Dynamic trees (Steator and Tarjan 83)

Operations that we do on the trees

Maketree(v)

w = findroot(v)

(v,c) = mincost(v)

addcost(v,c)

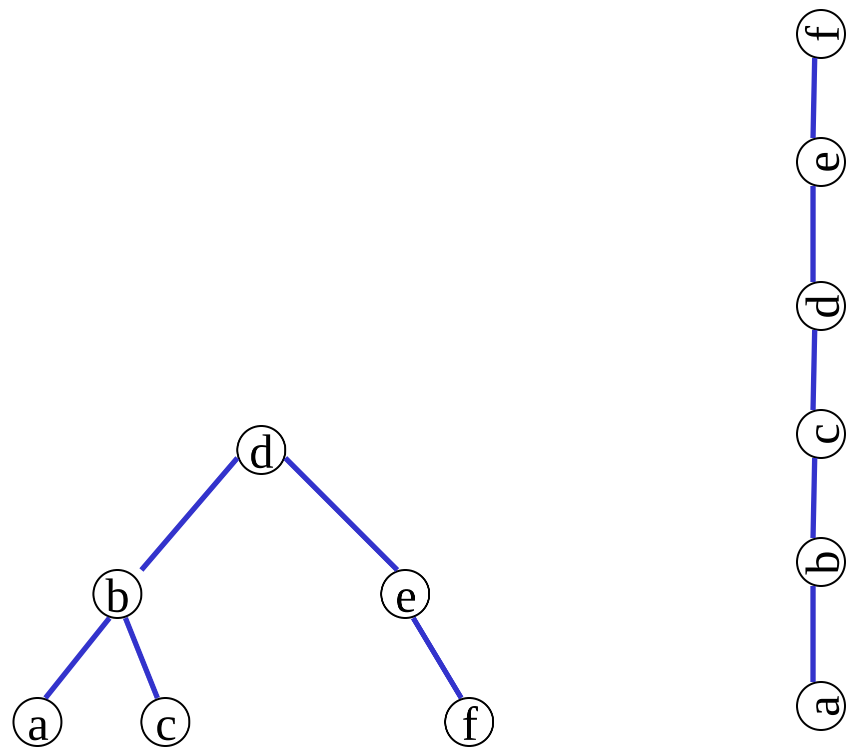
link(v,w,r(v,w))

cut(v)

findcost(v)

Simple case -- paths

Assume for a moment that each tree T in the forest is a path.
We represent it by a virtual tree which is a simple splay tree.



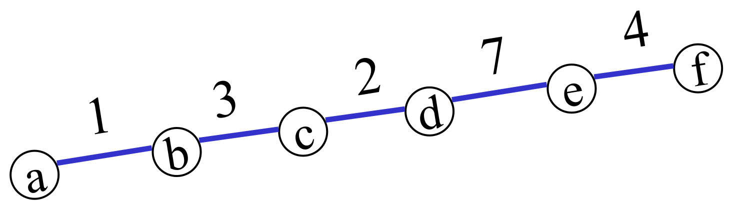
Findroot(v)

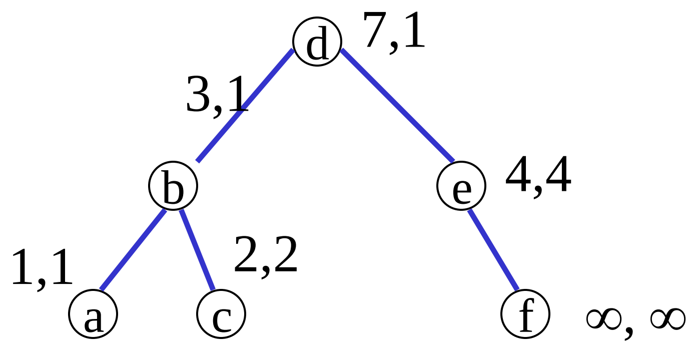
Splay at v, then follow right pointers until you reach the last vertex w on the right path. Return w and splay at w.

Mincost(v)

With every vertex x we record cost(x) = the cost of the edge (x,p(x))

We also record with each vertex x mincost(x) = minimum of cost(y) over all descendants y of x.





Mincost(v)

Splay at v and use mincost values to search for the minimum

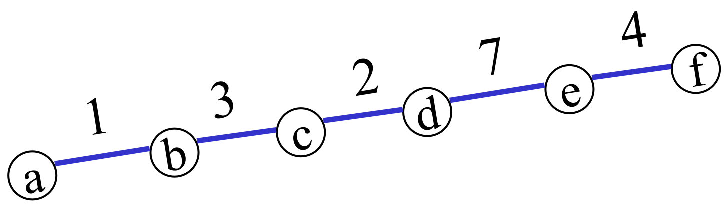
Notice: we need to update mincost values as we do rotations.

