Computational Learning Theory	**Spring Semester, 2003/4**

Lecture 2: March 9

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2.1 Coordination Ratio

Our main goal is to compare the "cost" of *Nash equilibrium* (*NE*) to the "cost" of a global optimum of our choice. The following examples will help us get a notion of the *Coordination Ratio*:

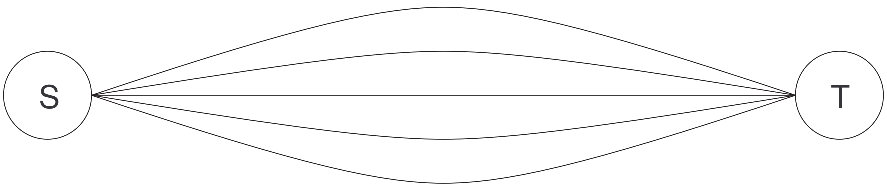

Figure 2.1: Routing on parallel lines

- Assume there is a network of parallel lines from an origin to a destination as shown in figure 2.1. Several agents want to send a particular amount of traffic along a path from the source to the destination. The more traffic on a particular line, the longer the traffic delay.
- Allocation jobs to machines as shown in figure 2.2. Each job has a different size and each machine has a different speed. The performance of each machine reduces as more jobs are allocated to it. An example for a global optimum function, in this case, would be to minimize the load on the most loaded machine.

In these scribes we will use only the terminology of the scheduling problem.

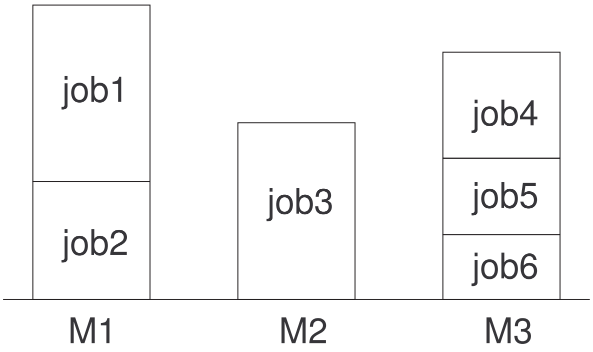

Figure 2.2: Scheduling jobs on machines

2.2 The Model

- Group of n users (or players), denoted $N = \{1, 2, ..., n\}$
- m machines: $M_1, M_2, ..., M_m$
- $\vec{s}$ speeds: $s_1, s_2, ..., s_m$ (in accordance to M_i)
- Each user i has a weight: $w_i > 0$
- ψ : mapping of users to machines:

$$\psi(i) = j$$

  where i is the user and j is the machine's index. Note that NE is a special type of ψ - one which is also an equilibrium.

- The load on machine M_j will be:

$$L_j = \frac{\sum_{i:\psi(i)=j} w_j}{s_j}$$

- The cost of a configuration will be defined as the maximal load of a machine:

$$cost(\psi) = \max_j L_j$$

Our goal is to minimize the cost. The minimal cost, sometimes referred to as the *social optimum* is denoted by OPT and defined as follows:

$$OPT = \min_{\psi} cost(\psi)$$

Definition We name the ratio between the worst NE and OPT the *Coordination Ratio* and define it to be:

$$\mathbf{CR} = \frac{\max_{NE} cost(NE)}{OPT}$$

2.3 Points of equilibria

In our discussion we will attend two types of equilibria:

- Deterministic: Each user i is assigned to one machine, M_j.
- Stochastic: Each user i has a distribution p_i over $\vec{M}$. Note that the deterministic model is a special case of the stochastic model where $p_i(j) = \begin{cases} 1 & \text{if } j = j_0 \\ 0 & \text{otherwise} \end{cases}$.

When each player chooses a certain distribution, the expected load on machine j is:

$$E[L_j] = \frac{\sum_{i=1}^{n} p_i(j) * w_i}{s_j}$$

Next we define for player i the cost of choosing machine j. This function represents the point of view of player i: we define it as if he chose the machine in a deterministic manner.

$$C_i(j) = \sum_{k \neq i} \frac{p_k(j) * w_k}{s_j} + \frac{w_i}{s_j} = E[L_j] + \frac{(1 - p_i(j)) * w_j}{s_j}$$

In other words, $C_i(j)$ is the load on M_j if player i moves to machine j.

