Computational Models

Introduction to the Theory of Computing

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Spring Semester, 2009. Mondays, 13–16.

http://www.cs.tau.ac.il/∼bchor/CM09/compute.html

Site is our sole means of disseminating messages (no mailing list or forum).

AdministraTrivia

Course Requirements:

- 6 problem sets (10% of grade, best 5-out-of-6).
  - Readable, concise, correct answers expected.
  - Late submition will not be accepted.
  - Assignments are 10% of grade, and are required. Solving assignments independently is highly recommended.
- Final exam is 90% of grade. Midterm exam (10% of midterm grade added to weighted average of final exam and homework).

AdministraTrivia II

- Midterm tentatively scheduled to Tue., April 7, 2009.
- Second final exam (Moed B): Same material and same averaging applies. Format may differ (*e.g.* proportion of closed/open problems).
- Prerequisites:
  - Extended introduction to computer science
  - Discrete mathematics course.
  - Students from other disciplines with mathematical background encouraged to contact the instructor.
- Textbook (extensively used, highly recommended):
  - Michael Sipser, *Introduction to the theory of computation*, 1st or 2nd edition.

Why Study Theory?

- Basic Computer Science Issues
  - What is a computation?
  - Are computers omnipotent?
  - What are the fundemental capabilities and limitations of computers?
- Pragmatic Reasons
  - Avoid intractable or impossible problems.
  - Apply efficient algorithms when possible.
  - Learn to tell the difference.

Course Topics

- Automata Theory: What is a computer?
- Computability Theory: What can computers do?
- Complexity Theory: What makes some problems computationally hard and others easy?
- Coping with intractability:
  - Approximation.
  - Randomization.
  - Fixed parameter algorithms.
  - Heuristics.

Automata Theory - Simple Models

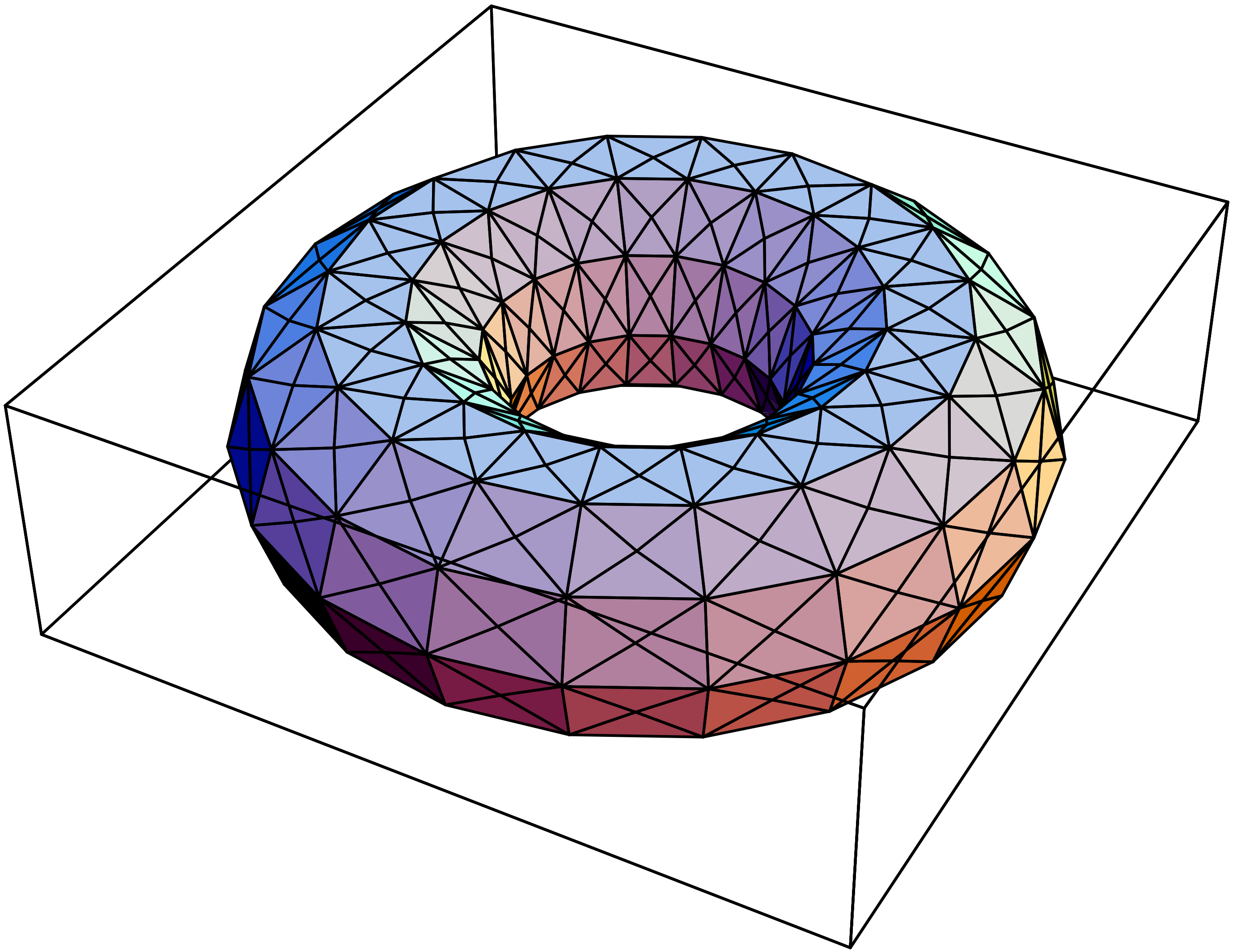
- Finite automata.
  - Related to controllers and hardware design.
  - Useful in text processing and finding patterns in strings.
  - Probabilistic (Markov) versions useful in modeling various natural phenomena (*e.g.* speech recognition).
- Push down automata.
  - Titely related to a family of languages known as context free languages.
  - Play important role in compilers, design of programming languages, and studies of natural languages.

