Lecture 9

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- Established Turing Machines as the gold standard of computers and computability …

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- and saw one problem, A_{TM}, that is computationally unsolvable.

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- Established Turing Machines as the gold standard of computers and computability …
- seen examples of solvable problems …
- and saw one problem, A_{TM}, that is computationally unsolvable.

In this lecture, we look at other computationally unsolvable problems, and establish the technique of mapping reducibilities for prove that languages are undecidable/non-enumerable.

Reducibility

Example:

- Finding your way around a new city

Reducibility

Example:

- Finding your way around a new city
- reduces to . . .

Reducibility

Example:

- Finding your way around a new city
- reduces to …
- obtaining a city map.

Reducibility, In Our Context

Always involves two problems, A and B.

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Desired Property: If A reduces to B, then any solution of B can be used to find a solution of A.

Reducibility, In Our Context

Always involves two problems, A and B.

Desired Property: If A reduces to B, then any solution of B can be used to find a solution of A.

Remark: This property says nothing about solving A by itself or B by itself.

Examples

Reductions:

- Traveling from Boshton to Paris . . .

Examples

Reductions:

- Traveling from Boshton to Paris . . .
- buying plane ticket . . .

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Reductions:

- Traveling from Boshton to Paris . . .
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- earning the money for that ticket . . .

Examples

Reductions:

- Traveling from Boshton to Paris . . .
- buying plane ticket . . .
- earning the money for that ticket . . .
- finding a job
  (or getting the $s from mom and dad. . .)

Examples

Reductions:

- Measuring area of rectangle . . .

Examples

Reductions:

- Measuring area of rectangle …
- measuring lengths of sides.

Examples

Reductions:

- Measuring area of rectangle . . .
- measuring lengths of sides.

Also:

Examples

Reductions:

- Measuring area of rectangle …
- measuring lengths of sides.

Also:

- Solving a system of linear equations …
- inverting a matrix.

Reducibility

If A is reducible to B, then

- A cannot be harder than B

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- if B is decidable, so is A.

Reducibility

If A is reducible to B, then

- A cannot be harder than B
- if B is decidable, so is A.
- if A is undecidable and reducible to B, then B is undecidable.

Undecidable Problems

We have already established that A_{TM} is undecidable.

Here is a related problem.

$H_{\text{TM}} = \{\langle M, w\rangle | M$ is a TM and M halts on input $w\}$

Clarification: How does H_{TM} differ from A_{TM}?

Undecidable Problems

$H_{\text{TM}} = \{\langle M, w\rangle | M \text{ is a TM and } M \text{ halts on input } w\}$

Theorem: H_{TM} is undecidable.

Proof idea:

- By contradiction.

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- So A_{TM} is reduced to H_{TM}.

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$H_{\text{TM}} = \{\langle M, w\rangle | M$ is a TM and M halts on input $w\}$

Theorem: H_{TM} is undecidable.

Proof idea:

- By contradiction.
- Assume H_{TM} is decidable.
- Let R be a TM that decides H_{TM}.
- Use R to construct S, a TM that decides A_{TM}.
- So A_{TM} is reduced to H_{TM}.
- Since A_{TM} is undecidable, so is H_{TM}.

Undecidable Problems

Theorem: H_{TM} is undecidable.

Proof: Assume, by way of contradiction, that TM R decides H_{TM}. Define a new TM, S, as follows:

- On input $\langle M, w \rangle$,

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- run R on $\langle M, w \rangle$.

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- If R rejects, reject.

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- run R on $\langle M, w \rangle$.
- If R rejects, reject.
- If R accepts (meaning M halts on w), simulate M on w until it halts.
- If M accepted, accept; otherwise reject. ♣

Undecidable Problems (2)

Does a TM accept any string at all?

$$E_{\text{TM}} = \{\langle M\rangle | M \text{ is a TM and } L(M) = \emptyset\}$$

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Proof structure:

- By contradiction.
- Assume E_{TM} is decidable.
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First attempt: When S receives input $\langle M, w\rangle$, it calls R with input $\langle M\rangle$.

- If R accepts, then reject, because M does not accept any string, let alone w.
- But what if R rejects?

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- If R accepts, then reject, because M does not accept any string, let alone w.
- But what if R rejects?

