,G}, D_{R,G})$ regular and H is $D_{L,H}$ left-regular. Also, if, further H is $(D_{L,H}, D_{R,H})$ regular then by Claim 2.23, $G \circ H$ is $(D_{L,G} D_{L,H}, D_{R,H})$ regular.

The main thing to prove is the fourth claim about expansion. Fix $S \subseteq L = L_G$ of cardinality at most $K_{L,G}$. It remains to be shown that $|\Gamma(S)| \geq (1 - \varepsilon_G)(1 - \varepsilon_H)|S| D_{L,G} D_{L,H}$. Now, in the construction of the routed product $G \circ H$, we put a gadget

$$H_z = (L_{H,z} = [D_{R,G}], R_{H,z}, E_{H,z})$$

for every $z \in R_G$, where a vertex in $L_{H,z}$ represents an incoming edge (x, z) in G. Let $S'_z \subseteq L_{H,z}$ be the set of vertices from $L_{H,z}$ that correspond to incoming edges $(x, z) \in E_G$ with $x \in S$.

Definition 3.7. *We say $z \in R_G$ is* heavy *if $|S'_z| \geq K_{R,G}$, and* light *otherwise. Let G_0 denote the set of all light elements in S_0.*

For every light z, $|S'_z| \leq K_{R,G} \leq K_H$. As H_z is a $(D_{L,H}, K_H, \varepsilon_H)$ bipartite expander,

$$|\Gamma_H(S'_z)| \geq (1 - \varepsilon_H) D_{L,H} |S'_z|$$

Thus,

$$|\Gamma(S)| \geq \sum_{z \in G_0} (1 - \varepsilon_H) D_{L,H} |S'_z| = (1 - \varepsilon_H) D_{L,H} \sum_{z \in G_0} |S'_z|.$$

To complete the proof we notice that as G is $(K_{L,G}, K_{R,G}, \varepsilon_G)$-scattering we have:

$$\sum_{z \in G_0} |S'_z| = |E(S, G_0)| \geq (1 - \varepsilon_G)|S| D_{L,G},$$

□

4 Explicitness

A D-regular graph $G = (V, E)$ is *explicit* if we can describe G with an algorithm running in time polynomial in the description size of G. The graph G is *fully explicit* if we can describe local components of G efficiently, e.g., if given $v \in V$ and $i \in [D]$ we can describe the i'th neighbor of v in time polynomial in the input length to the task, i.e., polynomial in $\log |V| + \log D$. Similarly, we say an object is explicit if we can describe the entire object in time polynomial in the description size, and we say it is fully explicit if we can locally describe small parts of the entire object in time polynomial in the input length to the task.

However, there is a subtle problem that is swept under the rug in the above discussion. Suppose we wish to construct a 2-dimensional HDX X. Suppose, as indeed is the case, we have a natural interpretation of X as group elements of $SL_3(R)$ over some ring R. Further suppose that given $s \in X(2)$ we can efficiently find the three vertices in it, and given a vertex we can find the $D_{0,2}$ triangles lying over it, using efficient group operations. The subtle problem that we face is that often the group G is naturally represented in a redundant way, e.g., $SL_3(R)$ might be represented as a 3×3 matrix, even though only some of the matrices are invertible and with determinant one. Thus, if for the applications (e.g., for constructing lossless condensers) we want to use the HDX on binary strings, we also need efficient procedures mapping strings to group elements (or triangles), and vertices back to strings. Sometimes, we need even more. For example, for the routed product, we need to map cloud names (which happen to be cosets of the group) to strings, and we need to index vertices within the cloud. Often, we need this indexing to be compressed, i.e., without much redundancy, and this is also the case when we want to construct a lossless condenser.

For this reason, we have two representations in mind: the first is a natural representation of the combinatorial object, and this representation is often redundant. The second is a compressed representation. The

way we treat this below, is that we first show (local) explicitness for the combinatorial object in the natural redundant representation, and then we show how to translate back and forth between the redundant and compressed representation. We divide our task into two parts, that allow us to talk separately about the combinatorial object in its own language (e.g. as a group) and in the language of the "outside observer" that does not need to understand the details of the combinatorial object but only needs to be able to easily access it in a compressed way.

To be more concrete, consider the case of a Cayley graph $C(G,S)$ of a group G with a set of generators S. Using the language of group theory, the graph is very easily described. The vertices are G and the edges correspond to pairs consisting of an element from the group and a generator. However, an "outside observer", which does not speak the group-theory language, also needs to know how to index (or represent) group elements, and how to carry on the group operations. Schematically, this looks like:

$$[n] \xrightarrow{\text{Indexing}} C \xrightarrow{\text{Explicit combinatorial description}} C \xrightarrow{\text{Compact representation}} [m] \tag{4.1}$$

Note that Definition 2.7 *does not* separate between the different parts of Equation (4.1), but captures the whole process at once.