Computability Theory

In the first half of the 20th century, mathematicians such as Kurt Göedel, Alan Turing, and Alonzo Church discovered that some fundemental problems cannot be solved by computers.

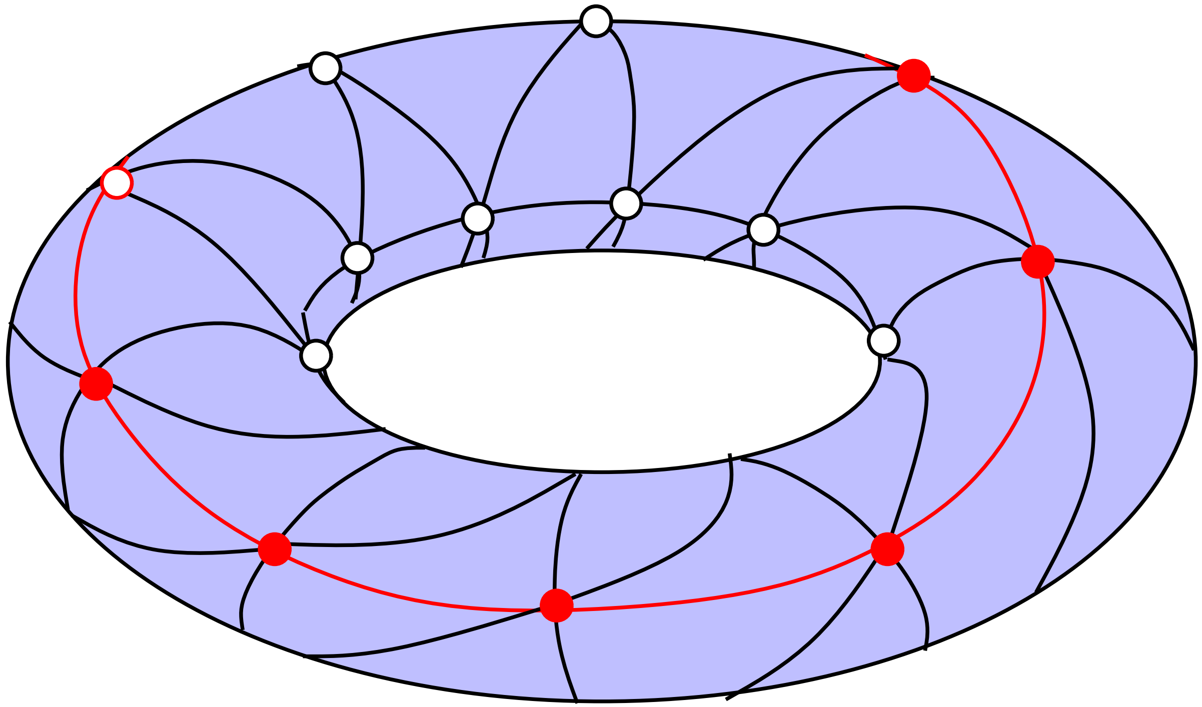
- Proof verication of statements can be automated.
- It is natural to expect that determining validity can also be done by a computer.
- Theorem: A computer cannot determine if mathematical statement true or false.
- Results needed theoretical models for computers.
- These theoretical models helped lead to the construction of real computers.

Computability Theory



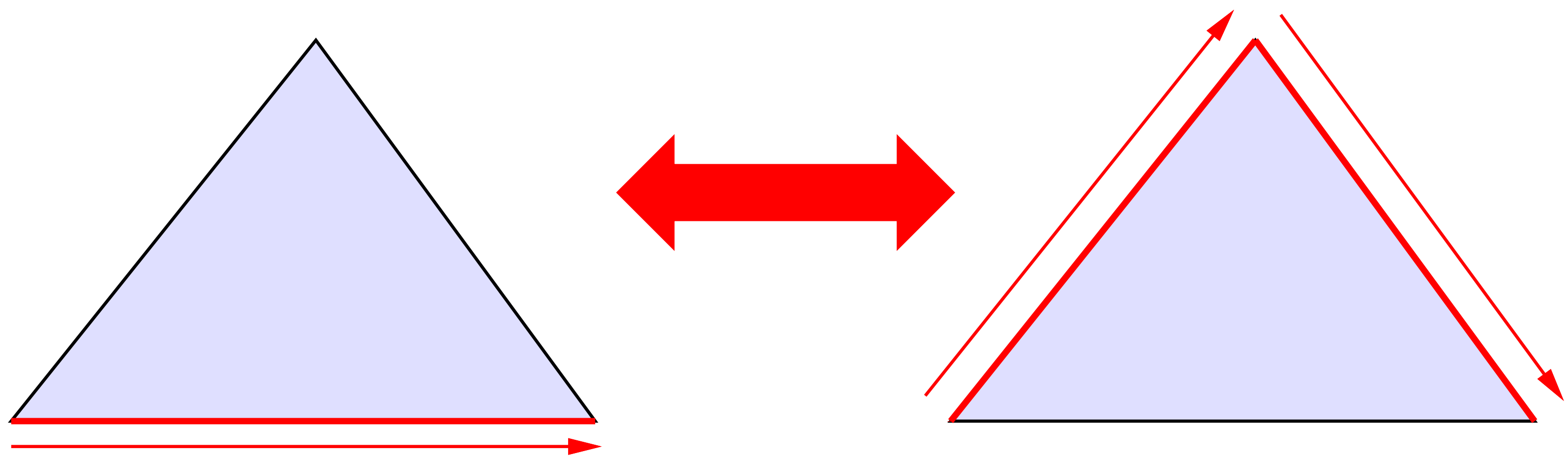
a simplicial complex

Paths and Loops



- a path is a sequence of vertices connected by edges
- a loop is a path that ends and ends at the same vertex