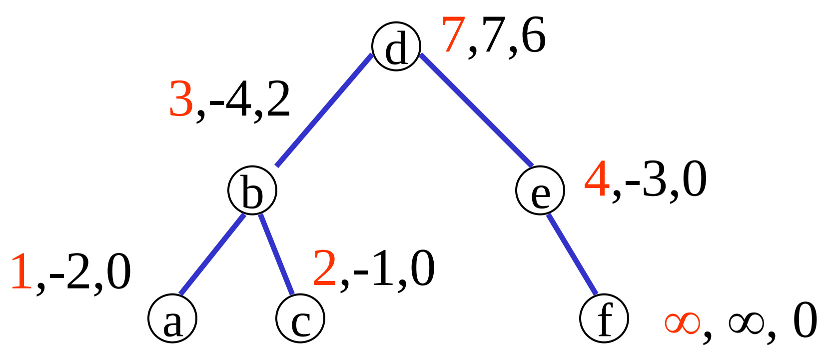
Addcost(v,c)

Rather than storing cost(x) and mincost(x) we will store

Δcost(x) = cost(x) - cost(p(x))

Δmin(x) = cost(x) - mincost(x)





Addcost(v,c) :
Splay at v,
Δcost(v) += c
Δcost(left(v)) -= c

similarly update
Δmin

Addcost(v,c) (cont)

Notice that now we have to update Δcost(x) and Δmin(x) through rotations

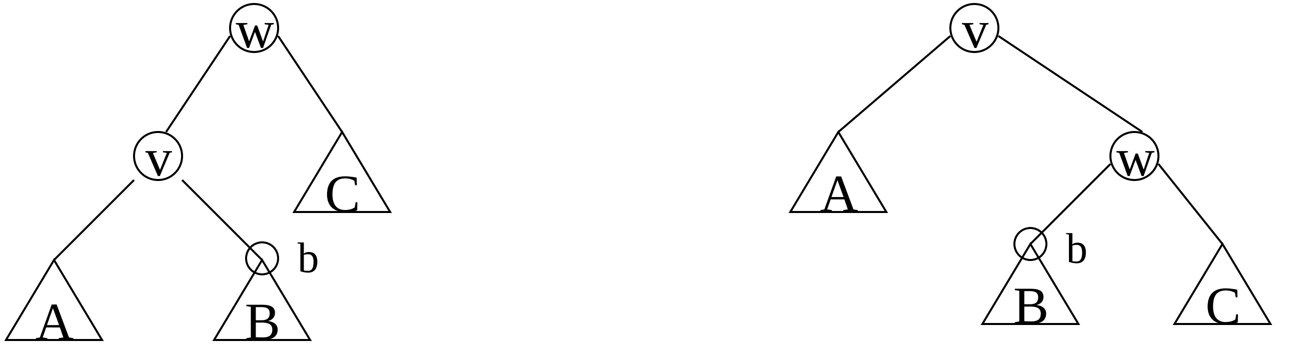
Δcost'(v) = Δcost(v) + Δcost(w)

Δcost'(w) = -Δcost(v)

Δcost'(b) = Δcost(v) + Δcost(b)

Addcost(v,c) (cont)

Update Δmin:



$\Delta\text{min}'(w) = \max\{0, \Delta\text{min}(b) - \Delta\text{cost}'(b), \Delta\text{min}(c) - \Delta\text{cost}(c)\}$

$\Delta\text{min}'(v) = \max\{0, \Delta\text{min}(a) - \Delta\text{cost}(a), \Delta\text{min}'(w) - \Delta\text{cost}'(w)\}$

Link(v,w,c), cut(v)

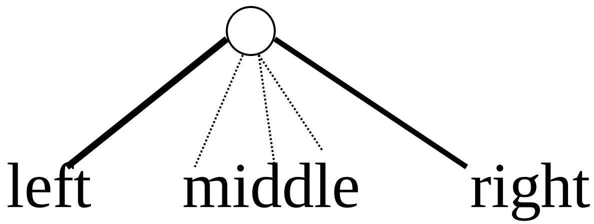
Translate directly into catenation and split of splay trees if we talk about paths.

Lets do the general case now.

The virtual tree

- We represent each tree T by a virtual tree V.