Definition 4.1. *(Combinatorial explicitness) Let $\mathcal{G}$ be a family of (D_L, D_R)-regular bipartite graphs $\mathcal{G} = \{G_N\}_{N\in\mathbb{N}}$ where $G_N = (L_N, R_N, E_N)$. We stress that L_N, R_N and E_N may represent their object in a redundant way, e.g., it may represent an invertible matrix with determinant one by its entries.*

We say $\mathcal{G}$ is combinatorially explicit *if there exists a polynomial-time algorithm that on input $N \in \mathbb{N}$, $\ell \in L_N$ and $k \in [D_L]$ outputs a pair (r,j) such that $r \in R_N$ and $j \in [D_R]$, and r is the k'th neighbor of ℓ and ℓ is the j'th neighbor of r.*

Definition 4.2. *Let $\mathcal{A} = \{A_N\}_{N\in\mathbb{N}}$ be a family of sets.*

- *We say $\mathcal{A}$ has* efficient indexing*, if there exists a polynomial time algorithm that on input $N \in \mathbb{N}$ and $i \in [|A_N|]$ outputs the i'th elements of A_N.*
- *Let $\Delta \in \mathbb{N}$. We say $\mathcal{A}$ has a* Δ-efficient encoding*, if there exists a polynomial time algorithm that for every $N \in \mathbb{N}$ injectively maps A_N to $[\Delta \cdot |A_N|]$.*

Definition 4.3. *Let $\mathcal{X} = \{X_N\}_{N\in\mathbb{N}}$ be a sequence of t-dimensional, $(0,t)$-regular simplicial complexes. We say $\mathcal{X}$ is combinatorially explicit if the bipartite inclusion graph family $\{(X_N(t), X_N(0))\}_{N\in\mathbb{N}}$ is combinatorially explicit.*

We now state a construction that parameter-wise is the same as Lemma 2.21, and in addition is also explicit.

Theorem 4.4. *For every $t \in \mathbb{N}$ and for every $\lambda < 1$ there exist constants $D_{t-1} < D_{t-2} \ldots < D_0$ and an combinatorially explicit sequence $\mathcal{X} = \{X_i\}_{i\in\mathbb{N}}$ such that*

- *X_i is a t-dimensional $(D_{-1}, D_0, \ldots, D_{t-1})$ hyper-regular pure simplicial complex,*
- *X_i has (-1)-level one-sided expansion $\lambda_{-1} \leq \lambda$*
- *$\{X_N(t)\}_{N\in\mathbb{N}}$ has efficient indexing, $\{X_N(0)\}_{N\in\mathbb{N}}$ has Δ-efficient encoding for $\Delta = ((t+1)!)^{(t+1)!}$ and the sequence $\{|X_N(t)|\}_{N\in\mathbb{N}}$ is polynomially-dense.*

The proof is given in the appendix in Proposition A.13, the construction is a variant of the Friedgut-Iluz construction. A direct consequence is:

Corollary 4.5. *Let $\mathcal{X}$ be as in Theorem 4.4 and let $\mathcal{G}$ denote the bipartite (D_L, D_R)-biregular inclusion graph family $\{(X_N(t), X_N(0))\}_{N\in\mathbb{N}}$. Let $H = (L', R', E')$ be a graph with $L' = [D_R]$ and $R' = [M]$ and left degree $D_{L'}$ then their routed product, $\{G_N \circ H\}_{N\in\mathbb{N}}$ is combinatorially explicit, the left hand side of $\{G_N \circ H\}_{N\in\mathbb{N}}$ has efficient indexing and the right hand side of $\{G_N \circ H\}_{N\in\mathbb{N}}$ has a Δ-efficient encoding for $\Delta = ((t+1)!)^{(t+1)!}$.*

The final bipartite graph is the same as $G_N \circ H$, except that we identify an element on the left hand size with its indexing, and we map an element of the right hand side to its encoding. This gives us a fully-explicit bipartite graph, with the same properties as $\{G_N \circ H\}_{N\in\mathbb{N}}$, except that the right hand size is larger by a factor of Δ. In the next section we analyze the parameters that we get.

5 Putting it together

We first write what we get in the combinatorial representation:

Theorem 5.1. *For every $\varepsilon > 0$ small enough and $\mu > 0$ there exists a polynomially-dense subset $A \subseteq \mathbb{N}$, and a combinatorially-explicit construction $\{G_N\}_{N\in A}$ s.t. for every $N \in A$, G_N is a bipartite expander (L_N, R_N, E_N) with $|L_N| = N$, $|R_N| = \mu N$, where*

- *G_N is D_L left-regular, where $D_L = \Theta(\frac{\log(\frac{\varepsilon}{\mu})}{\varepsilon^2})$, and,*
- *G_N is (K_N, ε) expanding, where $K_N = (\varepsilon\mu)^{O(t^t)}|L_N|$.*

Furthermore, given $x \in L_N$ and $i \in [D_L]$, the i'th neighbor of x can be computed in time polynomial in $\log(\frac{N}{\varepsilon})$.

Proof. Given ε and μ let $\varepsilon' = \frac{\varepsilon}{3}$, $t = \frac{1}{\varepsilon'}$ and $\mu_H = \frac{\mu}{t+1}$.

- Apply Theorem 4.4 with t and $\lambda_{-1} = c \cdot (\varepsilon')^4\mu_H^2$ for some small enough constant c. By Theorem 4.4 there exist constants $D_{t-1} < D_{t-2} < \ldots < D_0$ and a combinatorially explicit sequence $\{X_N\}_{N\in\mathbb{N}}$ with $\Delta = ((t+1)!)^{(t+1)!}$ such that X_N is a t-dimensional $(D_{-1}, D_0, \ldots, D_{t-1})$ hyper-regular pure simplicial complex with (-1)-level one-sided expansion λ_{-1}. Also each X_N is $(0,t)$-regular with $D_{0,t}$.
- Applying Theorem 3.2 with $\{X_N\}_{N\in\mathbb{N}}$ and ε', we see that the family of bipartite graphs $G(X_N) = (L_{G_N} = X_N(t), R_{G_N} = X_N(0), E_{G_N})$ is $(t+1, D_{0,t})$-regular and $(K_{L,G_N}, K_{R,G_N}, \frac{1}{t+1} + \varepsilon')$-scattering for:
  - $K_{L,G_N} = \frac{\lambda_{-1}}{D_{0,t}}|L_{G_N}|$,
  - $K_{R,G_N} = 2t(t + \frac{1}{\varepsilon'})\lambda_{-1}D_{0,t}$.