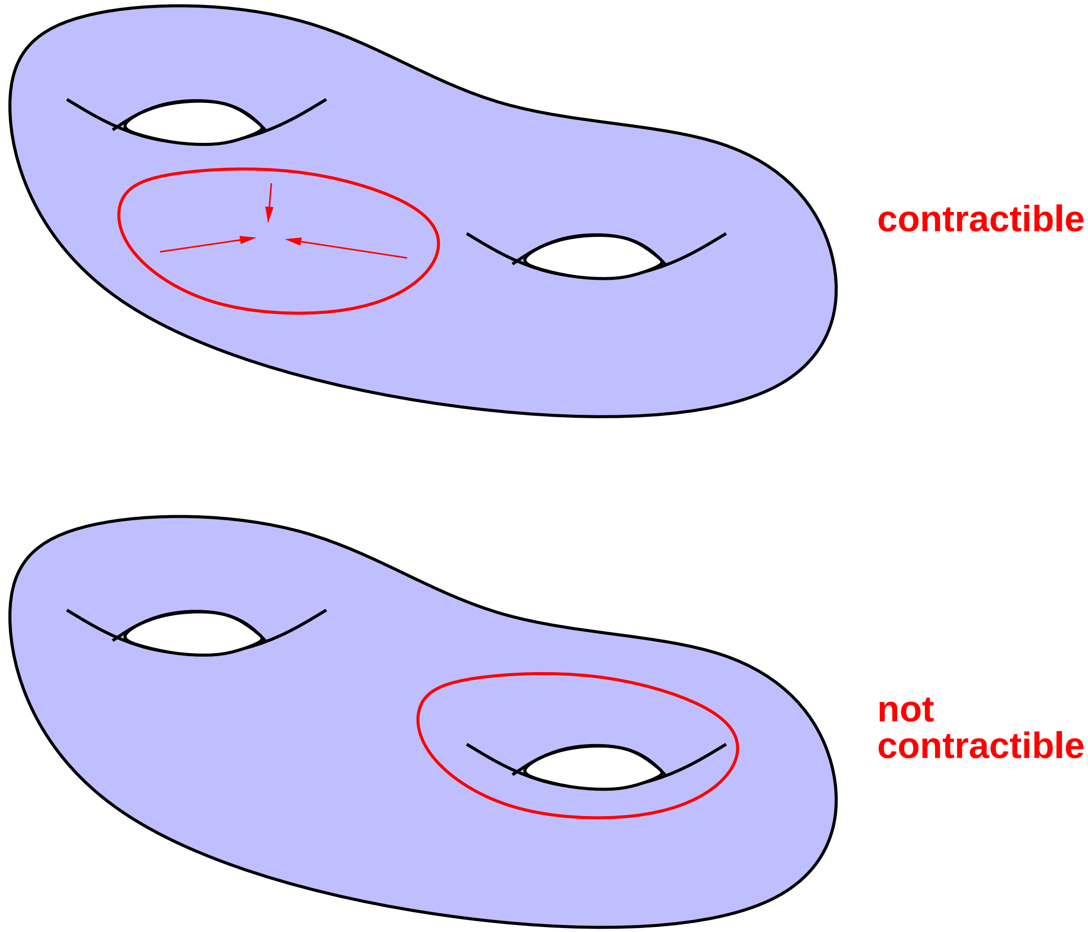
Paths and Loops can be Deformed



$$(v_0, v_1) \Leftrightarrow (v_0, v_2, v_1)$$

$$(v_0, v_0) \Leftrightarrow (v_0)$$

Contractibility



- No algorithm can determine whether an arbitrary loop in an arbitrary finite complex is contractible.

