Compilation

0368-3133 (Semester A, 2013/14)

Lecture 12: Abstract Interpretation

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Slides credit: Roman Manevich, Mooly Sagiv and Eran Yahav

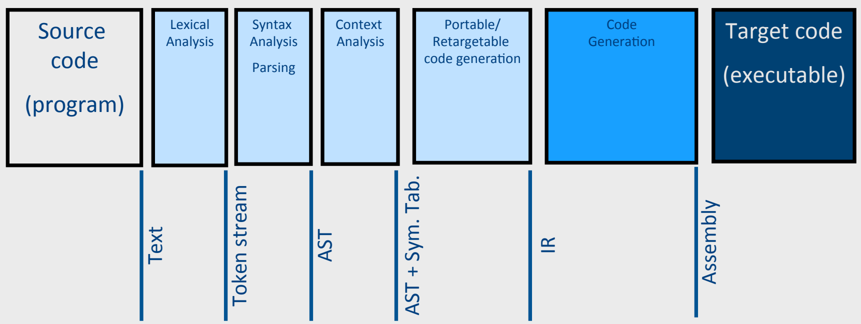
What is a compiler?

"A compiler is a computer program that transforms source code written in a programming language (source language) into another language (target language).

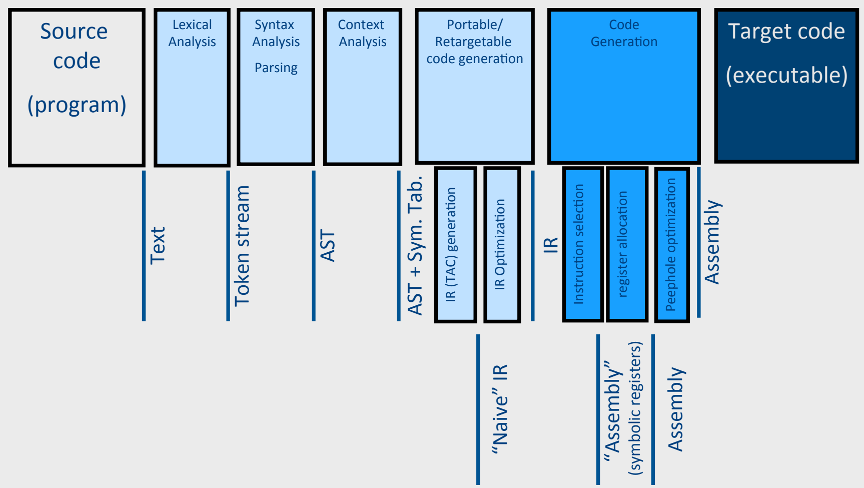
The most common reason for wanting to transform source code is to create an executable program."

--Wikipedia

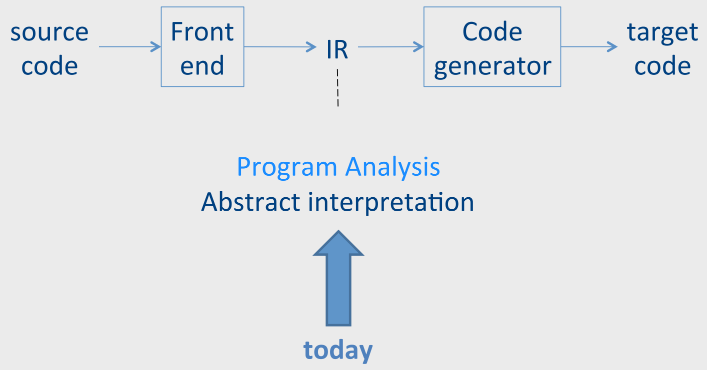
Stages of compilation



Stages of Compilation



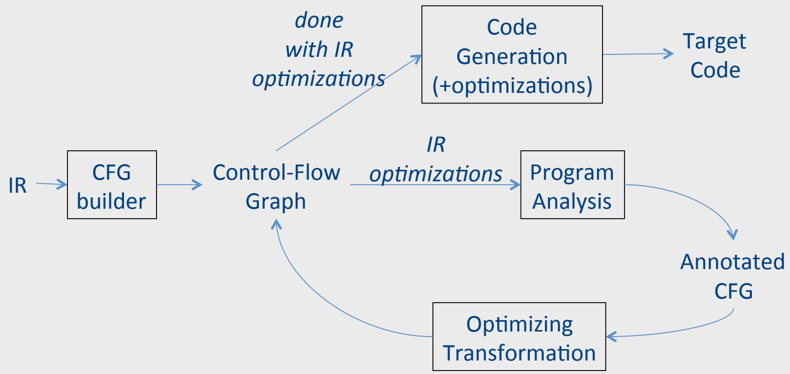
Optimization points



Program Analysis

- In order to optimize a program, the compiler has to be able to reason about the properties of that program
- An analysis is called **sound** if it never asserts an incorrect fact about a program
- All the analyses we will discuss in this class are sound
  - *(Why?)*

Optimization path



Soundness

```
int x;
int y;

if (y < 5)
    x = 137;
else
    x = 42;

Print(x);
```

"At this point in the program, **x** holds some integer value"

Soundness

```
int x;
int y;

if (y < 5)
    x = 137;
else
    x = 42;

Print(x);
```

"At this point in the program, **x** is either 137 or 42"

(Un) Soundness

```
int x;
int y;

if (y < 5)
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Print(x);
```

"At this point in the program, **x** is 137"

Soundness & Precision

```
int x;
int y;

if (y < 5)
    x = 137;
else
    x = 42;

Print(x);
```

"At this point in the program, **x** is either 137, 42, or 271"

Semantics-preserving optimizations

- An optimization is **semantics-preserving** if it does not alter the semantics (meaning) of the original program
  - ✓ Eliminating unnecessary temporary variables
  - ✓ Computing values that are known statically at compile-time instead of computing them at runtime
  - ✓ Evaluating iteration-independent expressions outside of a loop instead of inside
  - ✗ Replacing bubble sort with quicksort (why?)
- The optimizations we will consider in this class are all semantics-preserving

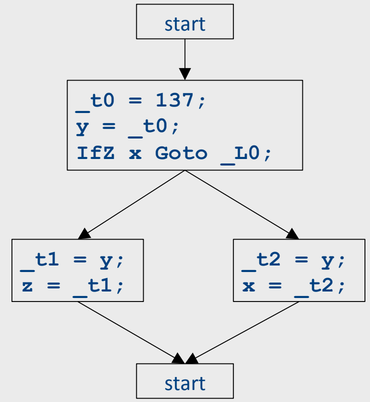
Types of optimizations

- An optimization is local if it works on just a single basic block
- An optimization is global if it works on an entire control-flow graph

Local optimizations

```
int main() {
   int x;
   int y;
   int z;

   y = 137;
   if (x == 0)
      z = y;
   else
      x = y;
}
```



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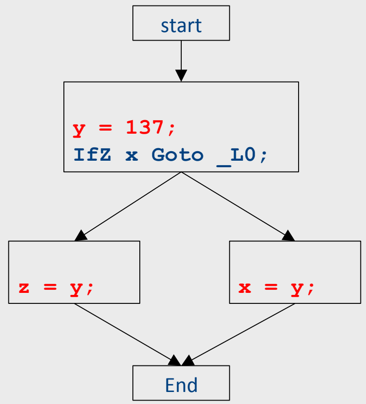

Global optimizations

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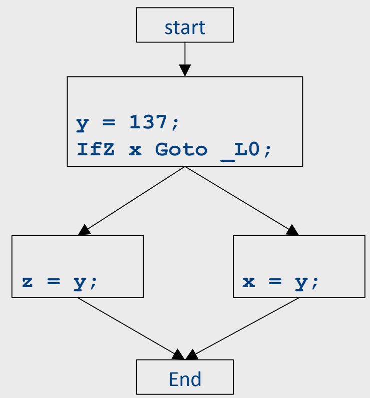

Global optimizations

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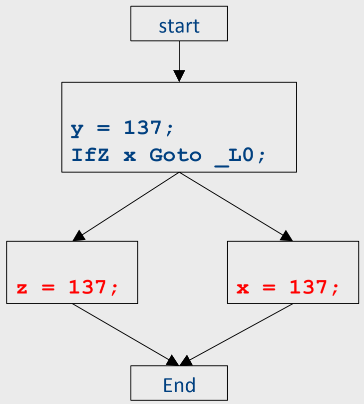
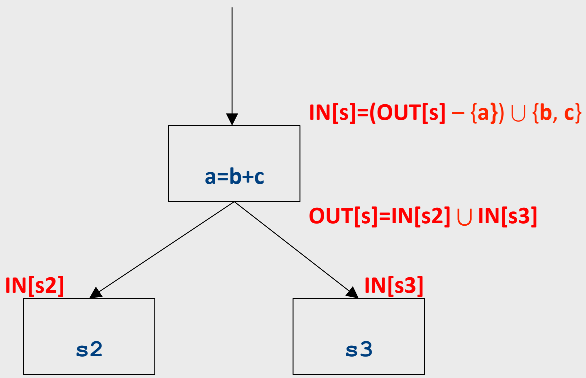