In an equilibrium player i will choose the machine with the minimal cost (and therefore he has no interest in changing to another machine). We define the cost to be:

$$Cost(i) = \min_{j} C_i(j)$$

Minimizing the cost function for player i means that $p_i(j) > 0$ only for machines that will have a minimal load after the player moves to them. For this reason, i actually shows *Best Response*. (As such, for each machine j: If $C_i(j) > Cost(i)$, then $p_i(j) = 0$. In such a case choosing M_j does not yield a Best Response).

2.4 Bounding CR

First we will show a simple bound on CR.

Claim 2.1 *For m machines, $CR \in [1, m]$.*

Proof: As any equilibrium point cannot be better than the global optimal solution, $CR >= 1$. Therefore we need only to establish the upper bound.
Let $S = \max_j s_j$. In the worst case any *Nash equilibrium* is bounded by:

$$Cost\_NE \leq \frac{\sum_{i=1}^{n} w_i}{S}$$

(Otherwise, the player can move to a machine with speed S for which its load is always less than $Cost\_NE$).
We also have that

$$OPT \geq \frac{\sum_{i=1}^{n} w_i}{\sum_{j=1}^{m} s_j}$$

(As if we can distribute each player's weight in an equal manner over all the machines).
Using the above bounds, we get:

$$CR = \frac{Cost\_NE}{OPT} \leq \frac{\frac{\sum_{i=1}^{n} w_i}{S}}{\frac{\sum_{i=1}^{n} w_i}{\sum_{j=1}^{m} s_j}} = \frac{\sum_{j=1}^{m} s_j}{S} \leq m$$

□

Note 2.2 *The bound now for CR is linear, but in Theorem 2.9 we will show that the bound is in fact logarithmic.*

Claim 2.3 *Finding OPT for m=2, is an NP-Complete problem.*

Proof: Given that $s1 = s2$, this problem becomes identical to dividing natural numbers into two disjoint sets such that the numbers in both sets yield the same sum. This problem (called partitioning) is known to be NP-C. □

Note 2.4 *We've seen models where the optimal solution was not an equilibrium (such the 'prisoner dilema'). In this example the optimal solution is a Nash Equilibrium.*

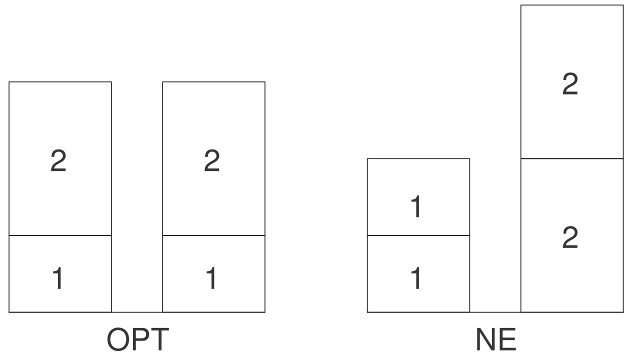

Figure 2.3: Example of $CR = \frac{4}{3}$

2.5 Two Identical Machines, Deterministic Model

As can be seen in figure 2.3, at a *Nash Equilibrium* point, the maximal load is 4. However, the maximal load of the *optimal solution* is only 3. Therefore $CR = \frac{4}{3}$.

Claim 2.5 *For 2 identical machines in the deterministic model,* $CR \geq \frac{4}{3}$.

Proof: Without loss of generality, let us assume that $L_1 > L_2$. We define $v = L_2 - L_1$.

a. If $L_2 \geq v$:
$L_1 = L_2 + v$. Therfore $Cost\_NE = L_2 + v$, and OPT is at least $\frac{L_1+L_2}{2} = L_2 + \frac{v}{2}$. Hence,

$$CR = \frac{NE}{OPT} = \frac{L_2 + v}{L_2 + \frac{v}{2}} = 1 + \frac{\frac{v}{2}}{L_2 + \frac{v}{2}} \leq 1 + \frac{\frac{v}{2}}{v + \frac{v}{2}} = \frac{4}{3}.$$

b. If $L_2 < v$:
As before $L_1 = L_2 + v$. Therefore $2L_2 < L_1 < 2v$. If L_1 consists of the weight of more than one player, we will define w to be the weight of the user with the smallest weight. Since this is a *Nash Equilibrium*, $w > v$. (Otherwise the player would rather move). However, $L_1 < 2v$, hence it is not possible to have two or more players on the same machine. Because of this, we will get one player on M_1 which is the optimal solution, and $CR = 1$ accordingly.

□

2.6 Two Identical Machines, Stochastic Model

For an example we'll look at 2 identical users, for which $w_1 = w_2 = 1$, as shown in figure 2.4. Each of the players chooses a machine at random.