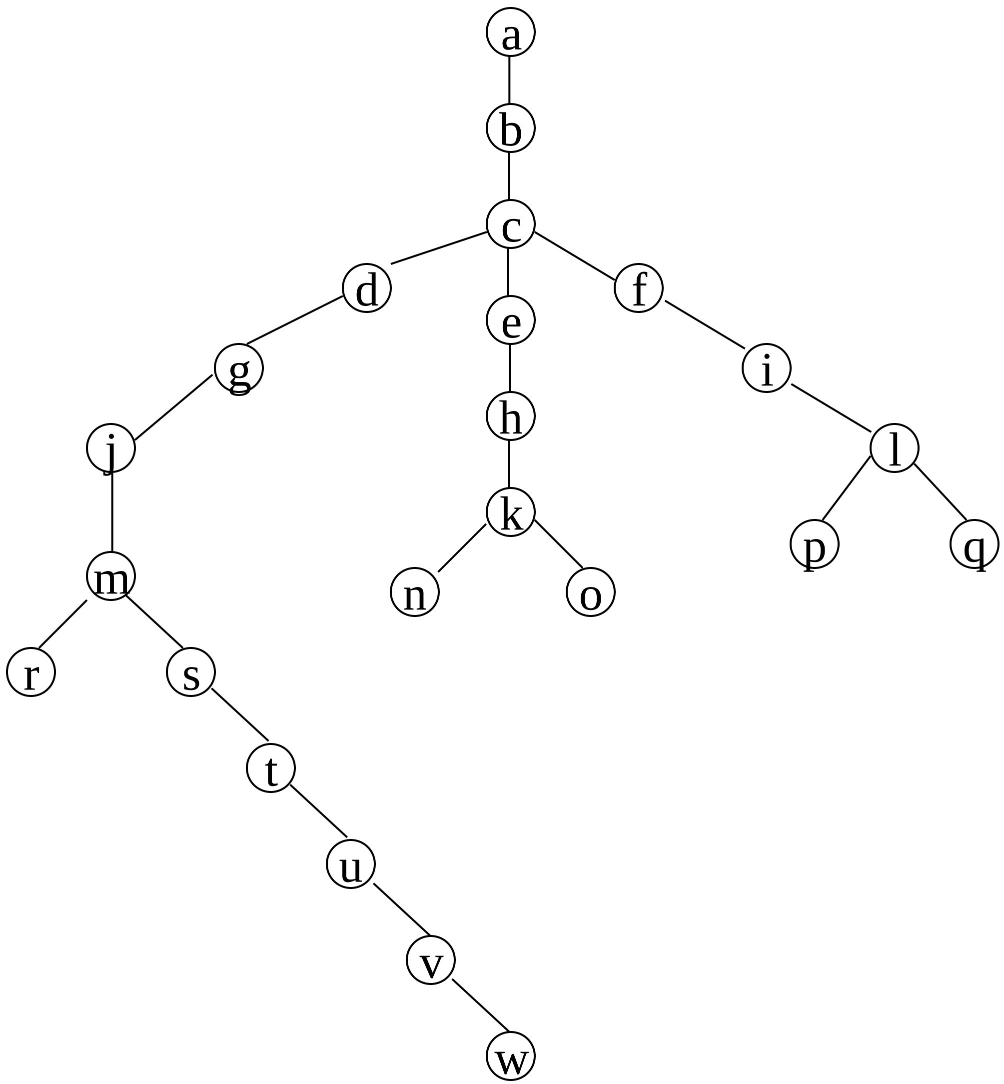
The virtual tree is a binary tree with middle children.



Think of V as partitioned into solid subtrees connected by dashed edges

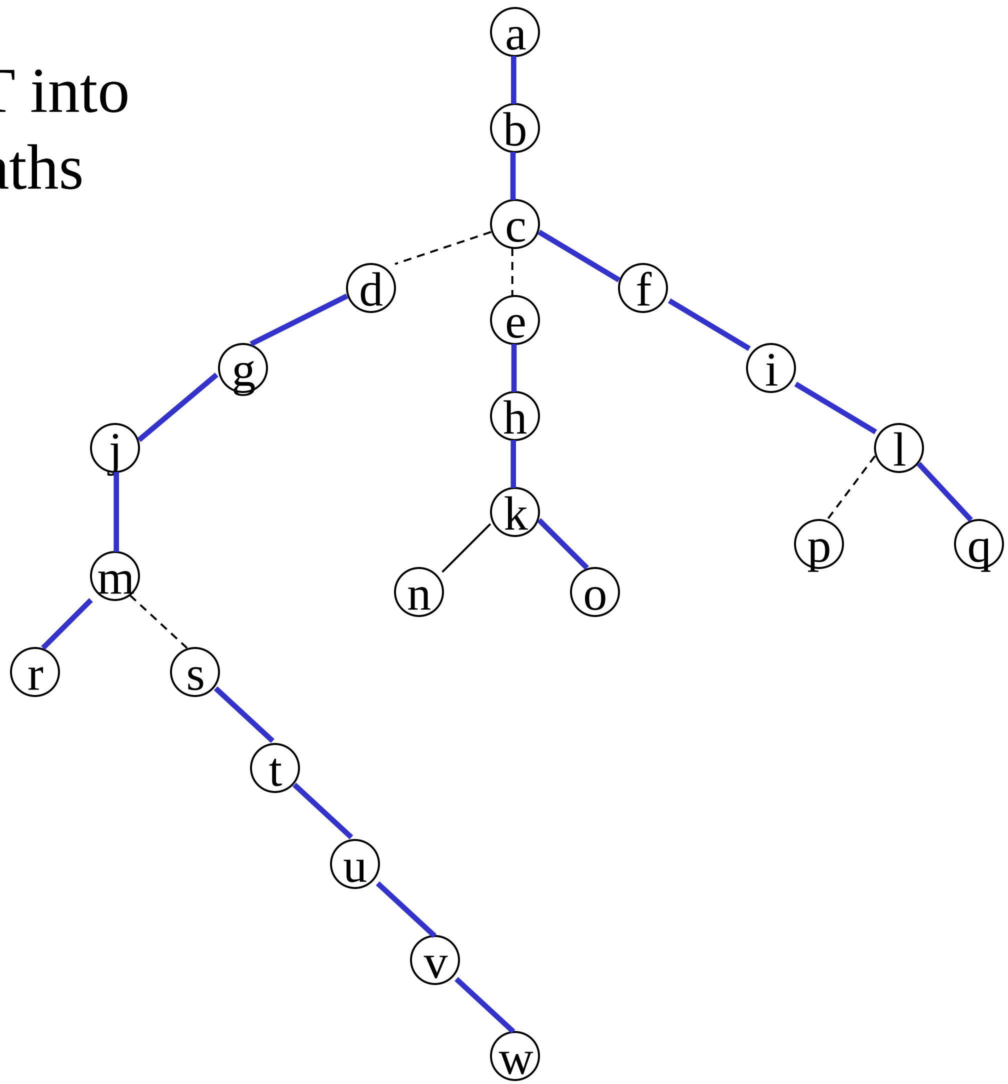
What is the relation between V and T ?