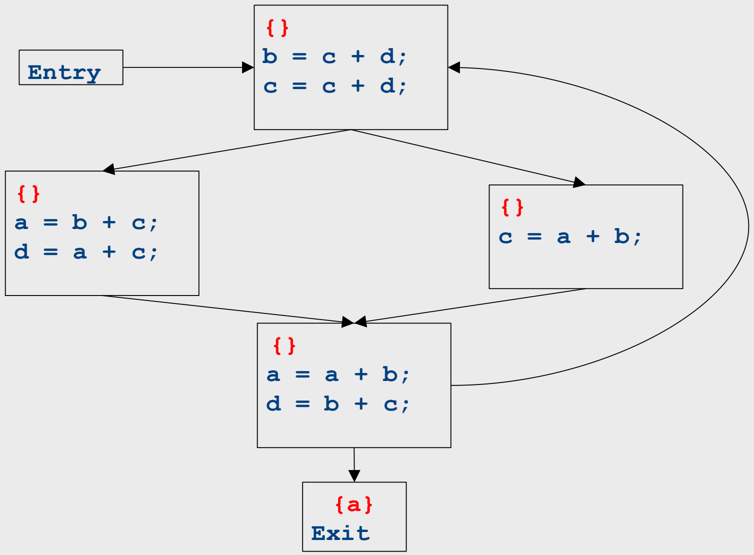
Global Analyses

- Common subexpression elimination
- Copy propagation
- Dead code elimination
- Live variables
  - Used for register allocation

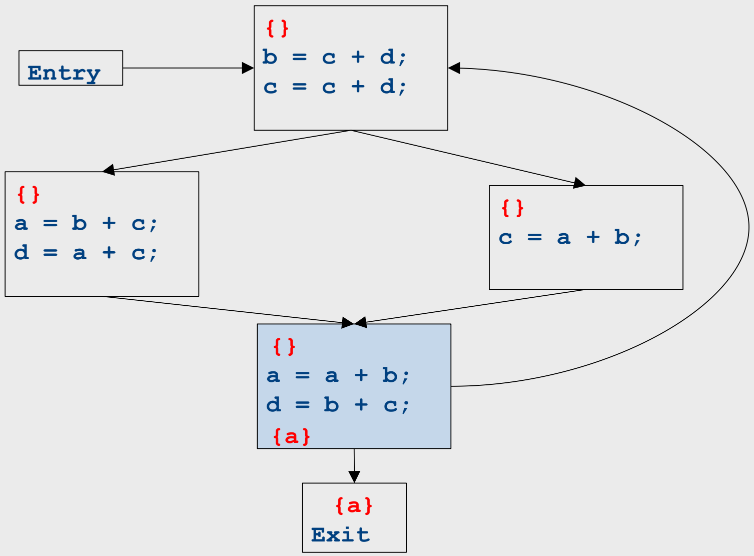
Global liveness analysis



Live variable analysis- initialization



Live variable analysis



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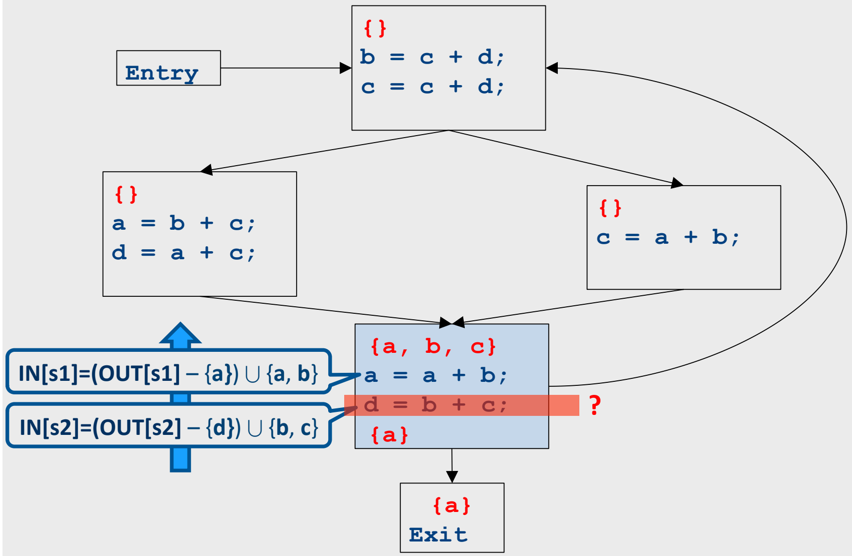

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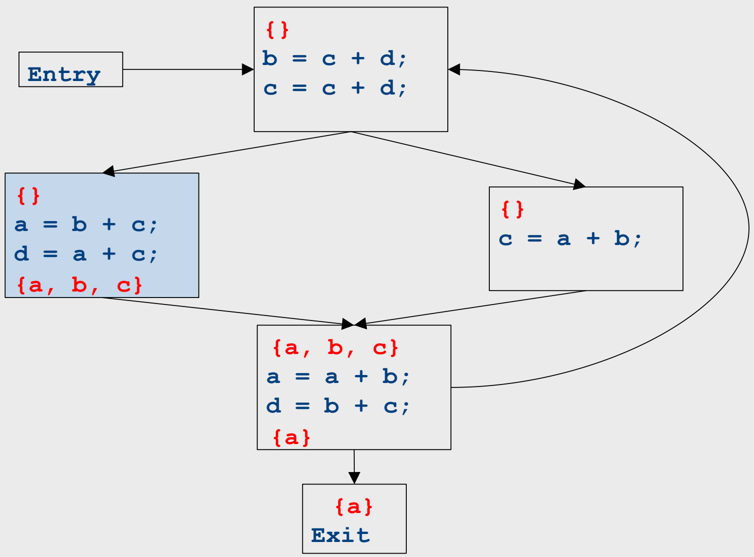
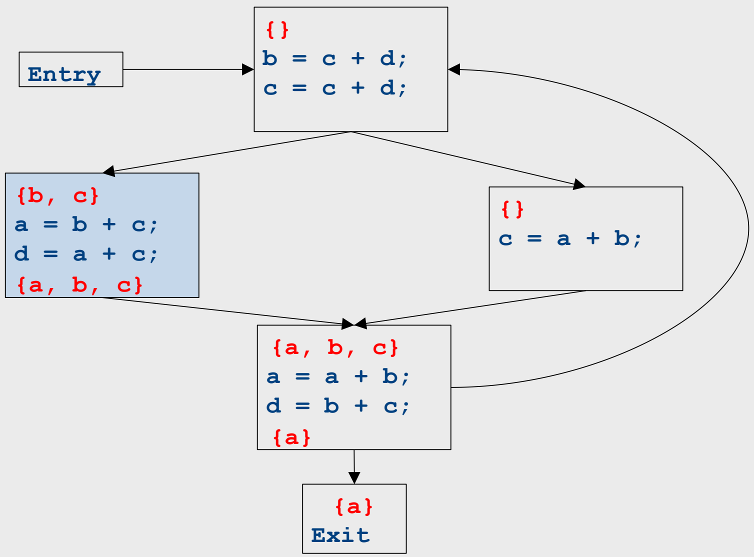


CFGs with loops - iteration



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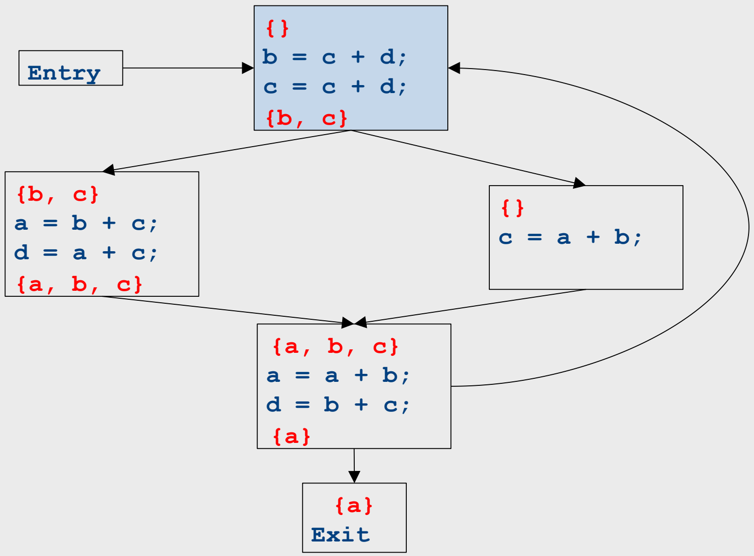

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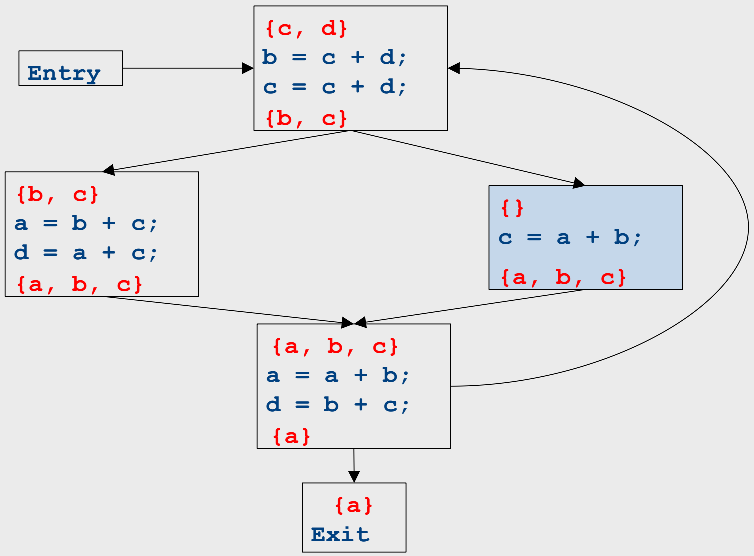