Next, we find the small (fixed) bipartite graph H by brute force. We apply Lemma 2.5 with $|L_H| = D_{0,t}$, ε' and μ_H to get a bipartite graph $H = (L_H, R_H, E_H)$ that is a $(D_{L,H}, K_H, \varepsilon')$ expander with

- $D_{L,H} = \Theta\Big(\frac{\log\frac{2}{\mu_H}}{\varepsilon'}\Big)$.
- $|K_H| = \Theta\big(\varepsilon'\frac{|R_H|}{D_{L,H}}\big)$.

We first notice that

$$K_{R,G_N} \le O\left(\frac{\lambda_{-1}D_{0,t}}{\varepsilon^2}\right) \le O(\varepsilon^2\mu_H^2)D_{0,t}$$
$$K_H = \Theta\left(\frac{(\varepsilon')^2\mu_H D_{0,t}}{\log(2/\mu_H)}\right) = \Theta(\varepsilon^2\mu_H^2)D_{0,t}.$$

Thus, $K_{R,G_N} \le K_H$ when the constant c chosen above is small enough.

Having $\{G(X_N)\}_{N\in\mathbb{N}}$ and H we let $G'_N = G(X_N) \circ H$ (for every $N \in \mathbb{N}$). By Theorem 3.6, $G'_N = G(X) \circ H = (L'_N, R'_N, E'_N)$, where $|L'_N| = |X_N(t)|$. Thus, the right-hand side R'_N of G'_N has cardinality:

$$|R'_N| = \mu_H D_{L,G_N}|L_N| = \mu_H(t+1)|L_N| = \mu|L_N|.$$

We also have

- G'_N is $D_L = (t+1)D_{L,H} = \Theta\Big(\frac{\log(\frac{\varepsilon}{\mu})}{\varepsilon^2}\Big)$ left-regular.
- G'_N is $(K = \frac{\lambda_{-1}}{D_{0,t}}|L_N|, (1 - \frac{1}{t+1} - \varepsilon')(1-\varepsilon'))$ expanding.

Notice that $1 - \frac{1}{t+1} = \frac{t}{t+1} = \frac{1}{1+\varepsilon'} \ge 1 - \varepsilon'$ and so $(1 - \frac{1}{t+1} - \varepsilon')(1-\varepsilon') \ge (1-2\varepsilon')(1-\varepsilon') \ge 1 - 3\varepsilon' = 1 - \varepsilon$. Also $K = \frac{\lambda_{-1}}{D_{0,t}}|L_N|$. From [FI20, sec 4.4.1.1] we can upper bound $D_{0,t} \le (1/\lambda_{-1})^{O(t^t)}$. This gives $K = \lambda_{-1}^{O(t^t)}|L_N|$. □

To get full-explicitness we need to map the vertices on the right-hand side of the bipartite graph of Theorem 5.1 to an efficient representation. This enlarges the right-hand size by a factor of $\Delta_t = (t+1)!^{(t+1)!}$, or equivalently, in the above theorem we shrink by a factor of $\Delta_t\mu$ instead of μ. Thus, if we want a constant $\varepsilon > 0$, this fixes a constant $t = O(1/\varepsilon)$ and we get:

Theorem 5.2. *For every constant $\varepsilon > 0$ and $\mu > 0$ there exists a polynomially-dense subset $A \subseteq \mathbb{N}$, and a fully-explicit construction $\{G_N\}_{N\in A}$ s.t. for every $N \in A$, G_N is a bipartite expander (L_N, R_N, E_N) with $|L_N| = N$, $|R_N| = \mu N$, where*

- *G_N is D_L left-regular, where $D_L = \Theta_\varepsilon(\log(\frac{1}{\mu}))$, and,*
- *G_N is (K_N, ε) expanding, where $K_N = \mu^{O_\varepsilon(1)}|L_N|$,*

where the constants hide the dependence on ε. Furthermore, given $x \in [L_N]$ and $i \in [D_L]$, the i'th neighbor of x can be computed in time polynomial in $\log N$.

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A Explicitness of the Friedgut-Iluz construction

In the t-dimensional KO construction (see Appendix B), the 1-skeleton $G = (X(0), X(1))$ is a $(t+1)$-partite graph where every t-face selects one vertex from each color. The regularity in the KO construction (e.g., the number of t-dimensional faces on an i-face ($i < t$) depends on the coloring of the i-face). The [FI20] construction may be seen as a symmetrization of the KO construction. In Appendix A.1 we define type-regularity for HDX like KO, where the regularity depends on the type. In Proposition A.13, we show that the [FI20] reduction from type-regular to hyper-regular preserves combinatorial explicitness, as well as the efficient indexing of the t-faces and the efficient encoding of the vertices. In Appendix B we show that the KO construction has all these properties, thus proving that the complete construction is combinatorially explicit with efficient indexing and encoding.