Some Other Undecidable Problems

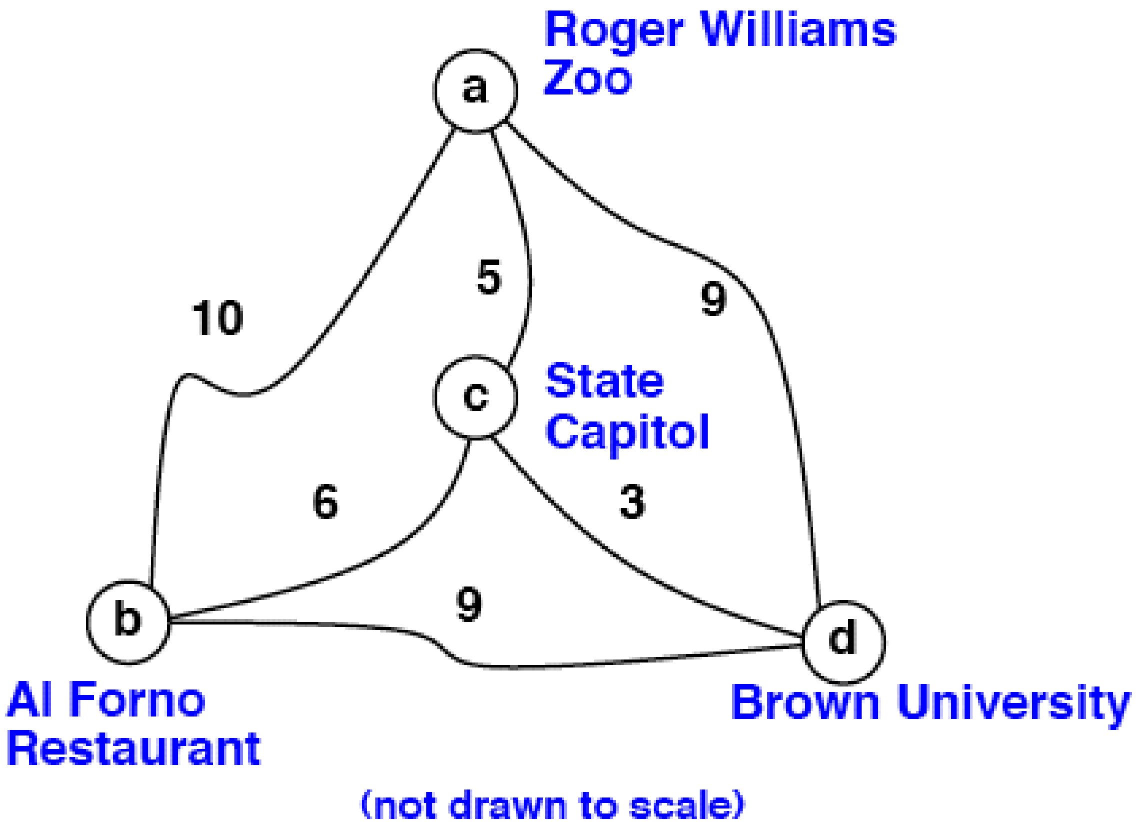
- Does a program run forever?
- Is a program correct?
- Are two programs equivalent?
- Is a program optimal?
- Does an equation with one or more variables and integer coefficients ($5x + 15y = 12$) have an integer solution (Hilbert's 10th problem).
- Is a finitely-presented group trivial?
- Given a string, x, how compressible is it?

Complexity Theory

Key notion: tractable vs. intractable problems.

- A problem is a general computational question:
  - description of parameters
  - description of solution
- An algorithm is a step-by-step procedure
  - a recipe
  - a computer program
  - a mathematical object
- We want the most efficient algorithms
  - fastest (usually)
  - most economical with memory (sometimes)
  - expressed as a function of problem size

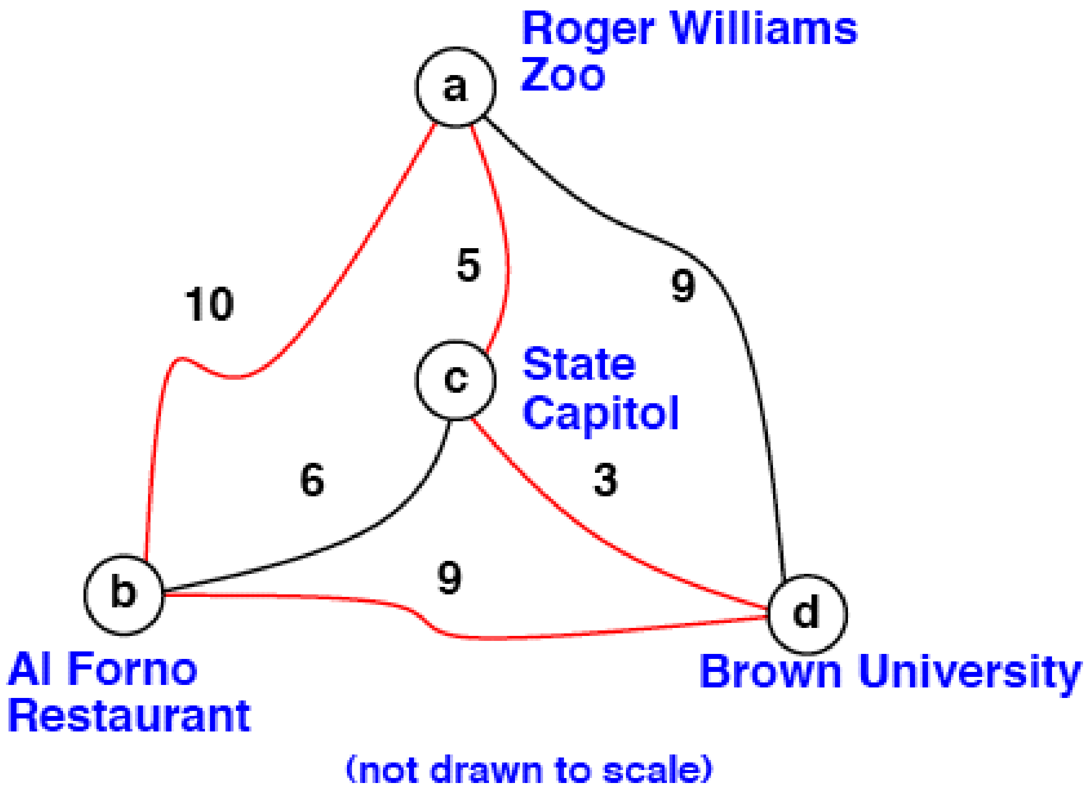
Example: Traveling Salesman Problem



Input:

- set of cities
- set of inter-city distances

Example: Traveling Salesman Problem



Goal:

- want the shortest tour through the cities
- example: $\mathtt{a}, \mathtt{b}, \mathtt{d}, \mathtt{c}, \mathtt{a}$ has length 27.