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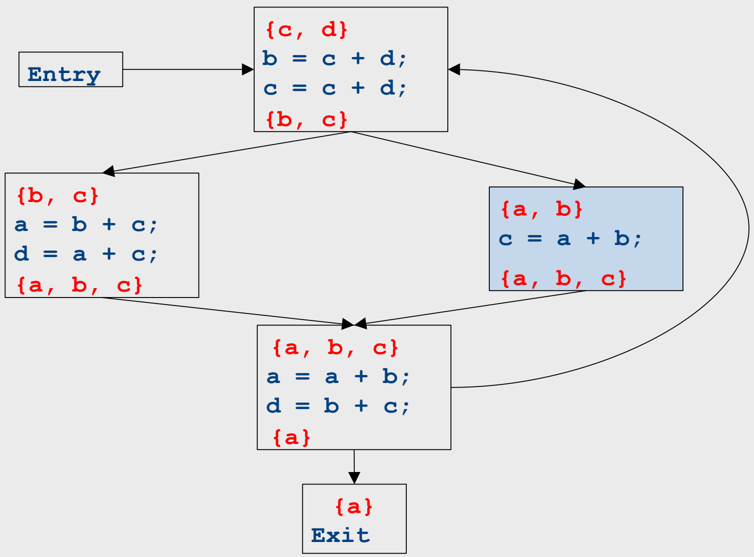

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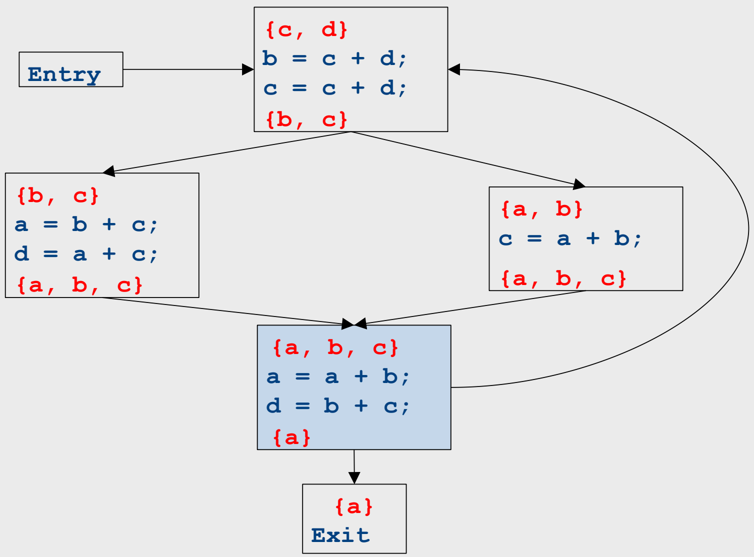

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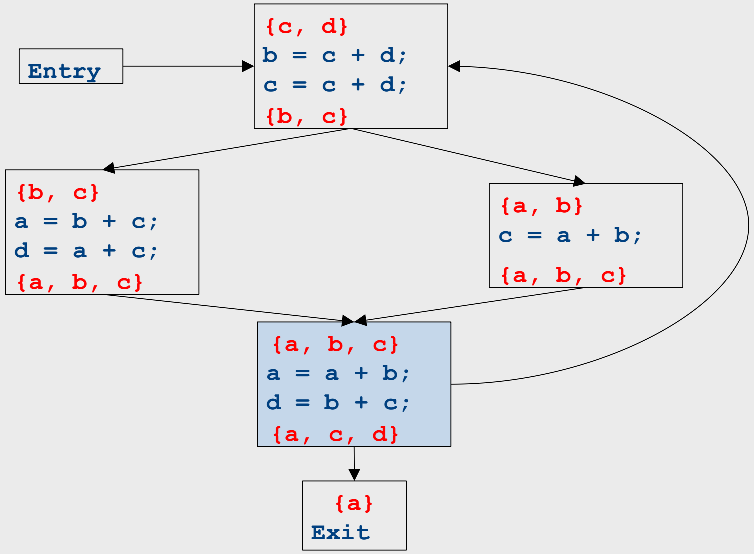

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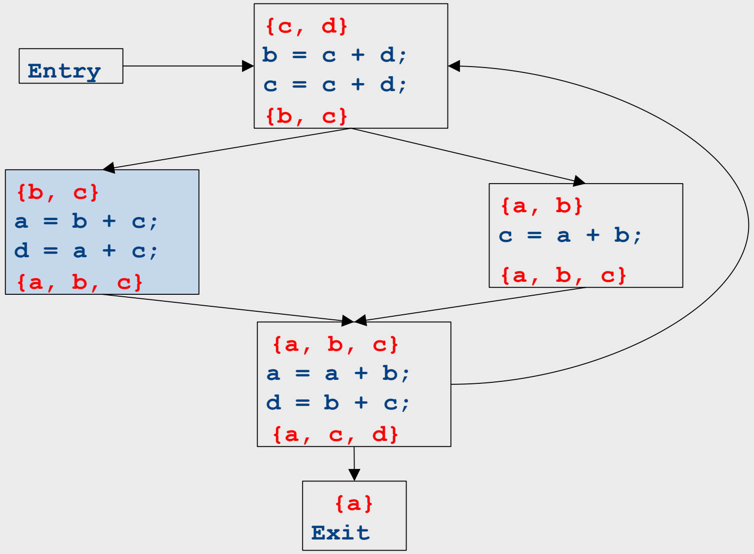

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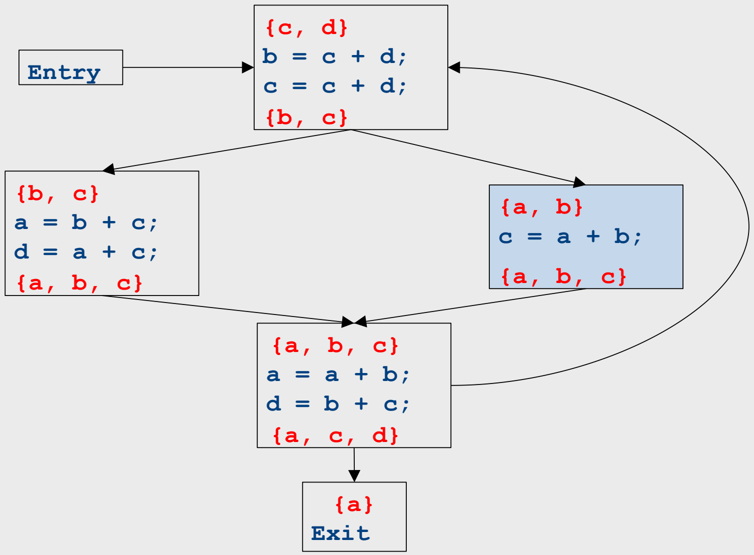

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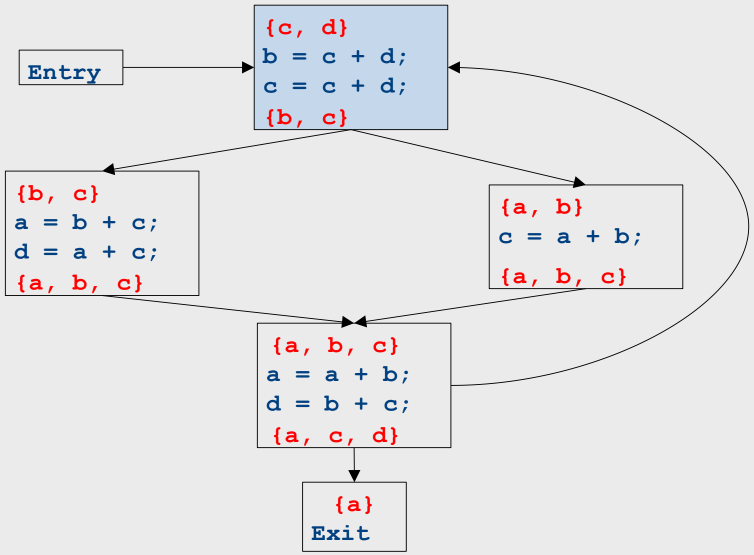

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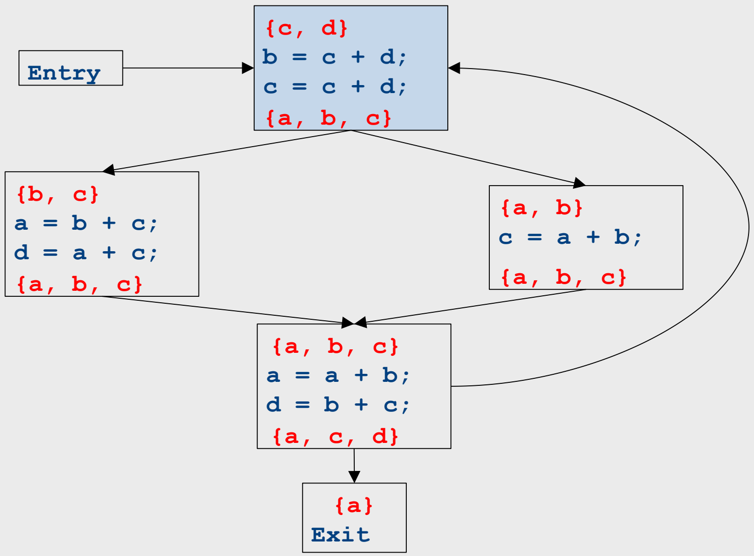