Actual tree



Path decomposition

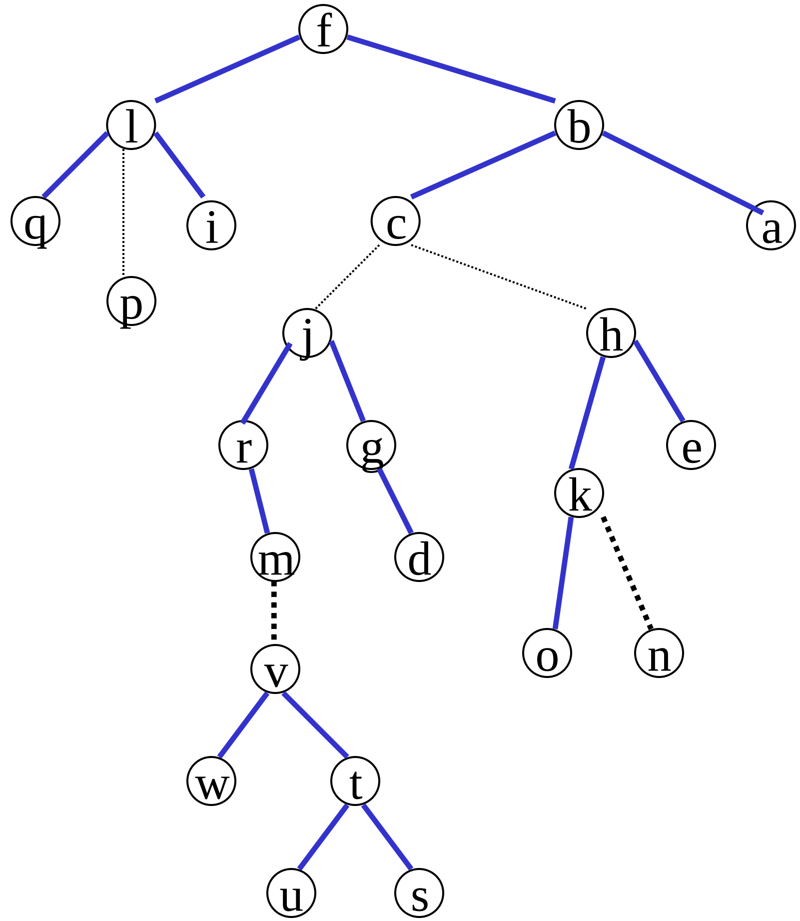
Partition T into disjoint paths



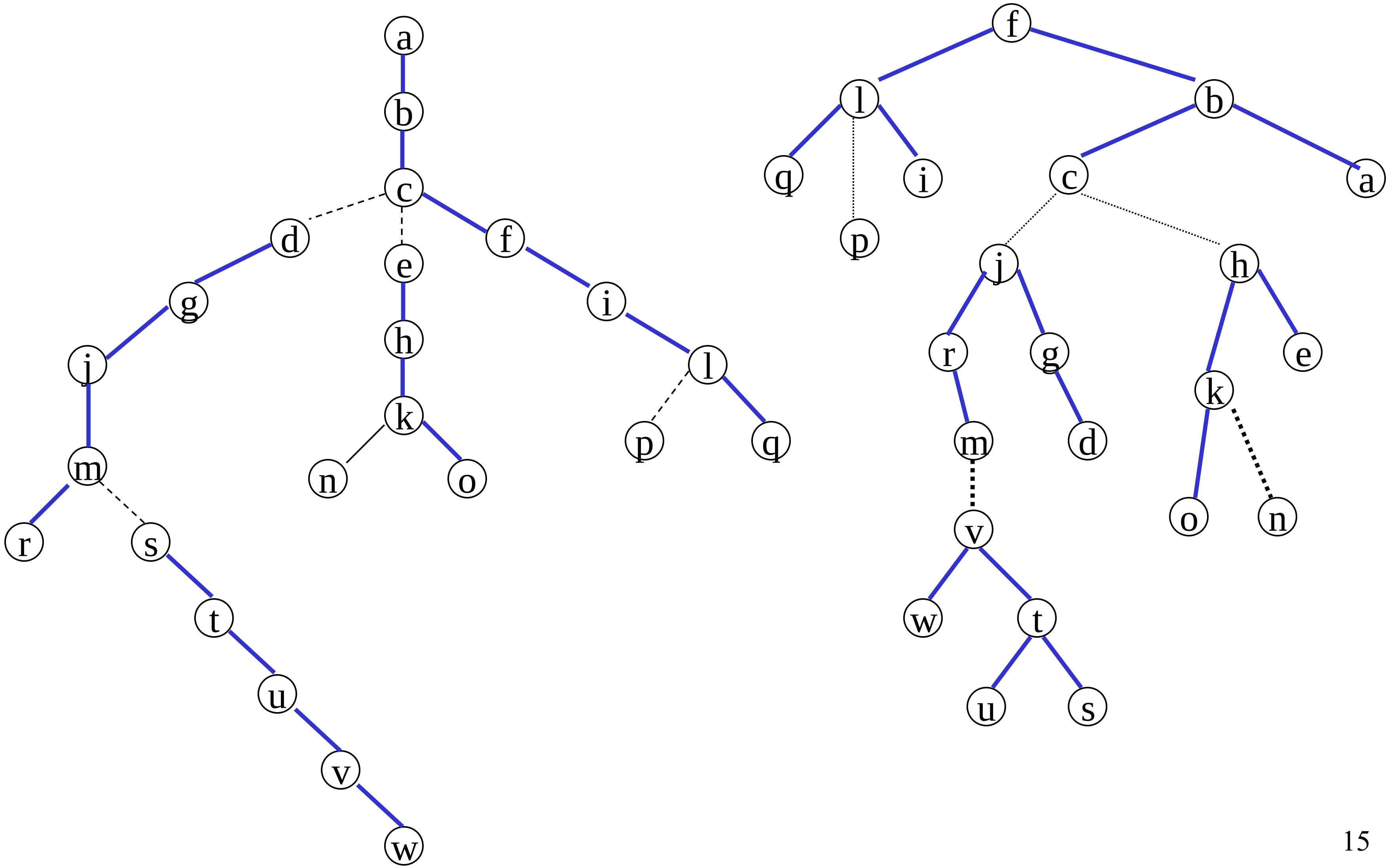
Virtual trees (cont)

Each path in T corresponds to a solid subtree in V

The parent of a vertex x in T is the successor of x (in symmetric order) in its solid subtree or the parent of the solid subtree if x is the last in symmetric order in this subtree



Virtual trees (cont)



Virtual trees (representation)

Each vertex points to p(x) to its left son l(x) and to its right son r(x).