Second attempt: Let's modify M.

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$$E_{\text{TM}} = \{\langle M \rangle | M \text{ is a TM and } L(M) = \emptyset\}$$

Define M_1: on input x,

1. if $x \neq w$, reject.
2. if $x = w$, run M on w and accept if M does.

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Define M_1: on input x,

1. if $x \neq w$, reject.
2. if $x = w$, run M on w and accept if M does.

M_1 either

- accepts just w, or
- accepts nothing.

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Question: Can a TM construct M_1 from M?

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Machine M_1: on input x,

1. if $x \neq w$, reject.
2. if $x = w$, run M on w and accept if M does.

Question: Can a TM construct M_1 from M?

Answer: Yes, because we need only hardwire w, and add a few extra states to perform the "$x = w?$" test.

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On input $\langle M, w \rangle$, where M is a TM and w a string,

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Theorem: E_{TM} is undecidable.

Define S as follows:

On input $\langle M, w\rangle$, where M is a TM and w a string,

- Construct M_1 from M and w.
- Run R on input $\langle M_1\rangle$,
- if R accepts, reject; if R rejects, accept. ♣

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Skeleton of Proof:

- By contradiction.
- Assume R_{TM} is decidable.
- Let R be a TM that decides R_{TM}.
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Skeleton of Proof:

- By contradiction.
- Assume R_{TM} is decidable.
- Let R be a TM that decides R_{TM}.
- Use R to construct S, a TM that decides A_{TM}.

But how?

Undecidable Problems (3)

$$R_{\text{TM}} = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is regular}\}$$

Modify M so that the resulting TM accepts a regular language if and only if M accepts w.

Undecidable Problems (3)

$$R_{\text{TM}} = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is regular}\}$$

Modify M so that the resulting TM accepts a regular language if and only if M accepts w.

Design M_2 so that

- if M does not accept w, then M_2 accepts $\{0^n 1^n | n \geq 0\}$ (non-regular)
- if M accepts w, then M_2 accepts Σ^* (regular).

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From M and w, define M_2:

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From M and w, define M_2:

On input x,

1. If x has the form $0^n 1^n$, accept it.
2. Otherwise, run M on input w and accept x if M accepts w.

Claim:

- If M does not accept w, then M_2 accepts $\{0^n 1^n | n \geq 0\}$.
- If M accepts w, then M_2 accepts Σ^*.

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Theorem: R_{TM} is undecidable.

Define S:

On input $\langle M, w \rangle$,

1. Construct M_2 from M and w.
2. Run R on input $\langle M_2 \rangle$.
3. If R accepts, accept; if R rejects, reject. ♣

Undecidable Problems (4)

Are two TMs equivalent?

$$\text{EQ}_{\text{TM}} = \{\langle M_1, M_2\rangle \quad | \quad M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: EQ_{TM} is undecidable.

Undecidable Problems (4)

Are two TMs equivalent?

$$\mathrm{EQ}_{\mathrm{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

Theorem: $\mathrm{EQ}_{\mathrm{TM}}$ is undecidable.

We are getting tired of reducing A_{TM} to everything.

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We are getting tired of reducing A_{TM} to everything.

Let's try instead a reduction from E_{TM} to EQ_{TM}.

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Idea:

- E_{TM} is the problem of testing whether a TM language is empty.

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Idea:

- E_{TM} is the problem of testing whether a TM language is empty.
- EQ_{TM} is the problem of testing whether two TM languages are the same.

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Idea:

- E_{TM} is the problem of testing whether a TM language is empty.
- EQ_{TM} is the problem of testing whether two TM languages are the same.
- If one of these two TM languages happens to be empty, then we are back to E_{TM}.

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Idea:

- E_{TM} is the problem of testing whether a TM language is empty.
- EQ_{TM} is the problem of testing whether two TM languages are the same.
- If one of these two TM languages happens to be empty, then we are back to E_{TM}.
- So E_{TM} is a special case of EQ_{TM}.

The rest is easy.

Undecidable Problems (4)

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Theorem: EQ_{TM} is undecidable.

Let M_{NO} be the TM: On input x, reject.

Let R decide EQ_{TM}.

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1. Run R on input $\langle M, M_{\text{NO}}\rangle$.
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Theorem: $\mathrm{EQ}_{\mathrm{TM}}$ is undecidable.