Problem Size

- What is an appropriate measure of problem size?
  - $\mathtt{m}$ nodes?
  - $\mathtt{m(m+1)/2}$ distances?
- Use an encoding of the problem
  - alphabet of symbols
  - strings: $\mathtt{a/b/c/d//10/5/9//6/9//3}$.
- Measures
  - Problem Size: length of encoding (here: 23 ascii characters).
  - Time Complexity: how long an algorithm runs, as function of problem size?

Time Complexity - What is tractable?

- We say that a function $f(n)$ is $O(g(n))$ if there is a constant c such that for large enough n, $|f(n)| \leq c \cdot |g(n)|$.
- A polynomial-time algorithm is one whose time complexity is $O(p(n))$ for some polynomial $p(n)$, where n denotes the length of the input.
- An exponential-time algorithm is one whose time complexity cannot be bounded by a polynomial (e.g., $n^{\log n}$).

Tractability – Basic distinction:

- Polynomial time $=$ tractable.
- Exponential time $=$ intractable.

	10	20	30	40	50	60
n	.00001 second	.00002 second	.00003 second	.00004 second	.00005 second	.00006 second
n^2	.00001 second	.00004 second	.00009 second	.00016 second	.00025 second	.00036 second
n^3	.00001 second	.00008 second	.027 second	.064 second	.125 second	.216 second
n^5	.1 second	3.2 seconds	24.3 seconds	1.7 minute	5.2 minutes	13.0 minutes
2^n	.001 second	1.0 second	17.9 minutes	12.7 days	35.7 years	366 centuries
3^n	.059 second	58 minutes	6.5 years	3855 centuries	$2 \cdot 10^8$ centuries	$1.3 \cdot 10^{13}$ centuries

Effect of Speed-Ups

Let's wait for faster hardware! Consider maximum problem size you can solve in an hour.

	present	100 times faster	1000 times faster
n	N_1	$100N_1$	$1000N_1$
n^2	N_2	$10N_2$	$31.6N_2$
n^3	N_3	$4.64N_3$	$10N_3$
n^5	N_4	$2.5N_4$	$3.98N_4$
2^n	N_5	$N_5 + 6.64$	$N_5 + 9.97$
3^n	N_6	$N_6 + 4.19$	$N_6 + 6.29$

NP-Completeness / NP-Hardness

Your boss says:

> "Get me an efficient traveling-salesman algorithm, or else..."

What are you going to do?

Response

"Yes Ma'am, expect it this afternoon!"

Problem is

- All known algorithms (essentially) check all possible paths.
- Exhaustive checking is exponential.
- Good luck!

Response

"Hah! I will prove that no such algorithm is possible"

Problem is, proving intractability is very hard.

Many important problems have

- no known tractable algorithms
- no known proof of intractability.

Response

"I can't find an efficient algorithm.
I guess I'm just a pathetic loser. "

- Bad for job security.

Response

“The problem is NP-hard. I can’t find
an efficient algorithm, but neither can any
of these famous people . . . ”

Advantage is:

- The problem is “just as hard” as other problems smart people can’t solve efficiently.
- So it would do your boss no good to fire you and hire a Technion/Hebrew Univ./MIT graduate.

Response

"Would you settle for a pretty good, but not the best, algorithm?"

Intractability isn't everything.

- Find an approximate solution (is a solution within 10% of optimum good enough, ma'am?).
- Use randomization.
- Fixed parameter algorithms may be applicable.
- Heuristics can also help.

- Approximation, randomization, etc. are among the hottest areas in complexity theory and algorithmic research today.

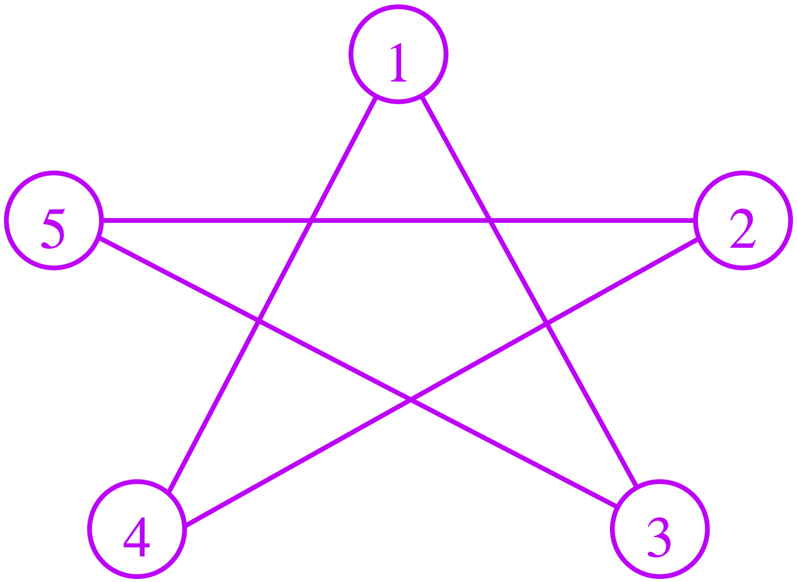
Next Subject

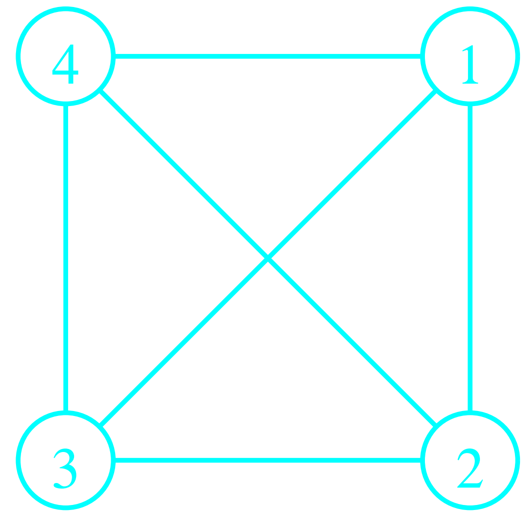
A Very Short Math Review

- Graphs
- Strings and languages
- Mathematical proofs
- Mathematical notations (sets, sequences, . . .) $\surd$
- Functions and predicates $\surd$

$\surd$ = will be done in recitation.