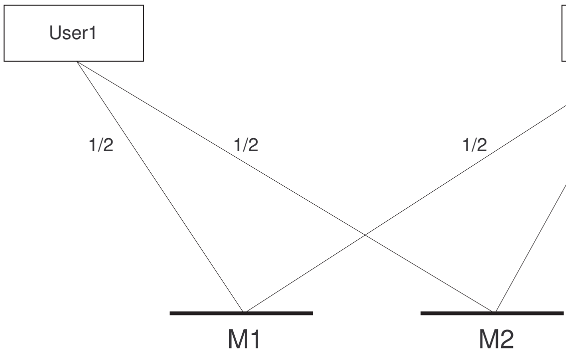

Figure 2.4: Stochastic model example

At a *Nash Equilibrium* point, with a probability of 1/2, the players will choose the same machine and with a probability of 1/2, each player will choose a different machine. Together we get $Cost\_NE = 1/2 * 2 + 1/2 * 1 = 3/2$. The cost of OPT is 1 and so it follows that $CR = 3/2$.

Theorem 2.6 *For 2 identical machines in the stochastic model,* $CR \leq \frac{3}{2}$

Proof: Let $p_i(b)$ be the probability that player i chooses machine M_b. We get that

$$\bar{L_b} = E[L_b] = \sum_{i=1}^{n}(p_i(b) * w_i).$$

And the cost of player i when he chooses machine M_b becomes:

$$(E[Cost_i(b)]) = w_i + \sum_{j \neq i}(p_j(b) * w_j) = C_i(b)$$

Since we have 2 machines, $Cost(i) = \min\{C_i(1), C_i(2)\}$.
Basically, the least loaded machine, when ignoring the weight of user i, is chosen. Since each user performs according to its optimal solution, we get that in a case of an equilibrium point, if $p_i(b) > 0$ then $C_i(b) = Cost(i)$.
On the other hand, if $C_i(b) > Cost(i)$ then $p_i(b) = 0$. In other words, the player chooses her *Best Response* according to what he sees.

We now define q_i to be the probability that player i chooses the most loaded machine. We get that

$$Cost\_NE = E[\max L_b] = \sum_{i=1}^{n}(q_i * w_i).$$

Furthermore, we will define the probability of a collision on a machine (both user i and user j choose the same machine) as t_{ij}.

Pay attention to the following properties:

1. In a *Nash Equilibrium* point, $\sum_{k\neq i}(t_{ik} * w_k) + w_i = Cost(i)$.

2. For m machines, $Cost(i) \leq \frac{1}{m}\sum_{k=1}^{n} w_k + \frac{m-1}{m}w_i$
Proof:

$$Cost(i) = \min_j C_i(j) \leq \frac{1}{m}\sum_{j=1}^{m} C_i(j)$$

$$= \frac{1}{m}\sum_{j=1}^{m}(E[L_j] + (1 - p_i(j)) * w_i) = \frac{1}{m}\sum_{j=1}^{m}\sum_{k=1}^{n}(p_k(j) * w_k) + \frac{m-1}{m}w_i$$

$$= \frac{1}{m}\sum_{k=1}^{n} w_k + \frac{m-1}{m}w_i$$

Substituting m for 2 machines, we get that

$$Cost(i) \leq \frac{1}{2}\sum_{k=1}^{n} w_k + \frac{w_i}{2}$$

3. $q_i + q_j \leq 1 + t_{ij}$
Proof:
$q_i + q_j - t_{ij} \leq Pr[i$ and j choose the most loaded machine$] \leq 1$.

4. $\sum_{k\neq i}(1 + t_{ik}) * w_k \leq \frac{3}{2}\sum_{k\neq i} w_k$
Proof:

$$\sum_{k\neq i}(1 + t_{ik}) * w_k = \sum_{k\neq i} w_k + \sum_{k\neq i} t_{ik}w_k$$

$$= \sum_{k\neq i} w_k + Cost(i) - w_i$$

using property 2:

$$\leq \sum_{k\neq i} w_k + \frac{1}{2}\sum_{k} w_k + \frac{w_i}{2} - w_i$$

$$= \frac{3}{2}\sum_{k\neq i} w_k + \frac{1}{2}w_i - \frac{1}{2}w_i$$

$$\leq \frac{3}{2}\sum_{k\neq i} w_k$$

To finish the proof of the theorem we now get:

$$Cost\_NE = \sum_{k=1}^{n} q_k w_k =$$

$$\sum_k (q_i + q_k)w_k - \sum_k q_i w_k$$

$$= 2q_i w_i + \sum_{k\neq i}(q_i + q_k)w_k - q_i \sum_k w_k$$

$$\leq 2q_i w_i + \sum_{k\neq i}(1 + t_{ik})w_k - q_i \sum_k w_k$$

$$\leq 2q_i w_i + \frac{3}{2}\sum_{k\neq i} w_k - q_i \sum_k w_k$$

$$= (2q_i - \frac{3}{2})w_i + (\frac{3}{2} - q_i)\sum_k w_k$$

As previously shown, $OPT \geq \max\{\frac{1}{2}\sum_{k=1}^{n} w_k, w_i\}$.
Realize that one of the following 2 situations may occur:

1. There exists a player i such that $q_i \geq \frac{3}{4}$.
   In such a case, $(2q_i - \frac{3}{2})w_i \leq (2q_i - \frac{3}{2}) * OPT$.
   Therefore,

$$Cost\_NE \leq (\frac{3}{2} - q_i) * 2OPT + (2q_i - \frac{3}{2}) * OPT$$

$$\leq [2q_i - \frac{3}{2} + 2(\frac{3}{2} - q_i)] * OPT$$

$$= \frac{3}{2} * OPT$$

2. For all i, $q_i \leq \frac{3}{4}$, therefore

$$Cost\_NE = \sum_{k=1}^{n} q_k w_k \leq \frac{3}{4} * \sum_{k=1}^{n} w_k$$

$$\leq \frac{3}{2} * OPT$$

In both cases we reach our desired result that $Cost\_NE \leq \frac{3}{2} * OPT$. □

2.7 Identical machines, deterministic users

First we define some variables:

$$w_{max} = \max_i w_i \tag{2.1}$$

$$L_{max} = \max_j L_j \tag{2.2}$$

$$L_{min} = \min_j L_j \tag{2.3}$$

Claim 2.7 *In a* Nash equilibrium, $L_{max} - L_{min} \leq w_{max}$

Proof: Otherwise there would be some user j s.t. $w_j \leq w_{max}$, which could switch to the machine with load L_{min}. □

Theorem 2.8 *Given identical machines and deterministic users,* $CR \leq 2$

Proof: There are two options:

- $L_{min} \leq w_{max}$

  Then $L_{max} \leq 2w_{max}$

  But since $OPT \geq w_{max}$ we get $CR \leq \frac{L_{max}}{OPT} \leq 2$

- $L_{min} > w_{max}$

  Then $L_{max} \leq L_{min} + w_{max} \leq 2L_{min}$, which results in $OPT \geq \frac{1}{m}\sum_k w_k \geq L_{min}$. Therefore $CR \leq \frac{L_{max}}{OPT} \leq \frac{2L_{min}}{L_{min}} = 2$

□

2.7.1 Example of $CR \to 2$

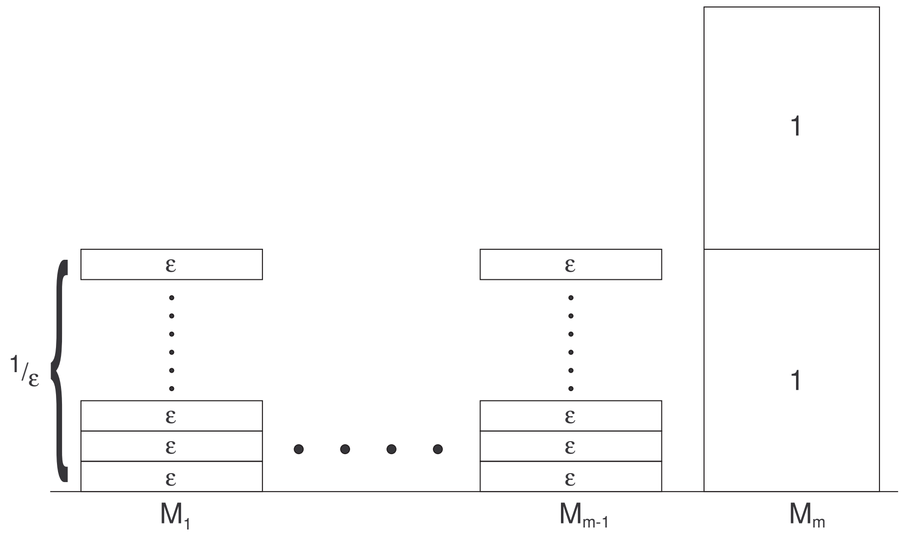

Figure 2.5: CR comes near to 2

Let's examine an example of a configuration with a CR that approaches 2. Consider m machines and $\frac{m-1}{\varepsilon}$ users with a weight of ε and 2 users with a weight of 1 as shown in figure 2.5. This is a *Nash equilibrium* with a cost of 2.