A vertex can easily decide if it is a left child a right child or a middle child.

Each solid subtree functions like a splay tree.

The general case

Each solid subtree of a virtual tree is a splay tree.

We represent costs essentially as before.

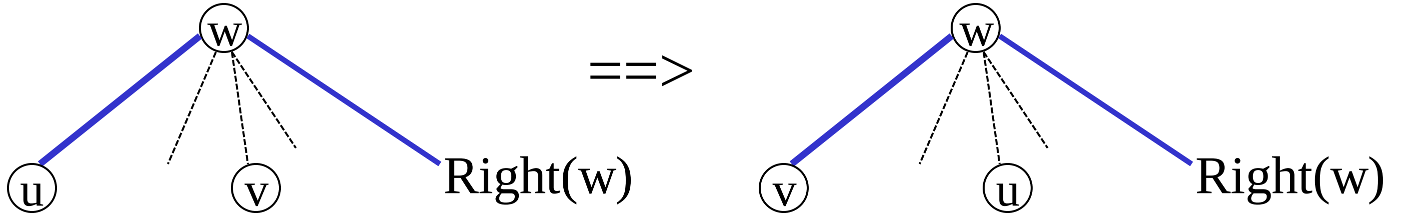
Δcost(x) = cost(x) - cost(p(x)) or cost(x) is x is a root of a solid subtree

Δmin(x) = cost(x) - mincost(x) (where mincost is the minimum cost within the subtree)

Splicing

Want to change the path decomposition such that v and the root are on the same path.