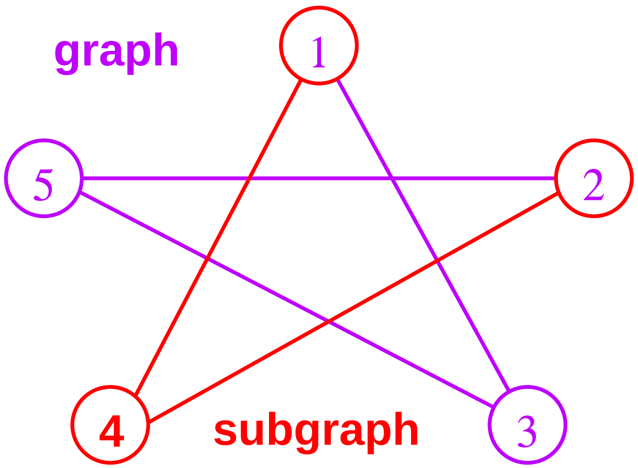
Graphs

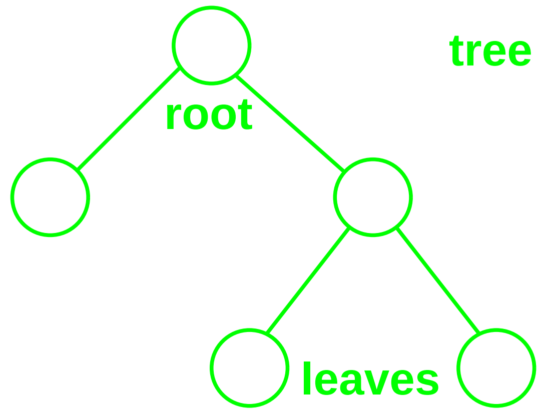


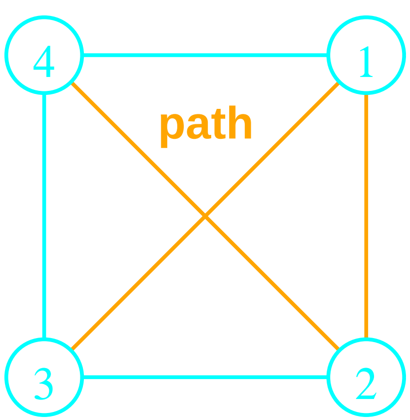


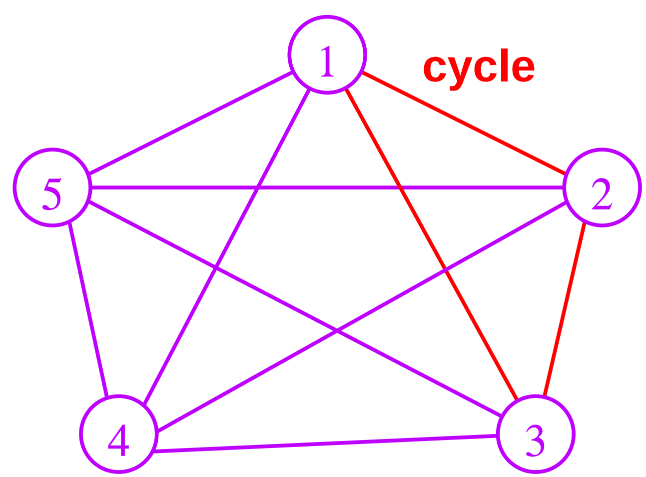
- $G = (V, E)$, where
- V is set of nodes or vertices, and
- E is set of edges
- degree of a vertex is number of edges