Let M_{NO} be the TM: On input x, reject.

Let R decide $\mathrm{EQ}_{\mathrm{TM}}$.

Let S be: On input $\langle M\rangle$:

1. Run R on input $\langle M, M_{\mathrm{NO}}\rangle$.
2. If R accepts, accept; if R rejects, reject.

If R decides $\mathrm{EQ}_{\mathrm{TM}}$, then S decides E_{TM}. ♣

Bucket of Undecidable Problems

Same techniques prove undecidability of

- Does a TM accept a decidable language?

Bucket of Undecidable Problems

Same techniques prove undecidability of

- Does a TM accept a decidable language?
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- Does a TM accept a decidable language?
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- Does a TM accept a finite language?
- Does a TM halt on all inputs?

Bucket of Undecidable Problems

Same techniques prove undecidability of

- Does a TM accept a decidable language?
- Does a TM accept a enumerable language?
- Does a TM accept a context-free language?
- Does a TM accept a finite language?
- Does a TM halt on all inputs?
- Is there an input string that causes a TM to traverse all its states?

Rice's Theorem

By now, some of you may have become cynical and embittered.

- Like, been there, done that, bought the T-shirt.

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By now, some of you may have become cynical and embittered.

- Like, been there, done that, bought the T-shirt.
- Looks like any non-trivial property of TMs is undecidable.

That is correct.

Rice's Theorem

Theorem: If $\mathcal{C}$ is a proper non-empty subset of the set of enumerable languages, then it is undecidable whether for a given TM, M, $L(M)$ is in $\mathcal{C}$.

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Proof by reduction from H_{TM} (does M halt on input x?).

Rice's Theorem

Theorem: If $\mathcal{C}$ is a proper non-empty subset of the set of enumerable languages, then it is undecidable whether for a given TM, M, $L(M)$ is in $\mathcal{C}$.

Proof by reduction from H_{TM}
(does M halt on input x?).

- Assume R decides if $L(M) \in \mathcal{C}$.
- Use R to implement S, which decides H_{TM}.

Further details of proof not given at the moment ...

Reducibility

So far, we have seen many examples of reductions from one language to another, but the notion was neither defined nor treated formally.

Reductions play an important role in

- decidability theory
- complexity theory (to come)

Time to get formal.

Computable Functions

A TM computes a function

$$f : \Sigma^* \longrightarrow \Sigma^*$$

if the TM

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if the TM

- starts with input w, and
- halts with only $f(w)$ on tape.

Computable Functions

Claim: All the usual arithmetic functions on integers are computable.

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Exercise: Design a TM that on input $\langle m, n \rangle$, halts with $\langle m + n \rangle$ on tape.

Computable Functions

A useful class of functions modifies TM descriptions. For example:

On input w:

- if $w = \langle M \rangle$ for some TM,

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- otherwise write ε and halt.

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Left as an exercise.

Mapping Reductions

Definition: Let A and B be two languages. We say that there is a mapping reduction from A to B, and denote