A.1 Type-regularity

The construction of hyper-regular high-dimensional expanders is highly non-trivial. As stated in Theorem 2.20, such complexes are constructed in [FI20]. Their construction starts with *clique complexes* with a weaker notion of regularity.

Definition A.1 (P-partite)**.** *Let P be an arbitrary finite set. We say that a graph $G = (\bigcup_{x\in P} V_x, E)$ is P-partite if V_i and V_j are disjoint for every $i \neq j \in P$, and each V_i is an independent set. We say that a simplicial complex X is P-partite if it is a clique complex and its 1-skeleton is a P-partite graph.*

In the following definitions, let X be a t-dimensional P-partite simplicial complex. Informally, a face s of type I is a face that touches exactly the parts I, formally,

Definition A.2 (Type)**.** *A face $s \in X$ is of type $I \subseteq P$ if for every $i \in I$ there exists a unique $v \in s \cap V_i$ and for every $i \notin I$: $s \cap V_i = \emptyset$. We let $X(I)$ denote the set of faces $s \in X$ of type I.*

Definition A.3 (Type-regularity)**.** *Let $I \subseteq J \subseteq P$. X is (I, J) regular, with degree $D_{I,J}$, if the every face s of type I in X is contained in exactly $D_{I,J}$ faces of type J. X is P-partite type-regular if for every $I \subseteq J \subseteq P$, X is (I, J) regular.*

We remark that in [FI20] type regularity is defined only with respect to types $I \subset J$ where J extends I with a single element. However, it is easy to see that the two definitions are equivalent, similarly to Claim 2.16.

We restrict out discussion to $P = [t+1]$ and to type regular complexes which are also $(0, t)$ regular in the sense of Definition 2.14. In the type notations, this means that for every i, the simplicial complex is $(\{i\}, [t+1])$-regular, and all these degrees are equal. For such objects, we already defined combinatorial explicitness in Definition 4.3.

A.2 Explicitness preservation under symmetrization

Definition A.4. *(Partite graph product [FI20]) Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be P-partite graphs, where $V_1 = (V_1^p)_{p\in P}$, $V_2 = (V_2^p)_{p\in P}$ are ordered tuples of mutually disjoint subsets. We define their P-partite product, denoted by $\circledast$, to be*

$$G_1 \circledast G_2 = G_{12} = (V_{12}, E_{12})$$

Where

$$V_{12} = \left(V_1^p \times V_2^p\right)_{p\in P}$$

$$E_{12} = \bigcup_{p_1 <_P p_2} \left\{ \left\{ (v_1^{p_1}, v_2^{p_1}), (v_1^{p_2}, v_2^{p_2}) \right\} \ : \ (v_1^{p_1}, v_1^{p_2}) \in E_1 \text{ and } (v_2^{p_1}, v_2^{p_2}) \in E_2 \right\}$$

Definition A.5. *[FI20] For clique complexes $X(G_1)$, $X(G_2)$, the product is naturally defined as follows:*

$$X(G_1) \circledast X(G_2) = X(G_1 \circledast G_2)$$

Let $\mathcal{X} = \{X_N\}_{N\in\mathbb{N}}$ $\mathcal{Y} = \{Y_N\}_{N\in\mathbb{N}}$ be sequences of $[t+1]$-partite t-dimensional simplicial complexes the P-partite product of $\mathcal{X}$ and $\mathcal{Y}$ is naturally defined as follows:

$$\mathcal{Z} = \mathcal{X} \circledast \mathcal{Y} = \{X_N \circledast Y_N\}_{N\in\mathbb{N}}.$$

Claim A.6. *Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two P-partite clique complexes. For every $I \subseteq P$, $\{v_1^i\}_{i\in I} = C_1 \subseteq V_1$ is a clique of type I in G_1 and $\{v_2^i\}_{i\in I} = C_2 \subseteq V_2$ is a clique of type I in G_2 if and only if $C_{12} = \{(v_1^i, v_2^i)\}_{i\in I}$ is a clique of type I in G_{12}.*

Proof. For all $p \in P$ denote $V_{12}^p \triangleq V_1^p \times V_2^p$.

($\implies$) Let C_1 and C_2 be as in Claim A.6. Fix two vertices $u_a = (v_1^{p_1}, v_2^{p_1})$ and $u_b = (v_1^{p_2}, v_2^{p_2})$ in C_{12}. Since C_1 is a clique of type I, there exists an edge $e = (v_1^{p_1}, v_1^{p_2}) \in E_1$. The same holds for C_2 with $e = (v_2^{p_1}, v_2^{p_2}) \in E_2$. Thus, by definition, $\{(u_a, u_b)\} \in E_{12}$. Since this is true for every $u_a, u_b \in C_{12}$, they form a clique.

($\impliedby$) Let C_{12} be as in Claim A.6. Fix two vertices in $v_1^{p_1}, v_1^{p_2} \in C_1$ and two vertices $v_2^{p_1}, v_2^{p_2} \in C_2$. Since C_{12} is a clique in G_{12}, there exists an edge $\left\{(v_1^{p_1}, v_2^{p_1}), (v_1^{p_2}, v_2^{p_2})\right\} \in E_{12}$. By the definition of partite graph product, this means that $(v_1^{p_1}, v_1^{p_2}) \in E_1$ and $(v_2^{p_1}, v_2^{p_2}) \in E_2$. Since this is true for every $v_1^{p_1}, v_1^{p_2} \in C_1$, C_1 forms a clique in G_1. The same holds for C_2 in G_2. □

Claim A.6 motivates the following notation. Let X and Y be two clique complexes and let $Z = X \circledast Y$. Let $C_Z = \left\{(v_X^{p_1}, v_Y^{p_1}), (v_X^{p_2}, v_Y^{p_2})\right\}$ be a clique in Z. We denote $C_Z|_X$ and $C_Z|_Y$ its corresponding cliques in X and in Y.