The optimal configuration is obtained by scheduling the two "heavy" users (with $w = 1$) on two separate machines and dividing the other users among the rest of the machines. In this configuration we get:

$C = OPT = 1 + \frac{1}{m} \to 1$

2.8 Identical machines, stochastic users

2.8.1 Example

Consider the following example: m machines, $n = m$ users, $w_i = 1$, $p_i(j) = \frac{1}{m}$. What is the maximal expected load?

This problem is identical to the following problem: m balls are thrown randomly into m bins; What is the expected maximum number of balls in a single bin? Let us first see what is the probability that k balls will fall into a certain bin:

$$Pr = \binom{m}{k} \left(\frac{1}{m}\right)^k \left(1 - \frac{1}{m}\right)^{m-k} \approx \left(\frac{c * m}{k}\right)^k \left(\frac{1}{m}\right)^k = \left(\frac{c}{k}\right)^k$$

The probability that there exists a bin with at least k balls is $1-(1-(\frac{c}{k})^k)^m$ which is constant for $k \sim \frac{\ln m}{\ln \ln m}$. Therefore the maximal load is roughly $\frac{\ln m}{\ln \ln m}$.

2.8.2 Upper bound

Using the Azuma-Hoeffding inequality we will establish a highly probable upper bound on the maximum expected load. Using theorem 2.8 from the deterministic part we know that:

$$\bar{L}_j = E[L_j] \leq 2OPT$$

We wish to prove that the probability of having a j for which $L_j \gg \bar{L}_j$ is negligible. The Azuma-Hoeffding inequality for some random variable $X = \sum x_i$, where x_i are random variables with values in the interval $[0, z]$, is:

$$P[X \geq \lambda] \leq \left(\frac{e * E[X]}{\lambda}\right)^{\frac{\lambda}{z}}$$

Let us define $\lambda = 2\alpha OPT$, $z = w_{max}$ and $x_i = \begin{cases} w_i & \text{if } p_i(j) > 0 \\ 0 & \text{otherwise} \end{cases}$

By applying the inequality we get:

$$P[L_j \geq 2\alpha OPT] \leq \left(\frac{e * E[L_j]}{2\alpha OPT}\right)^{\frac{2\alpha OPT}{w_{max}}} \leq \left(\frac{e}{\alpha}\right)^{2\alpha}$$

which results in

$$P[\exists j \ L_j \geq 2\alpha OPT] \leq m\left(\frac{e}{\alpha}\right)^{2\alpha}$$

Note that for $\alpha = \Omega(\frac{\ln m}{\ln \ln m})$ the probability is smaller than $\frac{1}{2m}$.

Theorem 2.9 *For m identical machines the worst case CR is* $O\left(\frac{\ln m}{\ln \ln m}\right)$

Proof: We shall calculate the expected cost including high loads which have a low probability, and see that their contribution is $O(1)$. For any random variable X and a natural number A we know that:

$$E[X] \leq A + \sum_{i=A}^{\infty} P[X \geq i]$$

In our case we get

$$E[\text{cost-NE}] \leq A * OPT + \sum_{\alpha=A}^{\infty} P[\text{cost-NE} \geq 2\alpha * OPT] * 2OPT$$

Therefore we define $A = 2 * c\frac{\ln m}{\ln\ln m}$ for some constant c and get

$$E[\text{cost-NE}] \leq 2 * c\frac{\ln m}{\ln\ln m} * OPT + m\sum_{\alpha}\left(\frac{e}{\alpha}\right)^{2\alpha} * OPT$$

But since $\frac{e}{\alpha} \leq \frac{1}{2m}$ we get

$$E[\text{cost-NE}] \leq 2 * c\frac{\ln m}{\ln\ln m} * OPT + O(1) * OPT$$

Resulting in

$$\text{CR} = O\left(\frac{\ln m}{\ln\ln m}\right)$$

□

2.9 Non-identical machines, deterministic users

We shall first examine a situation with a 'bad' *coordination ratio* of $\frac{\ln m}{\ln\ln m}$, then establish an upper bound.