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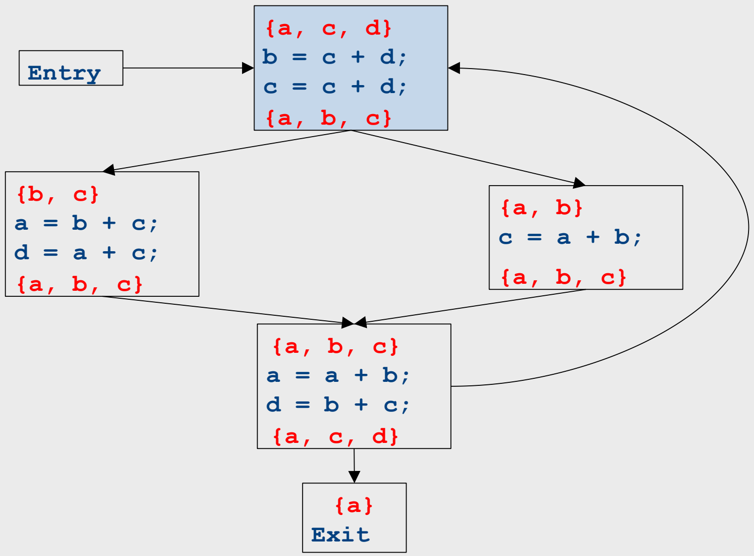

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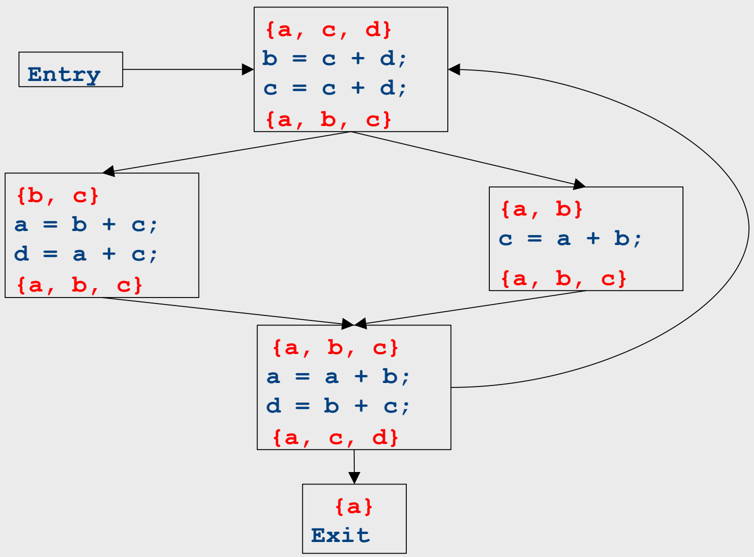
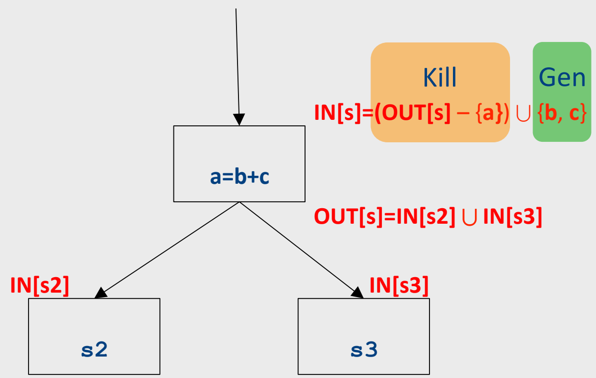


Global liveness analysis: Kill/Gen



Formalizing data-flow analyses

- Define an analysis of a basic block as a quadruple (D, V, ⊔, F, I) where
  - **D** is a direction (forwards or backwards)
  - **V** is a set of values the program can have at any point
  - ⊔ merging (joining) information
  - **F** is a family of transfer functions defining the meaning of any expression as a function $f : \mathbf{V} \rightarrow \mathbf{V}$
  - I is the initial information at the top (or bottom) of a basic block

Liveness Analysis

- **Direction:** Backward
- **Values:** Sets of variables
- **Transfer functions:** Given a set of variable assignments V and statement a = b + c:
- Remove a from V (any previous value of a is now dead.)
- Add b and c to V (any previous value of b or c is now live.)
- Formally: $V_{in} = (V_{out} \setminus \{\mathbf{a}\}) \cup \{\mathbf{b}, \mathbf{c}\}$
- **Initial value:** Depends on semantics of language
  - E.g., function arguments and return values (pushes)
  - Result of local analysis of other blocks as part of a global analysis
- **Merge:** ⊔ = ∪

Available Expressions

- **Direction:** Forward
- **Values:** Sets of expressions assigned to variables
- **Transfer functions:** Given a set of variable assignments V and statement a = b + c:
  - Remove from V any expression containing a as a subexpression
  - Add to V the expression a = b + c
  - Formally: $V_{out} = (V_{in} \setminus \{e \mid e \text{ contains } \mathbf{a}\}) \cup \{a = b + c\}$
- **Initial value:** Empty set of expressions
- **Merge:** ⊔ = ?

Why does this work? (Live Var.)

- To show correctness, we need to show that
  - The algorithm eventually terminates, and
  - When it terminates, it has a sound answer
- Termination argument:
  - Once a variable is discovered to be live during some point of the analysis, it always stays live
  - Only finitely many variables and finitely many places where a variable can become live
- Soundness argument (sketch):
  - Each individual rule, applied to some set, correctly updates liveness in that set
  - When computing the union of the set of live variables, a variable is only live if it was live on some path leaving the statement

Abstract Interpretation

- Theoretical foundations of program analysis
- Cousot and Cousot 1977
- Abstract meaning of programs
  - "Executed" at compile time
    - Execution = Analysis

Another view of program analysis

- We want to reason about some property of the runtime behavior of the program
- Could we run the program and just watch what happens?
- **Idea:** Redefine the semantics of our programming language to give us information about our analysis

Another view of program analysis

- We want to reason about some property of the runtime behavior of the program
- Could we run the program and just watch what happens?

Another view of program analysis

- The only way to find out exactly what a program will actually do is to run it
- Problems:
  - The program might not terminate
  - The program might have some behavior we didn't see when we ran it on a particular input

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  - Basic blocks contain no loops
  - There is only one path through the basic block

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- Inside a basic block, it is simpler
  - Basic blocks contain no loops
  - There is only one path through the basic block
  - But still …

Assigning new semantics (Local)

- Example: Available Expressions
- Redefine the statement **a = b + c to** mean "**a now holds the value of b + c,** and any variable holding the value **a is** now invalid"
- "Run" the program assuming these new semantics

Assigning new semantics (Global)

- Example: Available Expressions
- Redefine the statement **a = b + c to** mean "**a now holds the value of b + c,** and any variable holding the value **a is** now invalid"
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- Merge information from different paths

Abstract Interpretation

- Example: Available Expressions
- Redefine the statement **a = b + c to** mean "**a now holds the value of b + c,** and any variable holding the value **a is** now invalid"
- "Run" the program assuming these new semantics
- Merge information from different paths
- Treat the optimizer as an interpreter for these new semantics

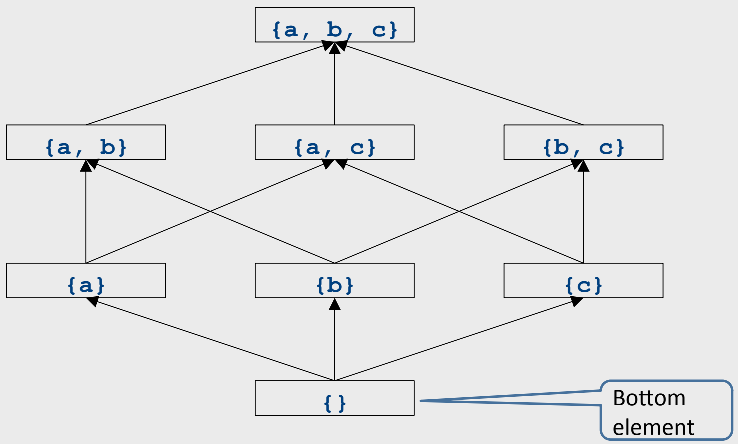
Theory of Program Analysis

- Building up all of the machinery to design this analysis was tricky
- The key ideas, however, are mostly independent of the analysis:
  - We need to be able to compute functions describing the behavior of each statement
  - We need to be able to merge several subcomputations together
  - We need an initial value for all of the basic blocks
- There is a beautiful formalism that captures many of these properties

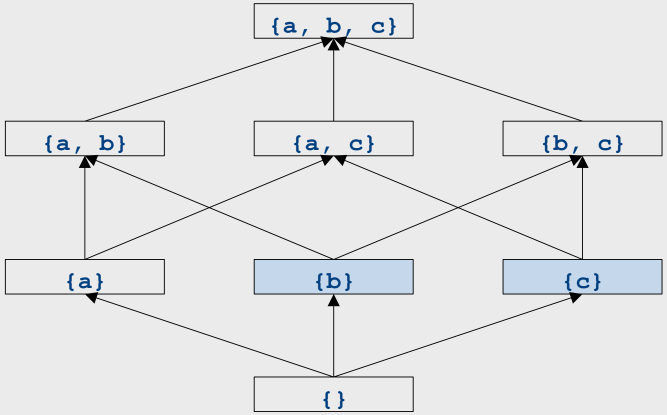
Join semilattices

- A join semilattice is a ordering defined on a set of elements
  - $0 \leq 1 \leq 2 \leq \ldots$
  - $\{\} \leq \{0\} \leq \{0,1\}$, $\{1,2\} \leq \{0,1,2\} \nleq \{1,2,3,4\}$
- Any two elements have some *join* that is the smallest element larger than both elements
- There is a unique *bottom* element, which is smaller than all other elements
  - The join of two elements represents combining (merging) information from two elements by an overapproximation
- The bottom element represents "no information yet" or "the least conservative possible answer"

Join semilattice for liveness



What is the join of {b} and {c}?