Corollary A.7. *If $\mathcal{X}$ has (I, J) degree D_X and $\mathcal{Y}$ has (I, J) degree D_Y then $\mathcal{Z}$ has (I, J) degree $D_X \cdot D_Y$. Particularly, if $\mathcal{X}$ and $\mathcal{Y}$ are type regular, then $\mathcal{Z}$ is type regular.*

Proof. This is trivial given Claim A.6 □

We now turn to prove that the partite graph product preserves the explicitness properties of type regular simplicial complexes.

Claim A.8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two sequences of type regular clique complexes. If $\mathcal{X}$ and $\mathcal{Y}$ are combinatorially explicit then $\mathcal{Z}$ is combinatorially explicit.*

Proof. First, note that the $(0, t)$ degree of $\mathcal{Z}$ is $D_{0,t}^Z = D_{0,t}^X \cdot D_{0,t}^Y$. We are given an input $N \in \mathbb{N}$, $s \in Z(t)$ and $i \in [t+1]$. We use the combinatorial explicitness of $\mathcal{X}$ and $\mathcal{Y}$ on $s|_X$ and s_Y to obtain (v_i^X, j_X) and (v_i^Y, j_Y). We output $((v_i^X, v_i^Y), j_X \cdot |D_{0,t}^Y| + j_Y)$. □

We proceed by proving that efficient indexing of top dimensional faces is preserved under the partite graph product.

Claim A.9. *Let $\mathcal{X} = \{X_N\}_{N\in\mathbb{N}}$ and $\mathcal{Y} = \{Y_N\}_{N\in\mathbb{N}}$ be two sequences of $[t+1]$-partite t-dimensional simplicial complexes. Let $\mathcal{Z} = \{Z_N\}_{N\in\mathbb{N}}$ denote their partite graph product. If $\mathcal{X}_t = \{X_N(t)\}_{N\in\mathbb{N}}$ and $\mathcal{Y}_t = \{Y_N(t)\}_{N\in\mathbb{N}}$ have efficient indexing then $\mathcal{Z}_t = \{Z_N(t)\}_{N\in\mathbb{N}}$ has efficient indexing.*

Proof. Let $A_\mathcal{X}$ and $A_\mathcal{Y}$ be algorithms for efficient indexing for $\mathcal{X}_t$ and $\mathcal{Y}_t$. We construct an algorithm for efficient indexing on $\mathcal{Z}_t$. We are given an input $N \in \mathbb{N}$ and $i \in |Z_N(t)| = |X_N(t)| \cdot |Y_N(t)|$. We divide it with a remainder by $|Y_N(t)|$, $i = a|Y_N(t)| + b$. We then output the t-face obtained by $A_\mathcal{X}(a), A_\mathcal{X}(b)$ and Claim A.6. □

The Δ-efficient encoding is slightly more difficult. The efficient encoding is preserved under $\circledast$, but with some deterioration in the parameters. In a $[t+1]$ partite graph, we index the vertices by their side, and by their index in their side. That is, we index a vertex by (i, j) where $i \in [t+1]$ and $j \in [|V_i|]$, where V_i are the vertices of type $\{i\}$.

Claim A.10. *Let $\mathcal{X} = \{X_N\}_{N\in\mathbb{N}}$ and $\mathcal{Y} = \{Y_N\}_{N\in\mathbb{N}}$ be two sequences of $[t+1]$-partite t-dimensional simplicial complexes. Let $\mathcal{Z} = \{Z_N\}_{N\in\mathbb{N}}$ denote their partite graph product. If $\mathcal{X}_0 = \{X_N(0)\}_{N\in\mathbb{N}}$ and $\mathcal{Y}_0 = \{Y_N(0)\}_{N\in\mathbb{N}}$ have Δ_X and Δ_Y efficient encoding then $\mathcal{Z}_0 = \{Z_N(0)\}_{N\in\mathbb{N}}$ has $\Delta_X \cdot \Delta_Y$ efficient encoding.*

Proof. Let $A_{\mathcal{X}}$ and $A_{\mathcal{Y}}$ be algorithms for Δ_X and Δ_Y efficient encoding for $\mathcal{X}_0$ and $\mathcal{Y}_0$. We construct an algorithm for $\Delta_X \cdot \Delta_Y$ efficient encoding for $\mathcal{Z}_0$. We are given an input $N \in \mathbb{N}$ and $v \in Z_N(0)$. By the definition of the partite graph product, v is of the form (v_X, v_Y) where $v_X \in X_N(0)$ and $v_Y \in Y_N(0)$ and both are of type $\{i\}$ for some $i \in [t+1]$. We compute $((i, j_X)) = A_{\mathcal{X}}(N, v_X)$ and $((i, j_Y)) = A_{\mathcal{Y}}(N, v_Y)$. We output (i, j_X, j_Y). □

Claim A.11. *If $\mathcal{X}_0 = \{X_N(0)\}_{N\in\mathbb{N}}$ and $\mathcal{Y}_0 = \{Y_N(0)\}_{N\in\mathbb{N}}$ have Δ_X and Δ_Y-efficient encoding then $\mathcal{Z}_0 = \{Z_N(0)\}_{N\in\mathbb{N}}$ has $\Delta_X \cdot \Delta_Y$ efficient encoding.*