2.9.1 Example

Let us have $k+1$ groups of machines, with N_j machines in group j. The total number of machines $m = N = \sum_{j=0}^{k} N_j$. We define the size of the groups by induction:

- $N_k = \sqrt{N}$
- $N_j = (j+1) * N_{j+1}$
- $N_0 = k! * N_k$

From the above it results that:

$$k \sim \frac{\ln N}{\ln\ln N}$$

the speed of the machines in group N_j is defined $s_j = 2^j$.

First we set up an equilibrium with a high cost. Each machine in group N_j receives j users, each with a weight of 2^j. It is easy to see that the load in group N_j is j and therefore the cost is k. Note that group N_0 received no users.

Claim 2.10 *This setup is a Nash equilibrium.*

Proof: Let us take a user in group N_j. If we attempt to move him to group N_{j-1} he will see a load of

$$(j-1) + \frac{2^j}{2^{j-1}} = j+1 > j$$

On the other hand, on group N_{j+1} the load is $j+1$ even without his job and therefore he has no reason to move there. □

To achieve the optimum we simply need to move all the users of group N_j to group N_{j-1} (for $j = 1...k$). Now there is a separate machine for each user and the load on all machines is $\frac{2^j}{2^{j-1}} = 2$.

Corollary 2.11 *The coordination ratio is* $\sim \frac{\ln m}{\ln \ln m}$

2.9.2 Upper bound

The machines have different speeds; Without loss of generality let us assume that $s_1 \geq s_2 \cdots \geq s_m$. The cost is defined $C = \max L_j$.

For $k \geq 1$, define J_k to be the smallest index in $\{0, 1, \ldots, m\}$ such that $L_{J_k+1} < k * OPT$ or, if no such index exists, $J_k = m$. We can observe the following:

- All machines up to J_k have a load of at least $k * OPT$
- The load of the machine with an index of $J_k + 1$ is less than $k * OPT$

Let C^* be defined:

$$C^* = \lfloor \frac{C - OPT}{OPT} \rfloor$$

Our goal is to show that $C^*! < J_1$ which will result in

$$C = O\left(\frac{\log m}{\log \log m}\right) * OPT$$

We will show this using induction.

Claim 2.12 *(The induction base)* $J_{C^*} \geq 1$

Proof: By the way of contradiction, assume $J_{C^*} = 0$. This implies (from the definition of J_k) that $L_1 < C^* * OPT \leq C - OPT$. Let q denote the machine with the maximum expected load. Then $L_1 + OPT < C = L_q$.

We observe that any user that uses q must have a weight w_i larger than $s_1 * OPT$. Otherwise he could switch to the fastest machine, reaching a cost of $L_1 + \frac{w_i}{s_1} \leq L_1 + OPT < L_q$, which contradicts the stability of the *Nash equilibrium.* □

We shall divide the proof of the induction step into two claims. Let S be the group of users of the machines $M_1, \ldots, M_{J_{k+1}}$.

Claim 2.13 *An optimal strategy will not assign a user from group S to a machine $r > J_k$.*

Proof: From the definition of J_k, the users in S have a load of at least $(k+1) * OPT$. Machine $J_k + 1$ has a load of at most $k * OPT$. No user from S will want to switch to $J_k + 1$ because the minimal weight in S is $s_{J_k+1} * OPT$. Switching to machine $r > J_k + 1$ will result in a load bigger than OPT because $s_r < s_{J_k+1}$. □

Claim 2.14 *If an optimal strategy assigns users from group S to machines $1, 2, \ldots, J_k$ then $J_k \geq (k+1)J_{k+1}$*

Proof: Let $W = \sum_{i \in S} w_i$.

$$W = \sum_{j \leq J_{k+1}} s_j * E[L_j] \geq (k+1)OPT(\sum_{j \leq J_{k+1}} s_j)$$

Since an optimal strategy uses only machines $1, 2, \ldots, J_k$ we get:

$$OPT(\sum_{j \leq J_k} s_j) \geq W$$

$$\sum_{j \leq J_k} s_j \geq (k+1) * \sum_{j \leq J_{k+1}} s_j$$

Since the sequence of the speeds is non-increasing, this implies that $J_k \geq (k+1)J_{k+1}$, the induction step. □

Now we can combine the two claims above using induction to obtain:

Corollary 2.15 $C^*! < J_1$

By definition $J_1 \leq m$. Consequently $C^*! \leq m$, which implies the following:

Corollary 2.16 *(Upper bound)* $C = O(\frac{\log m}{\log \log m})$