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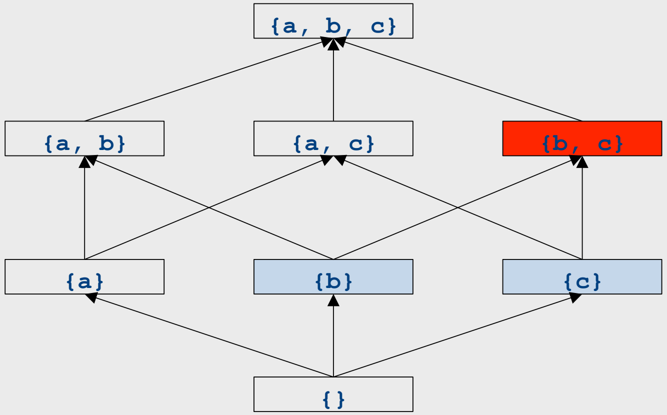
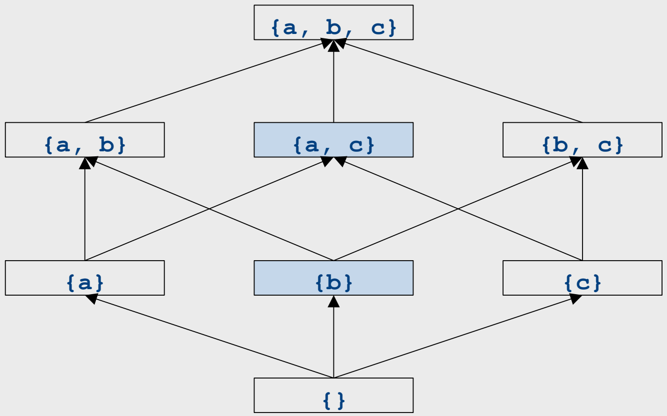


What is the join of {b} and {a,c}?



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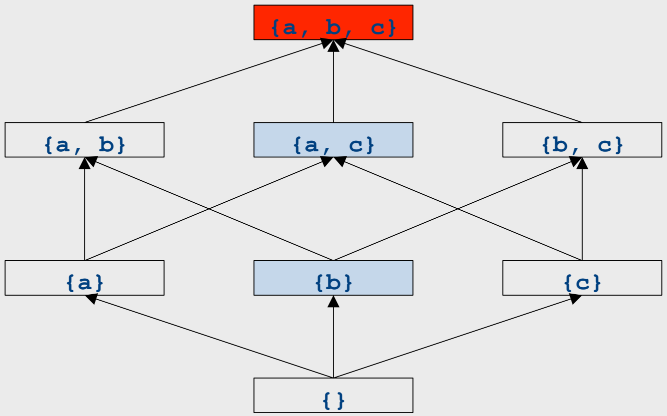

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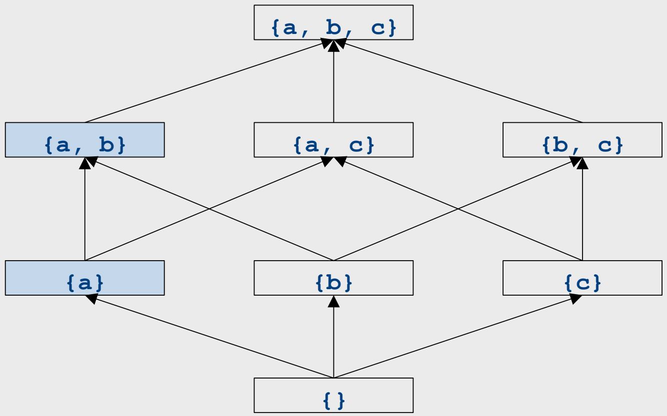

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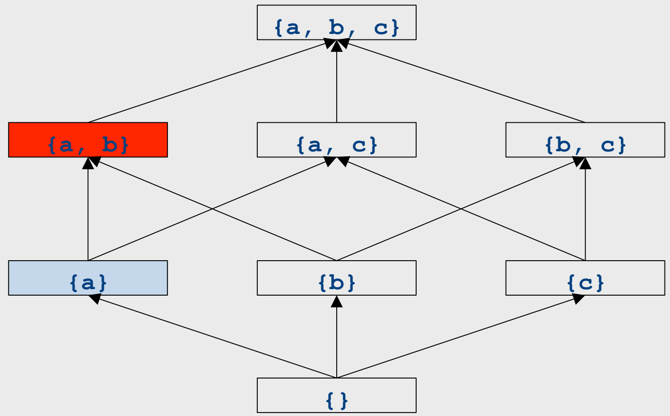
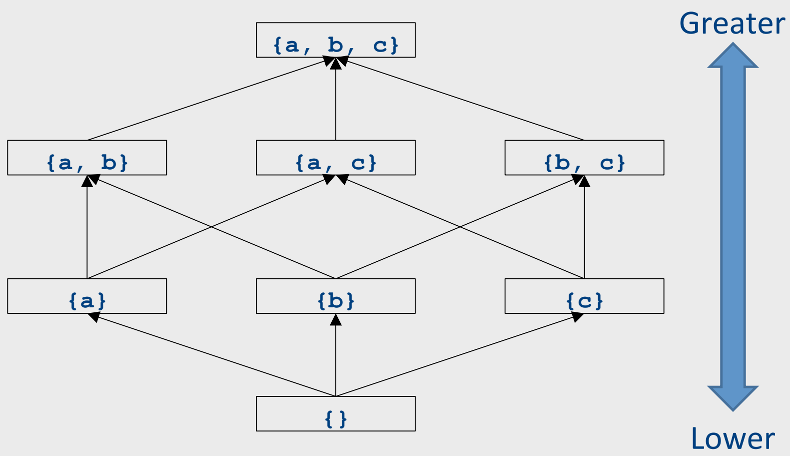


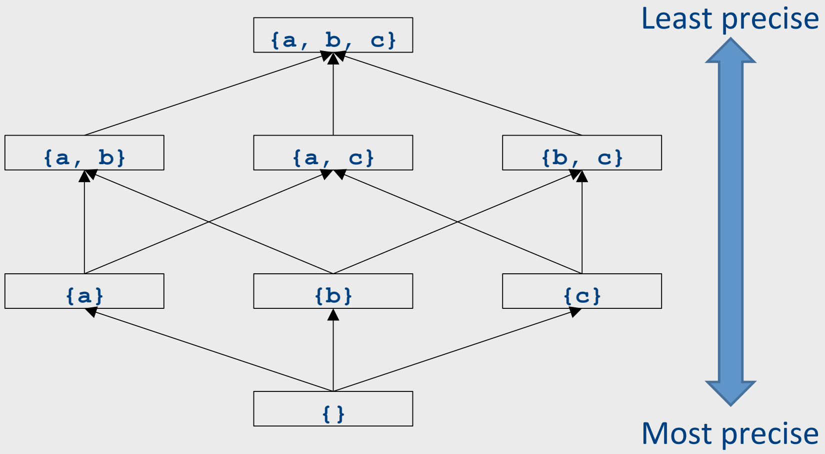
Formal definitions

- A join semilattice is a pair (V, $\sqcup$), where
- V is a domain of elements
- $\sqcup$ is a join operator that is
  - commutative: $x \sqcup y = y \sqcup x$
  - associative: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
  - idempotent: $x \sqcup x = x$
- If $x \sqcup y = z$, we say that z is the join or (**l**east **u**pper **b**ound) of x and y
- Every join semilattice has a bottom element denoted $\perp$ such that $\perp \sqcup x = x$ for all x

Join semilattices and ordering



Join semilattices and ordering



Join semilattices and orderings

- Every join semilattice $(V, \sqcup)$ induces an ordering relationship $\sqsubseteq$ over its elements
- Define $x \sqsubseteq y$ iff $x \sqcup y = y$
- Need to prove
  - Reflexivity: $x \sqsubseteq x$
  - Antisymmetry: If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$
  - Transitivity: If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$

An example join semilattice

- The set of natural numbers and the **max** function
- Idempotent
  - **max**{a, a} = a
- Commutative
  - **max**{a, b} = **max**{b, a}
- Associative
  - **max**{a, **max**{b, c}} = **max**{**max**{a, b}, c}
- Bottom element is 0:
  - **max**{0, a} = a
- What is the ordering over these elements?