Let w be the root of a solid subtree and v a middle child of w



Want to make v the left child of w. It requires:

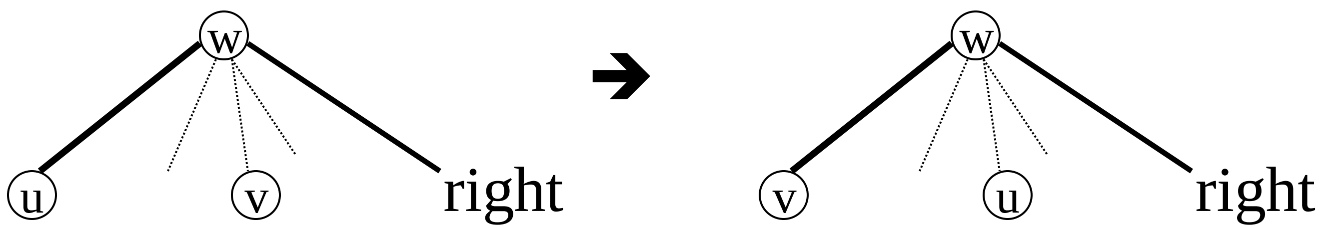
$\Delta cost'(v) = \Delta\ cost(v) - \Delta\ cost(w)$

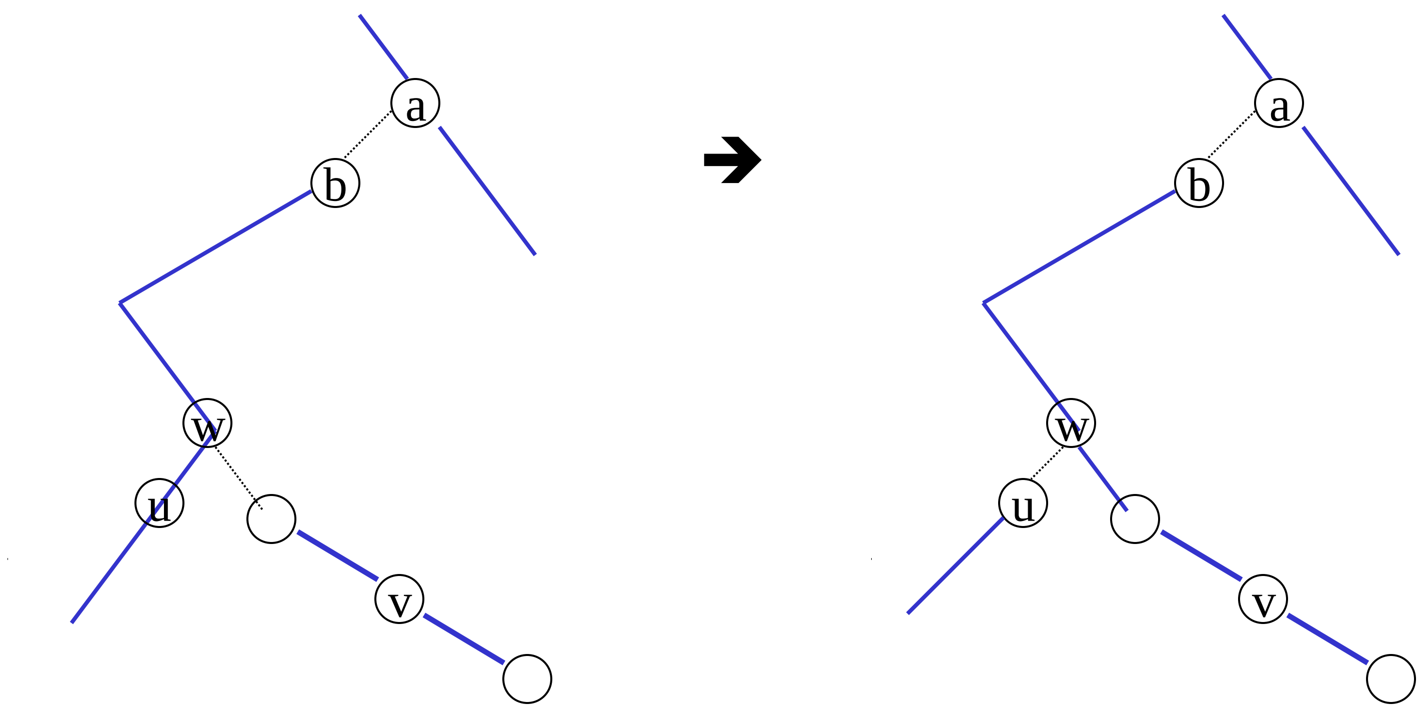
$\Delta cost'(u) = \Delta\ cost(u) + \Delta\ cost(w)$

$\Delta min'(w) = \max\{0, \Delta min(v) - \Delta cost'(v), \Delta min(right(w)) - \Delta cost(right(w))\}$

Splicing (cont)

What is the effect on the path decomposition of the real tree ?