The symmetrization process described in [FI20] goes as follows. For a $[t+1]$-partite graph G and a permutation π on $[t+1]$, denote $\pi(G)$ the graph obtained from X by changing the names of the sides. That is, if the vertices of G are $V = (V_0, \ldots, V_t)$, then the vertices of $\pi(G)$ are $(V_{\pi(0)}, \ldots, V_{\pi(T)})$ (recall that the names of the sides matter in the definition of the partite graph product, Definition A.4). The symmetrization of a graph is then given by $G^{\circledast S_{t+1}} = \underset{\pi\in S_{t+1}}{\circledast} \pi(G)$. The symmetrization of a clique complex and the symmetrization of a family of $[t+1]$-partite complexes extend this definition naturally.

Lemma A.12. *Fix a prime power q and an integer t. Let $\mathcal{X}$ be the family described in Appendix B.1 and let $\mathcal{S} = \{S_N\}_{N\in\mathbb{N}}$ denote its symmetrization. The set $\{|S_N(t)|\}_{N\in\mathbb{N}}$ is polynomially dense.*

Proof. Let $\mathcal{X} = \{X_N\}_{N\in\mathbb{N}}$ and $\mathcal{S} = \{S_N\}_{N\in\mathbb{N}}$ be as in the claim. Observe that $|X_N(t)| = |SL_{t+1}(\mathbb{F}_{q^N})|$. Using Claim A.6, for all $N \in \mathbb{N}$, the t-faces in S_N are exactly the sequences of $(t+1)!$ t-faces in X_N. Counting them, we have $|S_N(t)| = |X_N(t)|^{(t+1)!}$ and so the sizes of $S_N(t)$ are $\{|SL_{t+1}(\mathbb{F}_{q^N})|^{(t+1)!}\}_{N\in\mathbb{N}}$, which is polynomially dense.

□

Proposition A.13. *Fix a prime power q and an integer t. Let $\mathcal{X}$ be the family described in Appendix B.1 and let $\mathcal{S}$ denote its symmetrization. $\mathcal{S}$ is combinatorially explicit, $\mathcal{S}_t = \{S_N(t)\}_{N\in\mathbb{N}}$ has an efficient indexing and the sequence of their sizes is polynomially-dense. Additionally, $\mathcal{S}_0 = \{S_N(0)\}_{N\in\mathbb{N}}$ has a $((t+1)!)^{(t+1)!}$ efficient encoding.*

Proof. Note that changing the names of the sides of $\mathcal{X}$ trivially preserves the combinatorial explicitness, the efficient indexing and the $(t+1)!$ efficient encoding. The symmetrization is thus a repeated $\circledast$ operation ($(t+1)!$ times) on combinatorially explicit families with efficient indexing for their t-faces and efficient encoding for their vertices. The combinatorial explicitness of $\mathcal{S}$ is obtained by a repeated application of Claim A.8. The efficient indexing of $\mathcal{S}_t$ similarly follows Claim A.9. The polynomial density is by Lemma A.12. Finally, the efficient encoding of $\mathcal{S}_0$ is again a repeated application of Claim A.10 □

B Explicitness of the Kaufman-Oppenheim construction

The goal of this section is to prove the explicitness of the HDX construction of [KO23]. We begin with an overview of the construction. We then show the combinatorial description of its vertices and highest dimension faces. We then suggest a small change in construction and claim that the use of polynomial rings can be replaced with corresponding finite fields while preserving the HDX properties. This will allow us to construct a polynomial time algorithm to enumerate the faces of the highest dimension. Finally, we present a compressed representation for the vertices.

B.1 KO construction overview

The construction (theorem 4.10 of [KO23]) has 3 parameters: t, q and s where:

- t - controls the dimension of the highest dimensional face (denoted n in the original paper)
- q - controls the spectral gap
- s - controls the number of vertices, such that the number tends to infinity as s tends to infinity

Let $t \geq 2$ and let q be a prime power such that $q > (t-1)^2$ and $s \in \mathbb{N}$. We let

$$\mathcal{R}_s = \mathbb{F}_q[x]/\langle x^s \rangle$$

$\mathcal{R}_s$ is an $\mathbb{F}_q$ algebra with generating set $\{1, x\}$. We let

$$G = SL_{t+1}(\mathcal{R}_s)$$

i.e., G is the set of $(t+1) \times (t+1)$ matrices with determinant 1 and elements from $\mathcal{R}_s$.

Next, we define $t+1$ subgroups $K_{\{i\}}$ of G. For $0 \leq i, j \leq t$, $i \neq j$ and $r \in \mathcal{R}_s$, let $e_{i,j}(r)$ be the $(t+1) \times (t+1)$ matrix with 1's along the main diagonal, r in the (i,j) entry and 0's in all other entries. We define the subgroup $K_{\{i\}}$ for every $0 \leq i \leq t$ to be the subgroup that is generated by all the elements $e_{j,j+1}(r)$ where $j \neq i$ and $r \in \mathcal{R}_2$. It is a fact that $K_{\{0\}}$ is the subgroup of all upper triangular matrices with 1 on the diagonal, R_2 elements on the next diagonal, R_3 on the next one and so forth, and so it is a subgroup of order q^c for some constant that can be exactly computed and is $O(t^3)$. $K_{\{i\}}$ are the same excepts that the entries of the matrix are permuted.