A join semilattice for liveness

- Sets of live variables and the set union operation
- Idempotent:
  - $x \cup x = x$
- Commutative:
  - $x \cup y = y \cup x$
- Associative:
  - $(x \cup y) \cup z = x \cup (y \cup z)$
- Bottom element:
  - The empty set: $\emptyset \cup x = x$
- What is the ordering over these elements?

Semilattices and program analysis

- Semilattices naturally solve many of the problems we encounter in global analysis
- How do we combine information from multiple basic blocks?
- What value do we give to basic blocks we haven't seen yet?
- How do we know that the algorithm always terminates?

Semilattices and program analysis

- Semilattices naturally solve many of the problems we encounter in global analysis
- How do we combine information from multiple basic blocks?
  - Take the join of all information from those blocks
- What value do we give to basic blocks we haven't seen yet?
  - Use the bottom element
- How do we know that the algorithm always terminates?
  - Actually, we still don't! More on that later

A general framework

- A global analysis is a tuple (**D, V, ⊔, F, I**), where
  - **D** is a direction (forward or backward)
    - The order to visit statements within a basic block, not the order in which to visit the basic blocks
  - **V** is a set of values
  - ⊔ is a join operator over those values
  - **F** is a set of transfer functions $f : \mathbf{V} \rightarrow \mathbf{V}$
  - **I** is an initial value
- The only difference from local analysis is the introduction of the join operator

Running global analyses

- Assume that (**D, V, ⊔, F, I**) is a forward analysis
- Set OUT[**s**] = ⊥ for all statements **s**
- Set OUT[**entry**] = **I**
- Repeat until no values change:
  - For each statement **s** with predecessors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$:
    - Set IN[**s**] = OUT[$\mathbf{p}_1$] ⊔ OUT[$\mathbf{p}_2$] ⊔ … ⊔ OUT[$\mathbf{p}_n$]
    - Set OUT[**s**] = f_s (IN[**s**])
- The order of this iteration does not matter
  - This is sometimes called chaotic iteration

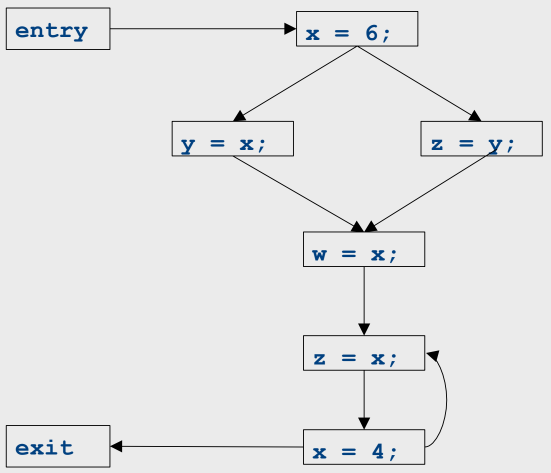
The dataflow framework

- This form of analysis is called the dataflow framework
- Can be used to easily prove an analysis is sound
- With certain restrictions, can be used to prove that an analysis eventually terminates
  - Again, more on that later

Global constant propagation

- Constant propagation is an optimization that replaces each variable that is known to be a constant value with that constant
- An elegant example of the dataflow framework

Global constant propagation



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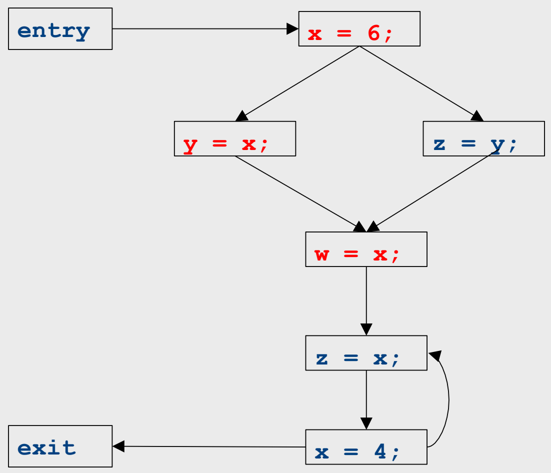

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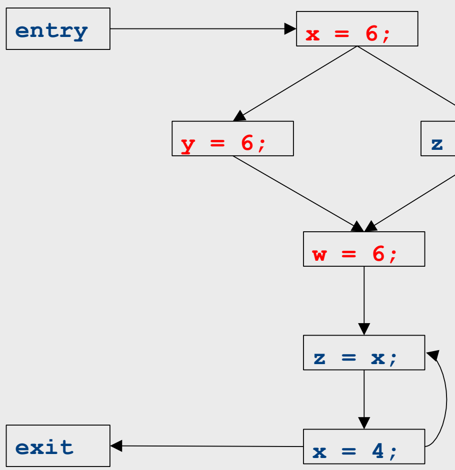


Constant propagation analysis

- In order to do a constant propagation, we need to track what values might be assigned to a variable at each program point
- Every variable will either
  - Never have a value assigned to it,
  - Have a single constant value assigned to it,
  - Have two or more constant values assigned to it, or
  - Have a known non-constant value.
  - Our analysis will propagate this information throughout a CFG to identify locations where a value is constant

Properties of constant propagation

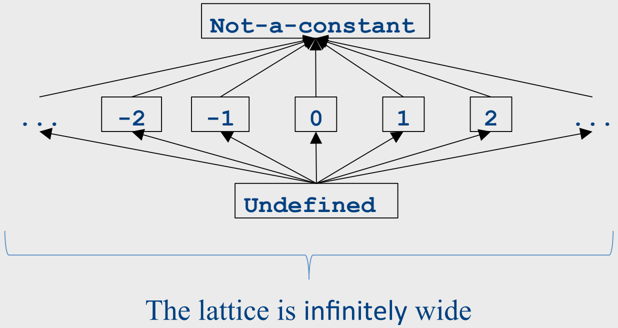
- For now, consider just some single variable **x**
- At each point in the program, we know one of three things about the value of **x**:
  - **x** is definitely not a constant, since it's been assigned two values or assigned a value that we know isn't a constant
  - **x** is definitely a constant and has value **k**
  - We have never seen a value for **x**
- Note that the first and last of these are **not** the same!
  - The first one means that there may be a way for **x** to have multiple values
  - The last one means that **x** never had a value at all

Defining a join operator

- The join of any two different constants is **Not-a-Constant**
  - (If the variable might have two different values on entry to a statement, it cannot be a constant)
- The join of **Not a Constant** and any other value is **Not-a-Constant**
  - (If on some path the value is known not to be a constant, then on entry to a statement its value can't possibly be a constant)
- The join of **Undefined** and any other value is that other value
  - (If **x** has no value on some path and does have a value on some other path, we can just pretend it always had the assigned value)

A semilattice for constant propagation

- One possible semilattice for this analysis is shown here (for each variable):



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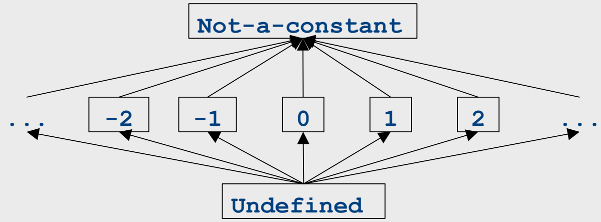
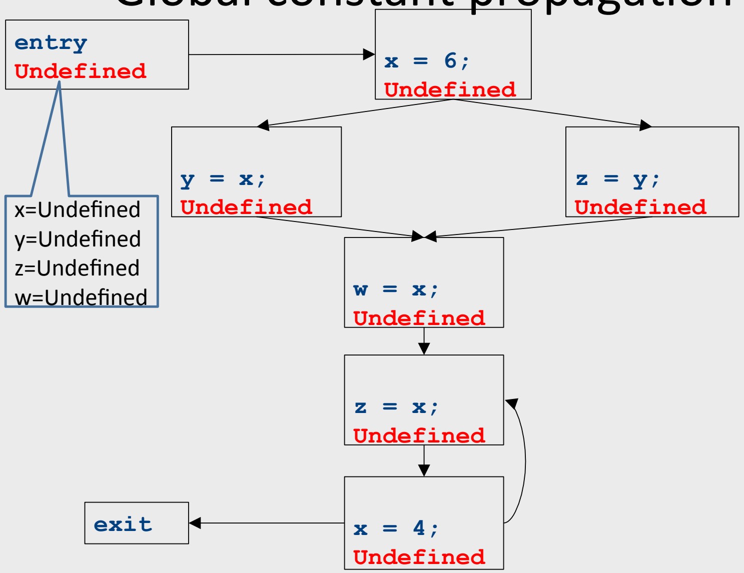
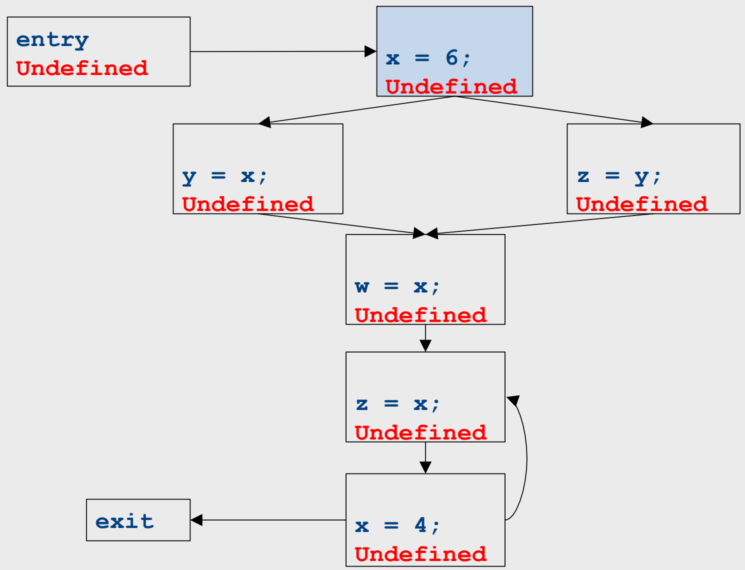