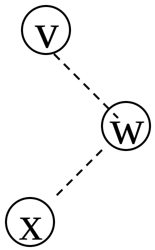
Splaying the virtual tree

Let x be the vertex in which we splay.

We do 3 passes:

1) Walk from x to the root and splay within each solid subtree

After the first pass the path from x to the root consists entirely of dashed edges



2) Walk from x to the root and splice at each proper ancestor of x.

Now x and the root are in the same solid subtree

3) Splay at x

Now x is the root of the entire virtual tree.

Dynamic tree operations

w = findroot(v) : Splay at v, follow right pointers until reaching the last node w, splay at w, and return w.

(v,c) = mincost(v) : Splay at v and use Δcost and Δmin to follow pointers to the smallest node after v on its path (its in the right subtree of v). Let w be this node, splay at w.

addcost(v,c) : Splay at v, increase Δcost(v) by c and decrease Δcost(left(v)) by c, update Δmin(v)

link(v,w,r(v,w)) : Splay at v and splay at w and make v a middle child of w

cut(v) : Splay at v, break the link between v and right(v), set Δcost(right(v)) += Δcost(v)

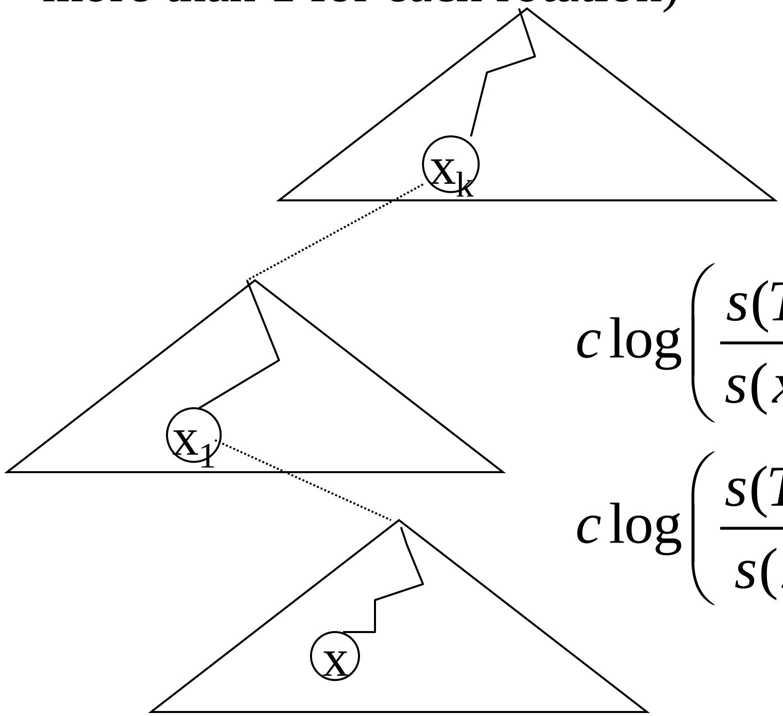
Dynamic tree (analysis)

It suffices to analyze the amortized time of splay.

An extension of the access lemma.

•Assign weight 1 to each node. The size of a node is the total number of descendants it has in the virtual tree. Rank is the log of the size.

Potential is c times the sum of the ranks for some constant c. (So we can charge more than 1 for each rotation)



$$c\log\left(\frac{s(T_k)}{s(x_k)}\right)+.....+c\log\left(\frac{s(T_1)}{s(x_1)}\right)+c\log\left(\frac{s(T_x)}{s(x)}\right)+k\leq$$

$$c\log\left(\frac{s(T_k)}{s(x)}\right)+k$$

Dynamic tree (analysis)