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$$A \ \leq_m B$$

if there is a computable function

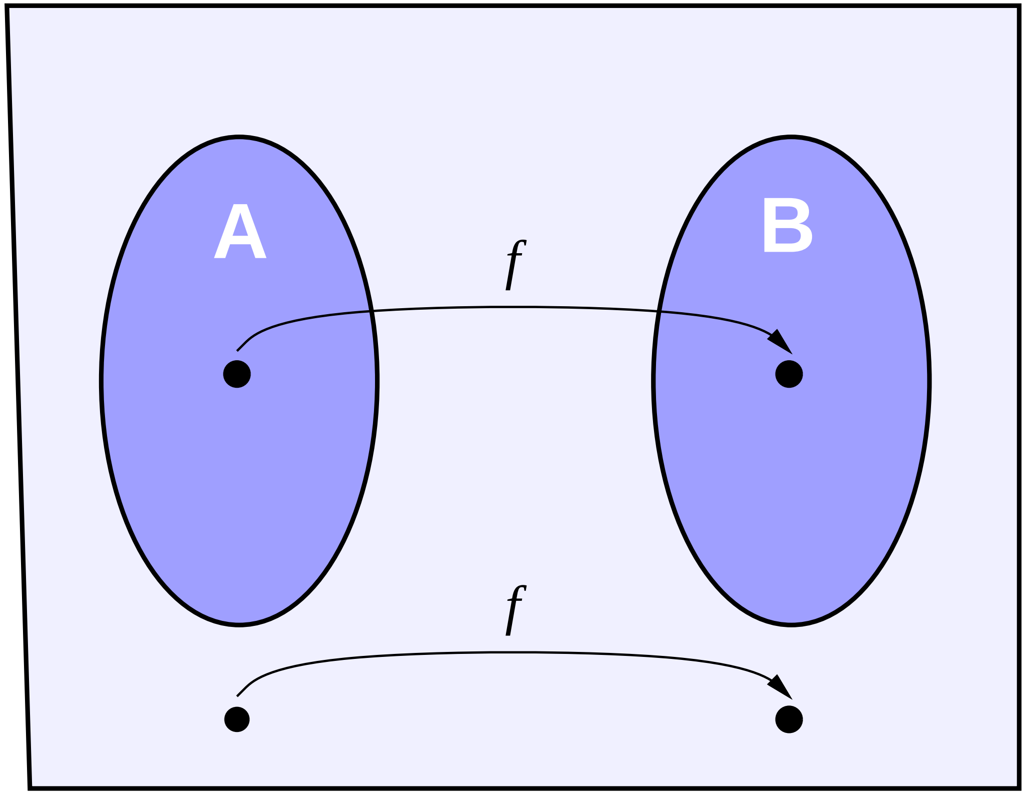
$$f : \Sigma^* \longrightarrow \Sigma^*$$

such that, for every w,

$$w \in A \Longleftrightarrow f(w) \in B.$$

The function f is called the reduction from A to B.

Mapping Reductions



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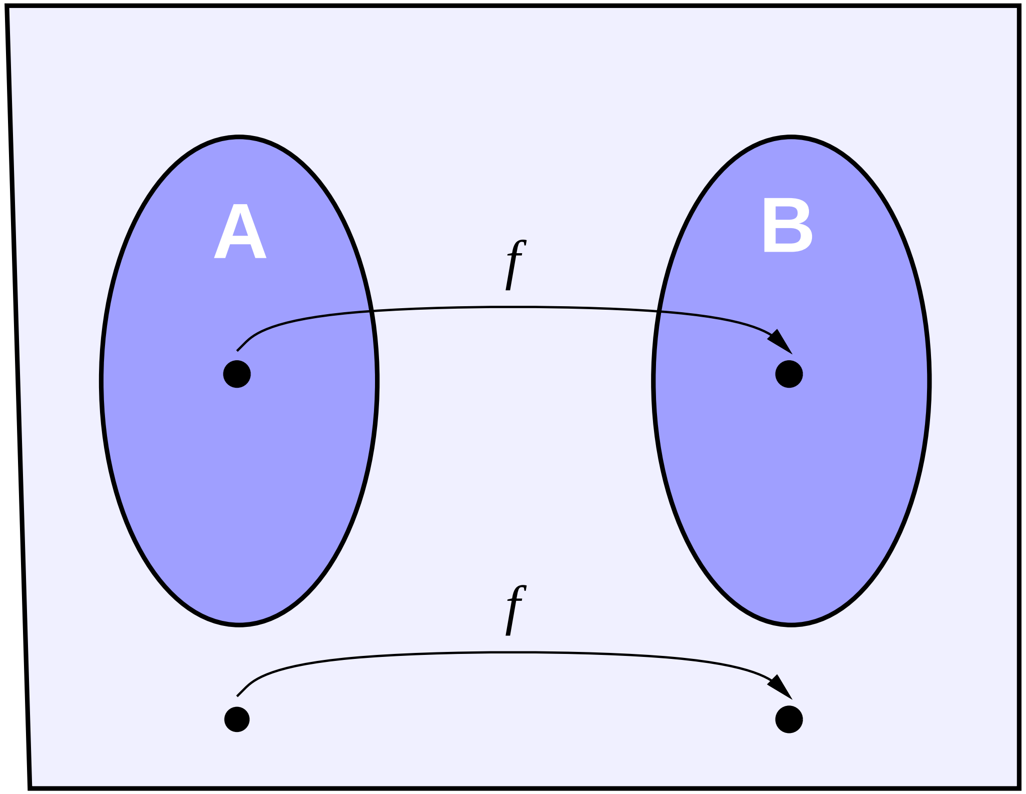

A mapping reduction converts questions about membership in A to membership in B

Mapping Reductions

Theorem: If $A \leq_m B$ and B is decidable, then A is decidable.