- Note:
  - The join of any two different constants is **Not-a-Constant**
  - The join of **Not a Constant** and any other value is **Not-a-Constant**
  - The join of **Undefined** and any other value is that other value

Global constant propagation



Global constant propagation



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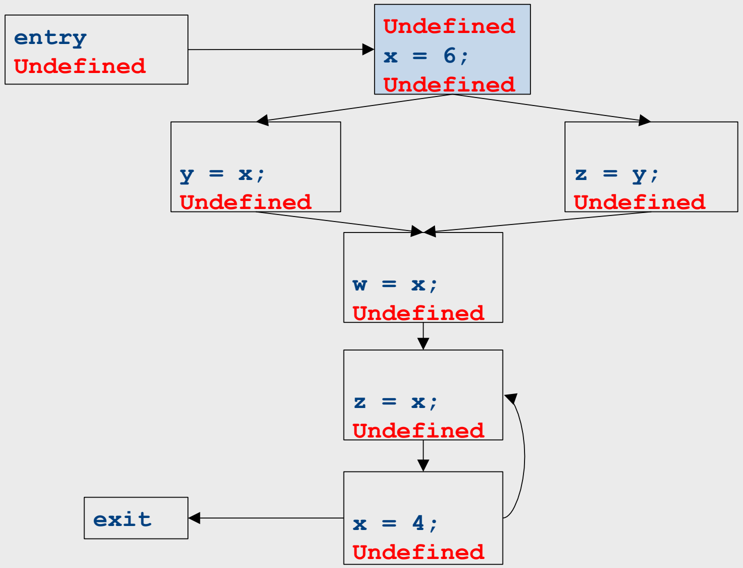
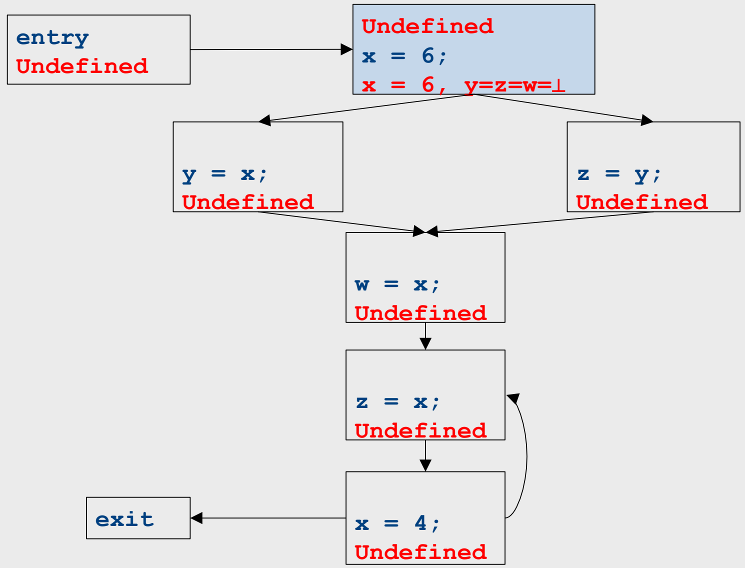
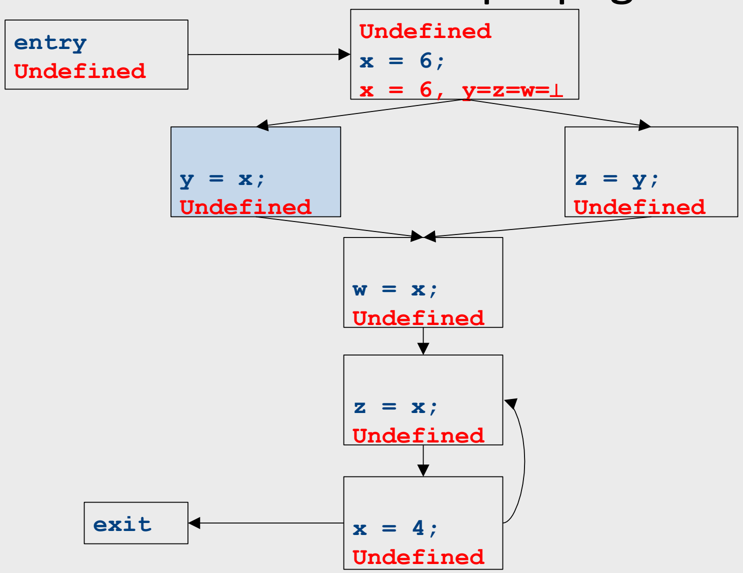