Graphs

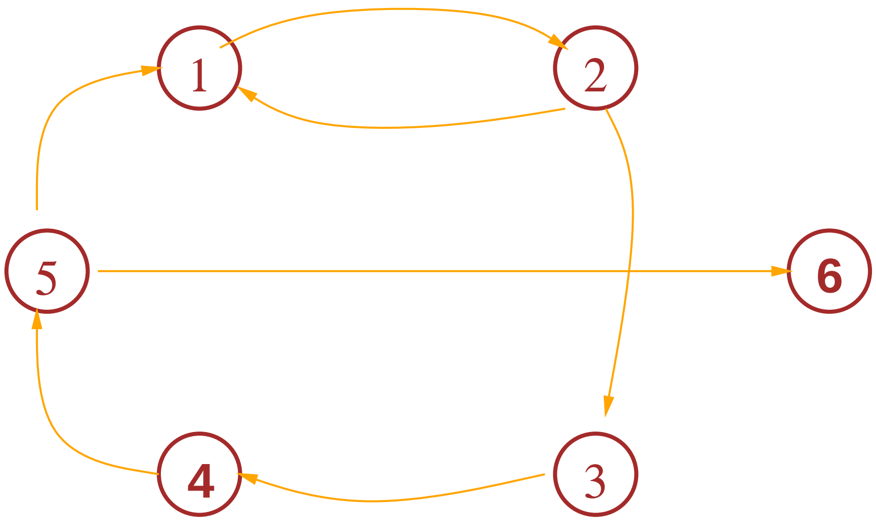

graph
1
2
3
4
5
subgraph

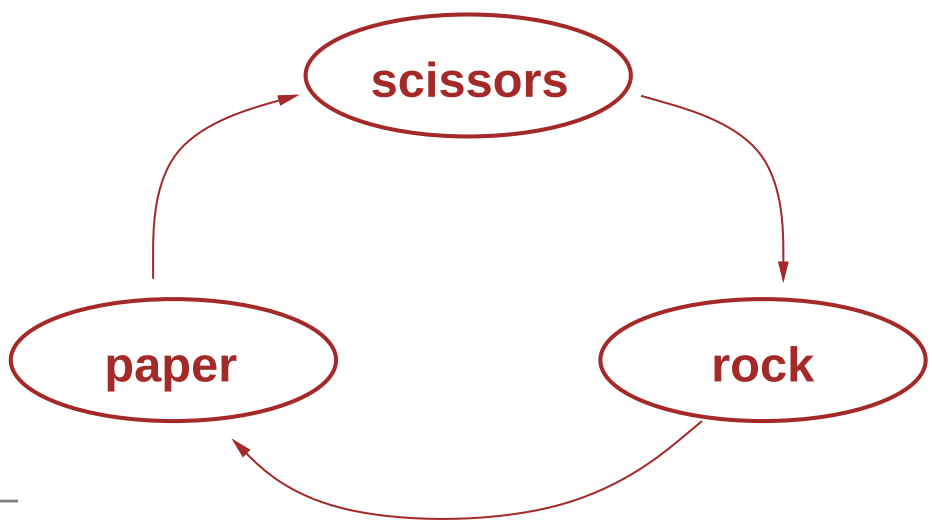

tree
root
leaves


4
1
3
2
path


1
cycle
5
2
4
3

Directed Graphs





Directed Graph and its Adjacency Matrix

Which directed graph is represented by the following 6-by-6 matrix?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Strings and Languages

- an alphabet is a finite set of symbols
- a string over an alphabet is a finite sequence of symbols from that alphabet.
- the length of a string is the number of symbols
- the empty string ε
- reverse: $abcd$ reversed is $dcba$.
- substring: xyz in $xyzzy$.
- concatenation of xyz and zy is $xyzzy$.
- x^k is $x \cdots x$, k times.
- a language L is a set of strings.

Proofs

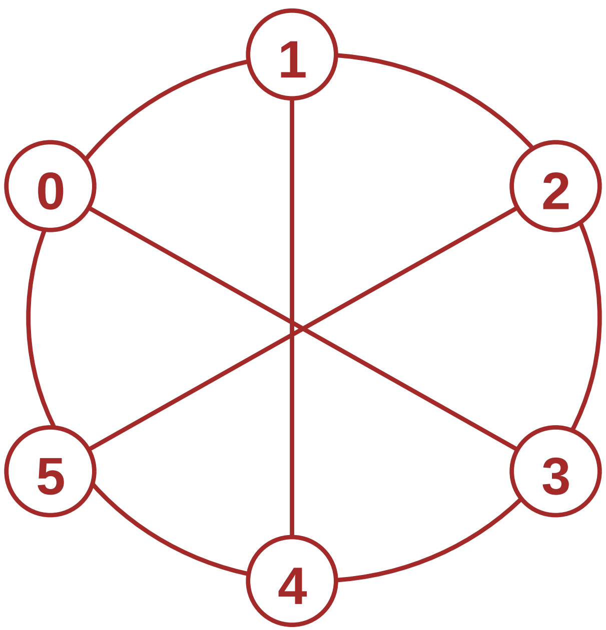
We will use the following basic kinds of proofs.

- by construction
- by contradiction
- by induction
- by reduction
  - we will often mix them.

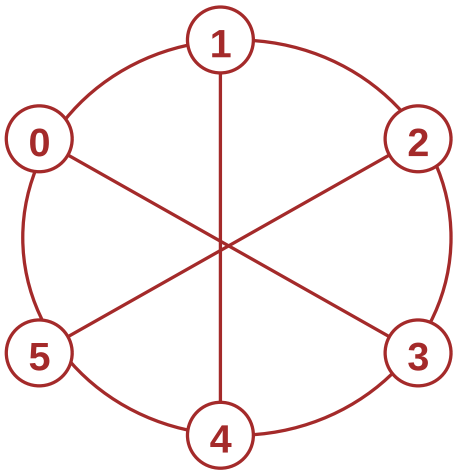
Proof by Construction

A graph is k-regular if every node has degree k.

Theorem: For every even $n > 2$, there exists a 3-regular graph with n nodes.



Proof by Construction



Proof: Construct $G = (V, E)$, where $V = \{0, 1, \ldots, n-1\}$ and

$$E = \{\{i, i+1\} \mid \textit{for } 0 \le i \le n-2\} \cup \{n-1, 0\} \cup \{\{i, i+n/2\} \mid \textit{for } 0 \le i \le n/2-1\}.$$

Note: A picture is helpful, but it is *not* a proof!

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof: Suppose not. Then $\sqrt{2} = \frac{m}{n}$, where m and n are relatively prime.

$$\begin{aligned} n\sqrt{2} &= m \\ 2n^2 &= m^2 \end{aligned}$$

Proof by Contradiction (cont.)