The set of vertices $X(0)$ of the complex X, is the collection of the cosets $gK_{\{i\}}$ where $g \in G$ and $0 \leq i \leq t$. The vertex $gK_{\{i\}}$ is said to have type i. Thus, the vertex set is a disjoint collection of $t+1$ sets. There is an edge between vertices v_1 and v_2 if their associated cosets have a non-empty intersection. Similarly, there is a k-face on $k+1$ vertices if their associated cosets intersect. The top-dimensional faces correspond to the intersection of $t+1$ cosets, and it is a fact that either this intersection is empty, or it has cardinally one, i.e., there is exactly one element $g \in G$ thas has the specified cosets. It can also be verified that this complex is a clique complex.

To make the HDX structure fully explicit we need to modify the [KO23] construction. Instead of defining $\mathcal{R}_s$ to be $\mathrm{F}_q[x]/\langle x^s \rangle$, we choose a degree s irreducible polynomial P over F_q and we let $\mathcal{R}_s = \mathrm{F}_q[x] \bmod P$. This makes $\mathcal{R}_s$ a field rather than a ring, and does not affect anything else.[3]

We now show full explicitness, i.e., combinatorial explicitness, efficient indexing for t-faces, efficient encoding for vertices and polynomial density of the sizes of the complexes in the family.

- (Combinatorial explicitness) We explain how to compute the rotation map of the inclusion graph of the HDX. Given a t-face g and $i \in [t+1]$, $Rot(g,i) = (v,j)$ where v is the i'th vertex on g, and g is the j'th t-face on v. To compute the rotation map we represent a t-face by a group element $g \in G$. Given g and $0 \leq i \leq t$ we compute the coset $gK_{\{i\}}$ and order it lexicographically. The output vertex v is the coset $gK_{\{i\}}$ (represented as an ordered set of group elements). We also let j be the index of g in the coset.

- (Enumerating group elements) Our next goal is to show a mapping from $[|G|]$ to group elements. We first recall what $|G|$ is. We first count the number of $t \times t$ non-singular matrices over a field F_b, the field with b elements. This is equivalent to counting the number of $t \times t$ matrices with linearly independent columns. For the first column out of all b^t potential columns, only the all-zeros column is invalid resulting in $b^t - 1$ options for the first column. For the $i \in \{2, ..., t\}$'s column, we need to choose a column that is independent of the previous $i-1$ columns, which results in $b^t - b^{i-1}$ valid options. Choosing the columns one by one results in $\prod_{i=0}^{t-1}(b^t - b^i)$ distinct non-singular matrices. In order to count the number non-singular matrices with determinant 1, notice that multiplying the last column of any non-singular matrix M by a constant $0 \neq a \in \mathbb{F}_b$ results in a matrix with determinant $a \cdot det(M)$, thus dividing the set of matrices into equivalence classes of size $(b-1)$. Then for the class of non-singular matrices with determinant 1 we have $|SL_t(\mathbb{F}_b)| = \frac{1}{b-1}\prod_{i=0}^{t-1}(b^t - b^i)$.

  Claim B.1. *(Enumerating $SL_t(\mathbb{F}_b)$) Let b be a prime power and $SL_t(\mathbb{F}_b)$ be the special linear group over a field $\mathbb{F}_b$. There exists a polynomial time computable bijection $f : [|SL_t(\mathbb{F}_b)|] \to SL_t(\mathbb{F}_b)$.*

[3] Theorem 4.10 in Kaufman and Oppenheim states the properties we require from $\mathcal{R}_s$ and all of them hold: It is still the case that $\mathbb{F}_b$ is finite and $|\mathbb{F}_b|$ tends to infinity with s. For every $s > t$, $T^t/P = T^t$ where $T = \{a_0 + a_1 x \ : \ a_0, a_1 \in \mathbb{F}_q\}$. Theorem 2.4, Corollary 3.3 and Theorem 3.10 are invariant to the replacement of $\langle x^s \rangle$ by (P). They also need the conditions for corollary 4.9 which are: The construction still uses coefficients from $R = \mathbb{F}_q$. $\mathbb{F}_q[x]/(P)$ is a finitely generated R-algebra. the smallest proper submodule of $T = \{a_0 + a_1 x \ : \ a_0, a_1 \in \mathbb{F}_q\}$ is of index q.

Proof. Following the size analysis of $|SL_t(\mathbb{F}_b)|$, in order to represent matrices from $SL_t(\mathbb{F}_b)$, one has $(b^t - b^i)$ options for the i'th column when $i \in [t-1]$, and $\frac{b^t - b^{t-1}}{b-1}$ options for the last column.

A number $c \in [|SL_t(\mathbb{F}_b)|]$ can be represented as a vector $(a_1, ..., a_t)$ where for $i \in [t-1]$: $a_i \in [b^t - b^i]$ and for $i = t$: $a_t \in [\frac{b^t - b^{t-1}}{b-1}]$. In order to compute f it remains to map the vector $(a_1, ..., a_t)$ into a distinct matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_t \\ | & & | \end{pmatrix}$. Notice that the encoding can be done iteratively, at the i'th step we need to encode the i'th column v_i given a_i and the previous $i-1$ encoded columns $v_1, ..., v_{i-1}$ which are linearly independent. Since the vectors are linearly independent, by using elementary row operations one can find an invertible matrix M of size $t \times t$ such that

$$M \cdot \begin{pmatrix} | & & | \\ v_1 & \cdots & v_{i-1} \\ | & & | \end{pmatrix} = \begin{pmatrix} I_{(i-1)\times(i-1)} \\ \mathbf{0}_{t-(i-1)\times(i-1)} \end{pmatrix} \tag{B.1}$$

Note that there are multiple solutions to M in Equation (B.1). In order for the encoding and decoding process to be consistent, M should be calculated in any *deterministic* way from $(v_1, ..., v_{i-1})$.