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Proof: Let

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Proof: Let

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Define N: On input w

1. compute $f(w)$
2. run M on input $f(w)$ and output whatever M outputs.

Mapping Reductions

Corollary: If $A \leq_m B$ and A is undecidable, then B is undecidable.

Mapping Reductions

Corollary: If $A \leq_m B$ and A is undecidable, then B is undecidable.

In fact, this has been our principal tool for proving undecidability of languages other than A_{TM}.

Example: Halting

Recall that

$$\begin{aligned} A_{\text{TM}} &= \{\langle M, w\rangle | \text{TM } M \text{ accepts input } w\} \\ H_{\text{TM}} &= \{\langle M, w\rangle | \text{TM } M \text{ halts on input } w\} \end{aligned}$$

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Earlier we proved that

- H_{TM} undecidable
- by (de facto) reduction from A_{TM}.

Let's reformulate this.

Example: Halting

Define a computable function, f:

- input of form $\langle M, w \rangle$

Example: Halting

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- where $\langle M, w \rangle \in A_{\text{TM}} \iff \langle M', w' \rangle \in H_{\text{TM}}$.

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- output $\langle M', w \rangle$

Enumerability

Theorem: If $A \leq_m B$ and B is enumerable, then A is enumerable.

Proof is same as before, using accepters instead of deciders.

Enumerability

Corollary: If $A \leq_m B$ and A is not enumerable, then B is not enumerable.

TM Equality

Theorem: Both EQ_{TM} and its complement, $\overline{EQ_{TM}}$, are not enumerable. Stated differently, EQ_{TM} is neither enumerable nor co-enumerable.

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Claim: A_{TM} is reducible to $\overline{\text{EQ}_{\text{TM}}}$.

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- $\langle M, w \rangle \in A_{\text{TM}} \iff \langle M_1, M_2 \rangle \in \text{EQ}_{\text{TM}}$.

♣

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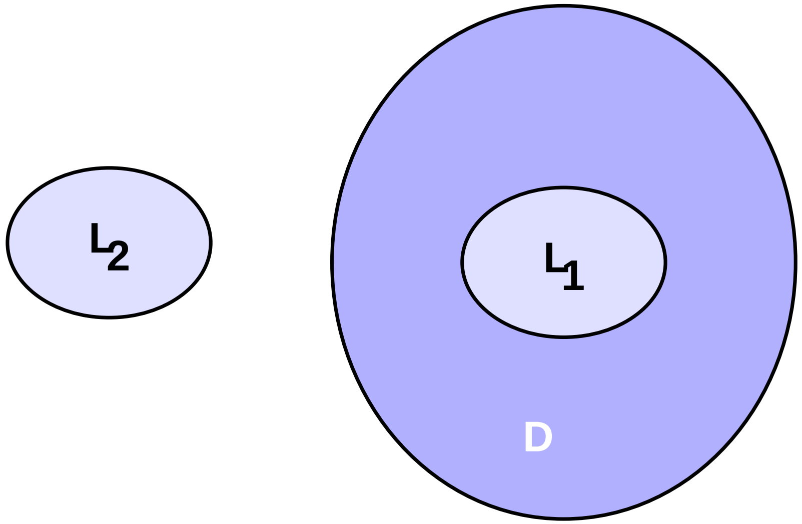
Example of recursively separable languages:



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Are there non-trivial examples?

Recursive Inseparability

Define

$$A_{\text{yes}} = \{\langle M\rangle | M \text{ is a TM that accepts } \langle M\rangle\}$$

and

$$A_{\text{no}} = \{\langle M\rangle | M \text{ is a TM that halts and rejects } \langle M\rangle\}$$

Theorem: A_{yes} and A_{no} are recursively inseparable.

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Proof by reduction and contradiction.

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- Claim:
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  - D separates A_{yes} and A_{no}, contradiction. ♣