So m^2 is even, and so is $m = 2k$.

$$\begin{aligned} 2n^2 &= (2k)^2 \\ &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

Thus n^2 is even, and so is n.

Therefore both m and n are even, and not relatively prime!
Reductio ad absurdam.

Proof by Induction

Prove properties of elements of an infinite set.

$$\mathcal{N} = \{1, 2, 3, \ldots\}$$

To prove that $\wp$ holds for each element, show:

- *base step:* show that $\wp(1)$ is true.
- induction step: show that if $\wp(i)$ is true (the induction hypothesis), then so is $\wp(i+1)$.

Induction Example

Theorem: All cows are the same color.
Base step: A single-cow set is definitely the same color.
Induction Step: Assume all sets of i cows are the same color. Divide the set $\{1, \ldots, i+1\}$ into $U = \{1, \ldots, i\}$, and $V = \{2, \ldots, i+1\}$.

All cows in U are the same color by the induction hypothesis.
All cows in V are the same color by the induction hypothesis.
All cows in $U \cap V$ are the same color by the induction hypothesis.

Induction Example (cont.)

Ergo, all cows are the same color.

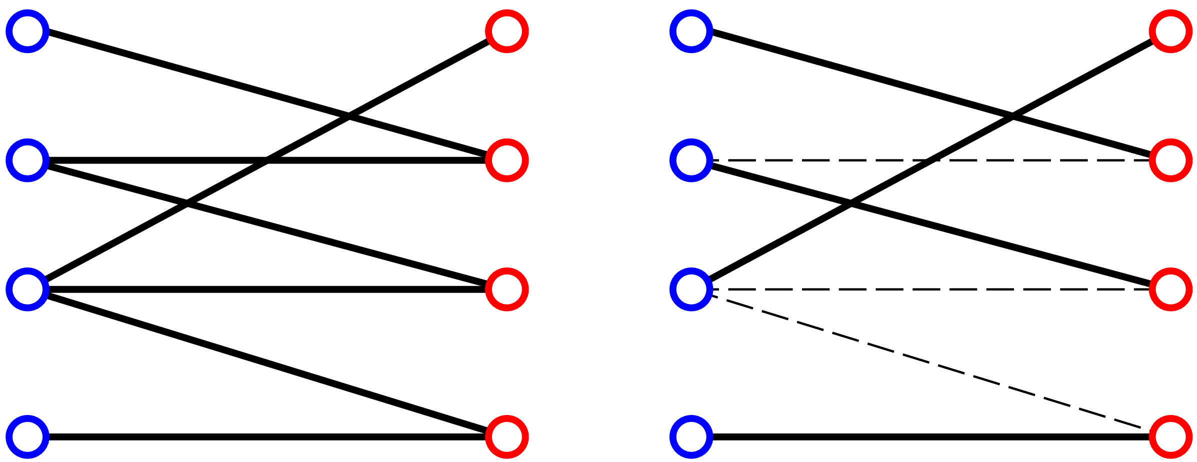
Quod Erat Demonstrandum (QED).

(cows' images courtesy of www.crawforddirect.com/ cows.htm)

Proof by Reduction

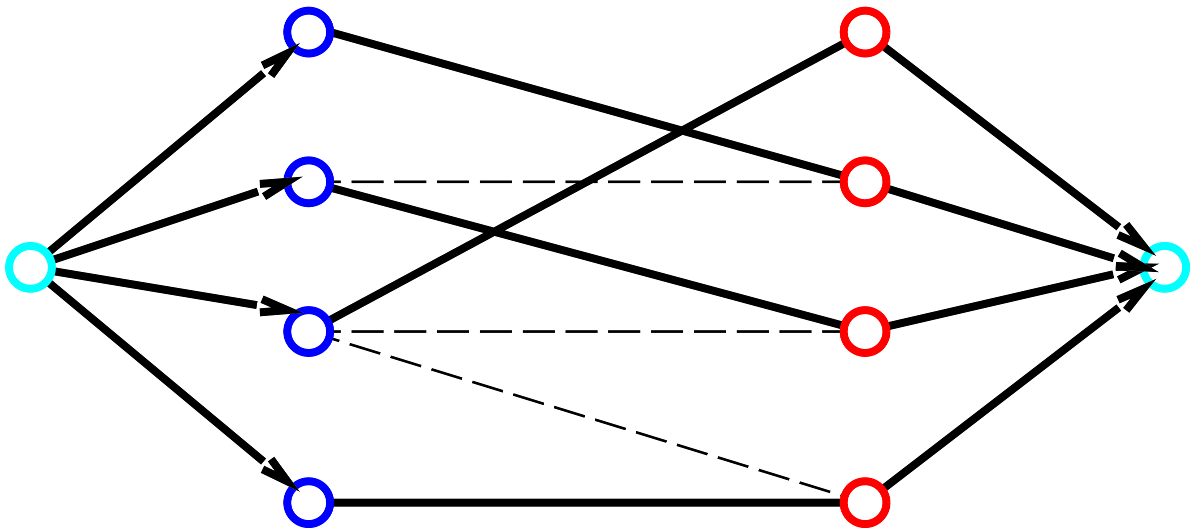
We can sometime solve problem A by reducing it to problem B, whose solution we already know.

Example: Maximal matching in bipartite graphs:



Proof by Reduction

Reducing bipartite matching to MAX FLOW:



Reduction: Put capacity 1 on each edge.

Maximum flow corresponds to maximum matching. So if we have an algorithm that produces max flow, we can easily derive a maximum bipartite matching from it.