Then in order to map a_i to a new linearly independent vector in $v_1, ..., v_{i-1}$: first map a_i to a vector with a non zero element in the indices $\{t-(i-1), ..., t\}$ and then apply M^{-1}. a_i can be simply mapped to such vector by enumerating all b-ary vectors starting from b^i.

For the last index $i = t$, one can map a_t to all vectors with 1 in their last non-zero index. While not all such mappings result in encoded matrices with determinant > 1, an equivalence class of each matrix can be efficiently computed by multiplying the last column by any scalar - allowing to find an equivalent unique matrix with determinant 1.

□

- (Representing a vertex) We now get a vertex, which is a coset $gK_{\{i\}}$, represented by an ordered set of group elements, and we want to map it to a compressed set. The mapping will be one-to-one but not a bijection, mapping the cosets to a slightly larger than optimal domain.

  Given a coset $gK_{\{i\}}$, let m be a member of the coset. As m is invertible with determinant one, the matrix can be decomposed using PLU-factorization into $Pm = LDU$ where P is permutation matrix, L is lower-unitriangular, U is upper-unitriangular and D is diagonal with determinant one. While P is not unique, the encoding is consistence as long as P is calculated in a deterministic way given m.

  PLU factorization For an invertible matrix m, its $m = LU$ decomposition is calculated using Gaussian elimination without row swapping where U is the result of the elimination and L matches the elimination process. While not every elimination process is possible due to zero division, one can always find a row permutation matrix P such that the factorization $Pm = LU$ exists (and is unique for Pm). Then one can easily find a diagonal matrix D that normalizes L, U such that they are unitriangular and $Pm = LDU$. Such a D is unique. Furthermore, the factorization can be calculated in polynomial time. For more information we refer the reader to [Lyc20].

  We begin by encoding vertices from the coset $gK_{\{0\}}$. Since the multiplication of two upper-unitriangular matrices results in an upper-unitriangular matrix and $K_{\{0\}}$ is a subgroup of upper-unitriangular matrices, every matrix in $gK_{\{0\}}$ differs up to a multiplication in an upper-triangular matrix U'. Then for a fixed ordering of the rows, either all elements of $gK_{\{0\}}$ have an LDU decomposition with the same L and D or do not have such a decomposition at all. Furthermore for a fixed ordering of the rows where LDU decomposition exists, in every coset there is exactly one matrix U where each polynomial $U_{i,j} \in \mathbb{F}_b = \mathbb{F}_q[t]/(P)$ for $0 < i < j \leq t$: the first $j-i+1$ monomials have zero coefficient. We illustrate this for $t = 2$:

Let U be any unitriangular matrix $U = \begin{pmatrix} 1 & u_1 & u_2 \\ 0 & 1 & u_3 \\ 0 & 0 & 1 \end{pmatrix}$ for $u_1, u_2, u_3 \in \mathbb{F}_b$. According to the construction $K_{\{0\}}$ is of the form $K_{\{0\}} = \begin{pmatrix} 1 & a_0 + a_1 t & c_0 + c_1 t + c_2 t^2 \\ 0 & 1 & b_0 + b_1 t \\ 0 & 0 & 1 \end{pmatrix}$ with $a_0, a_1, b_0, b_1, c_0, c_1, c_2 \in \mathbb{F}_q$. Then the multiplication of U and the subgroup results in a matrix of the form:

$$UK_{\{0\}} = \begin{pmatrix} 1 & a_0 + a_1 t + u_1 & c_0 + c_1 t + c_2 t^2 + (b_0 + b_1 t)u_1 + u_2 \\ 0 & 1 & b_0 + b_1 t + u_3 \\ 0 & 0 & 1 \end{pmatrix}$$

Then there is always a single member in $K_{\{0\}}$ such that:

- $a_0, a_1 t$ cancel the two lowest monomials of index $(1, 2)$
- $b_0, b_1 t$ cancel the two lowest monomials of index $(2, 3)$
- $c_0, c_1 t, c_2 t^2$ cancel the three lowest monomials of index $(1, 3)$

Thus, given $gK_{\{0\}}$, one can find and fix a row ordering P such that every member of the coset has an LDU decomposition and encode the selected permutation out of the $\Delta = (t+1)!$ possible permutations. Then, find the member $m \in gK_{\{0\}}$ where for $Pm = LDU$, U has zero leading monomials as described above. Encode the non-trivial polynomials coefficients of L, D, U where:

- In D there is no need to encode the last item $D_{t,t}$ since the matrix is of determinant 1 and $\prod_{i=0}^{t} D_{i,i} = 1$.
- In U do not encode the appropriate zero coefficients. This ensures that the encoding of the cosets is smaller by a $|K_{\{0\}}|$ factor in comparison to the previous encoding of $SL_t(\mathbb{F}_b)$ elements.

When given a coset of $K_{\{i\}}$ for $i \neq 0$, one can always permute the coset members to the form of $K_{\{0\}}$ and encode it as described above. As part of the representation of each vertex the index i is encoded.