pass 1 takes 3clogn + k

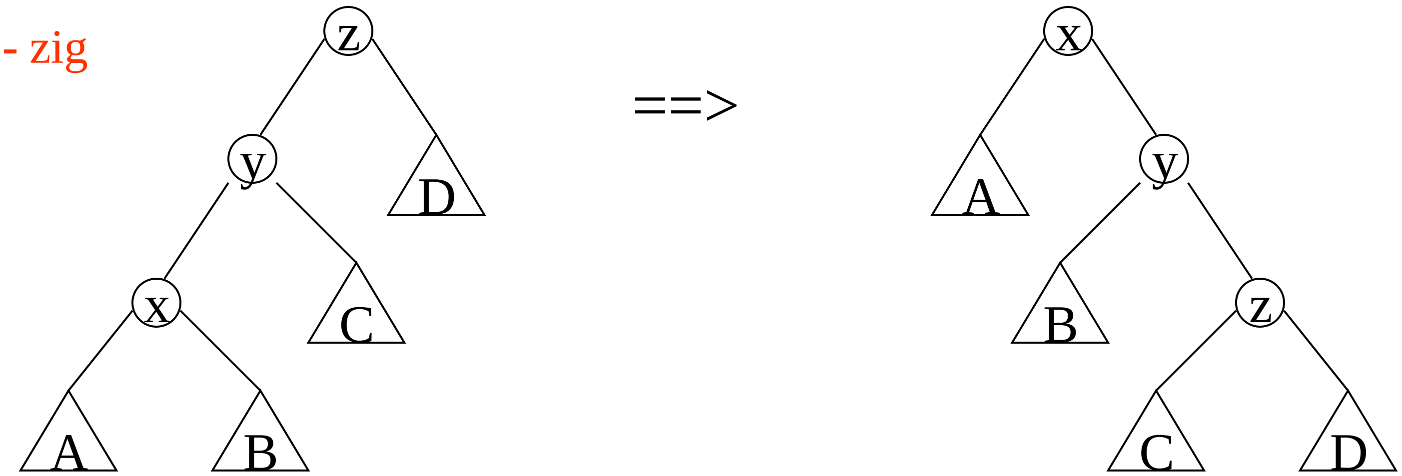
pass 2 takes k

pass 3 takes 3clogn + 1 – (c-1)(k-1)

k=#dashed edges on the path

Proof of the access lemma (cont)

(1) zig - zig



amortized time(zig-zig) = 2 + $\Delta\Phi$ =

$$2 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \le$$

$$2 + r'(x) + r'(z) - r(x) - r(y) \quad \le \quad 2 + r'(x) + r'(z) - r(x) - r(x) =$$

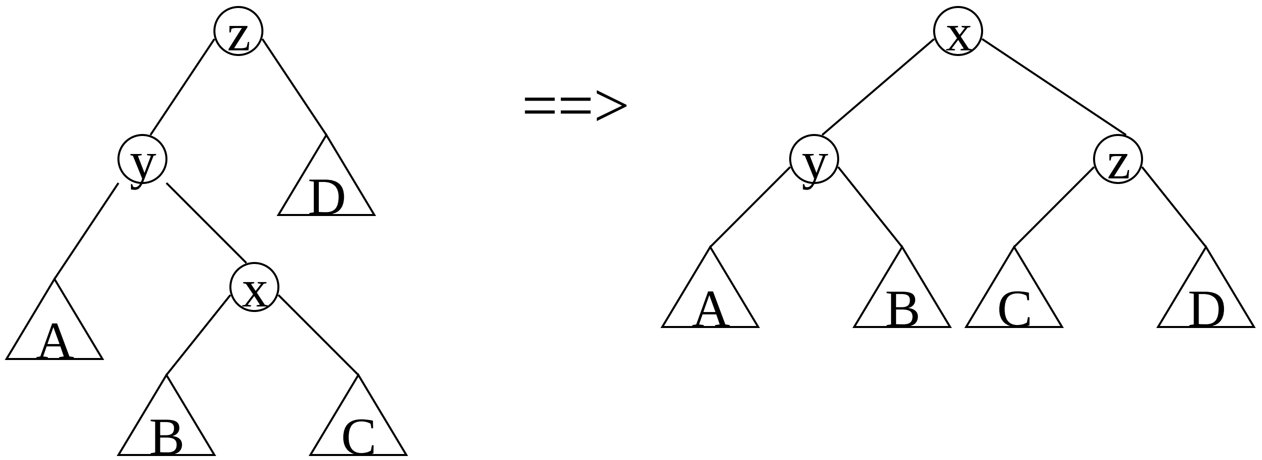
$$2 + r(x) - r'(x) + r'(z) - r'(x) + 3(r'(x) - r(x)) \le$$

$$2 + \log(s(x)/s'(x)) + \log(s'(z)/s'(x)) + 3(r'(x) - r(x)) \le$$

$$2 + \log([s'(x)/2]/s'(x)) + \log([s'(x)/2]/s'(x)) + 3(r'(x) - r(x)) = 3(r'(x) - r(x))$$

Proof of the access lemma (cont)

(2) zig - zag



amortized time(zig-zig) = 2 + $\Delta\Phi$ =

$2 + r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z) \leq$

$2 + r'(y) + r'(z) - r(x) - r(y) \quad \leq \quad 2 + r'(y) + r'(z) - r(x) - r(x) =$

$2 + r'(y) - r(x) + r'(z) - r(x) + 2(r'(x) - r(x)) \leq$

$2 + \log(s'(y)/s(x)) + \log(s'(z)/s(x)) + 2(r'(x) - r(x)) \leq$

$2 + \log([s(x)/2]/s(x)) + \log([s(x)/2]/s(x)) + 2(r'(x) - r(x)) \leq 3(r'(x) - r(x))$