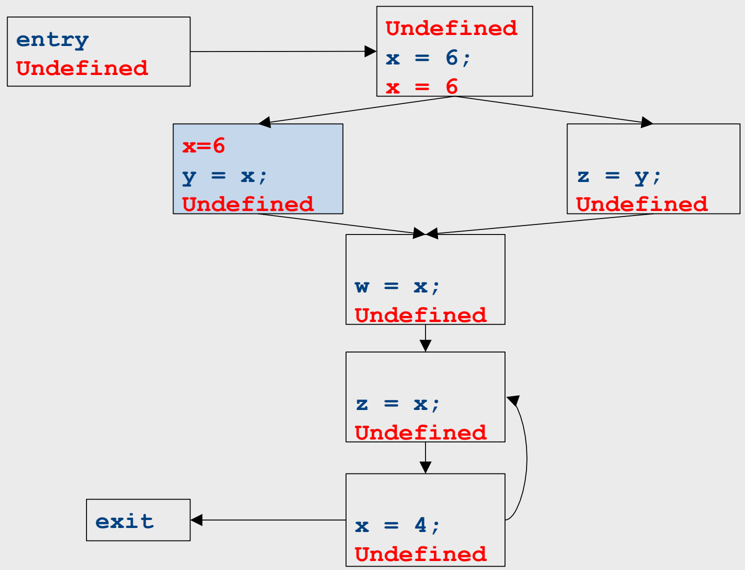
Global constant propagation



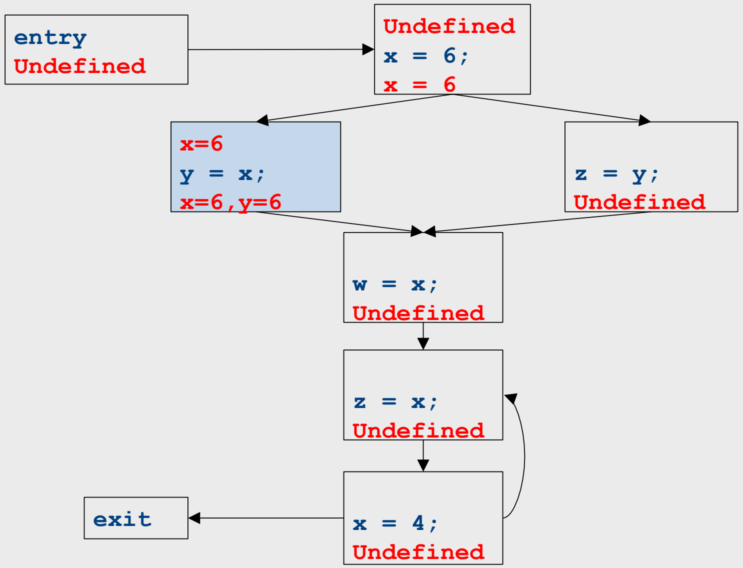
Global constant propagation



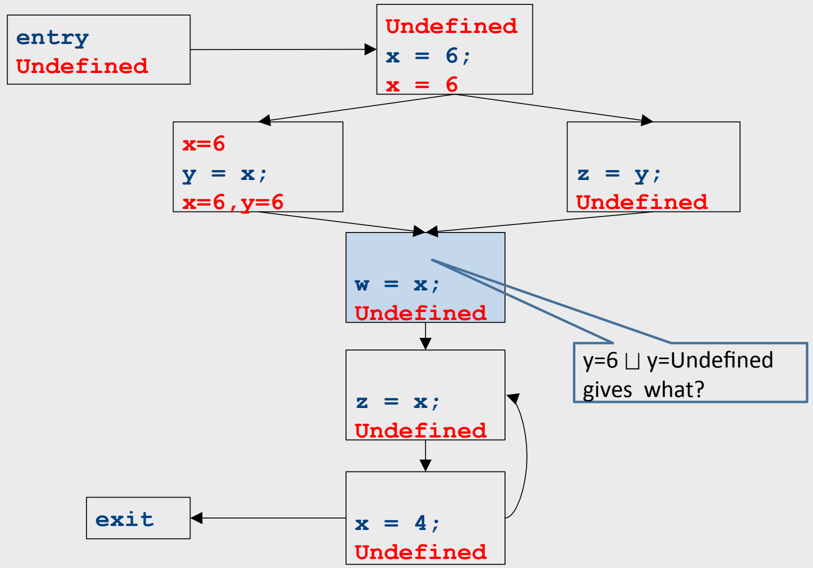
Global constant propagation



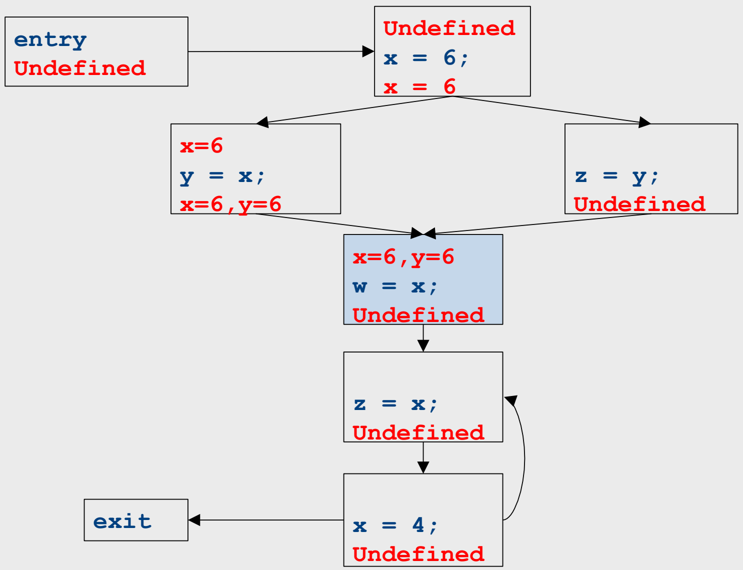
Global constant propagation



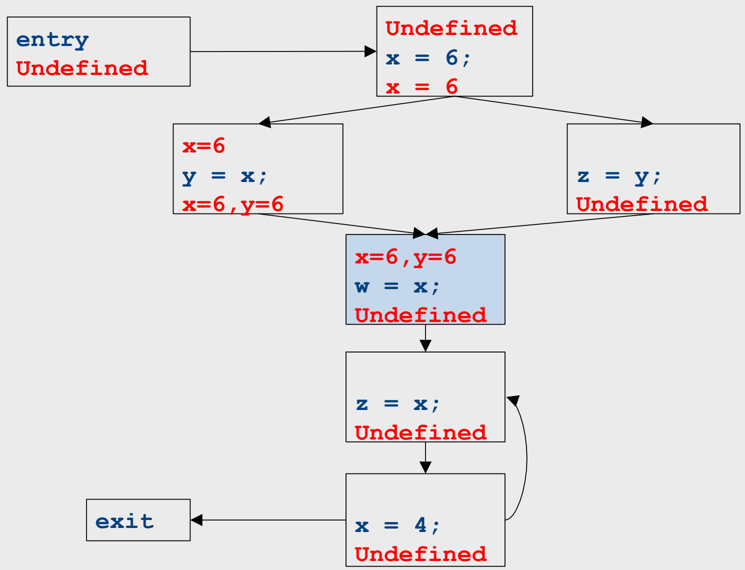
Global constant propagation



Global constant propagation



Global constant propagation



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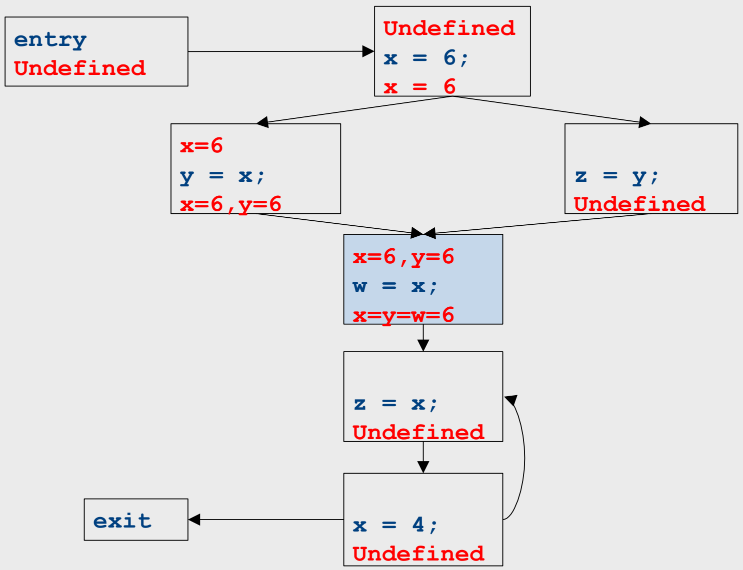
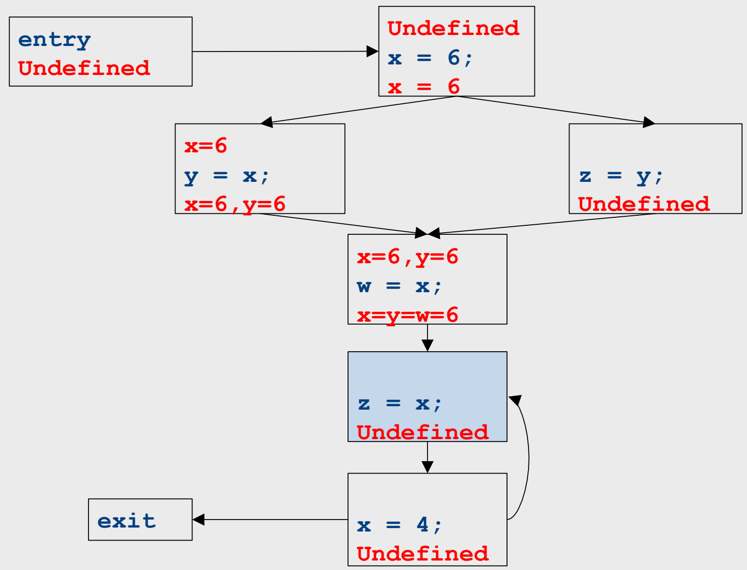


Global constant propagation



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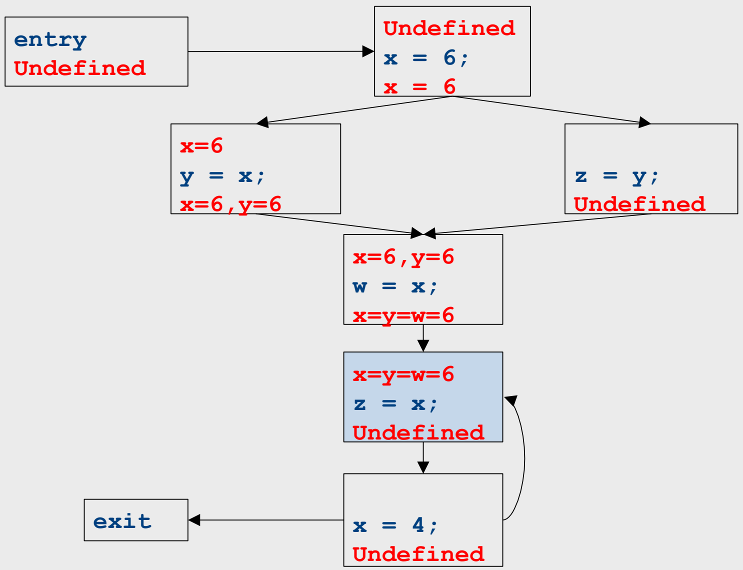

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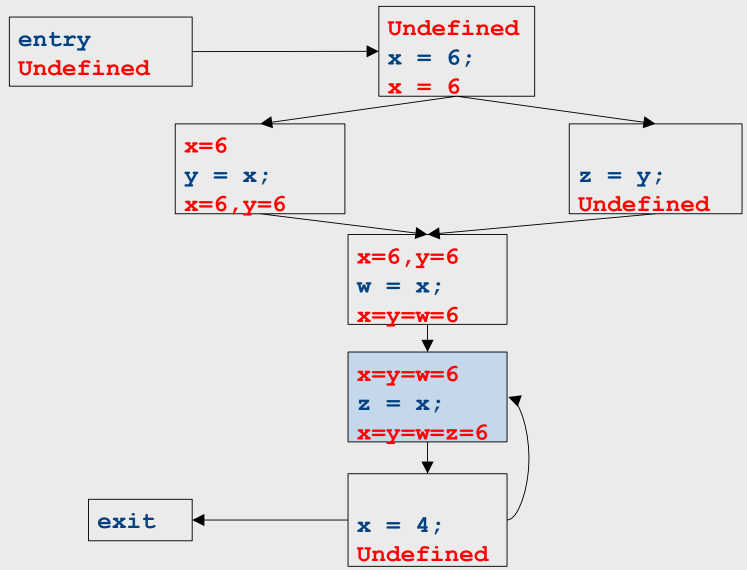

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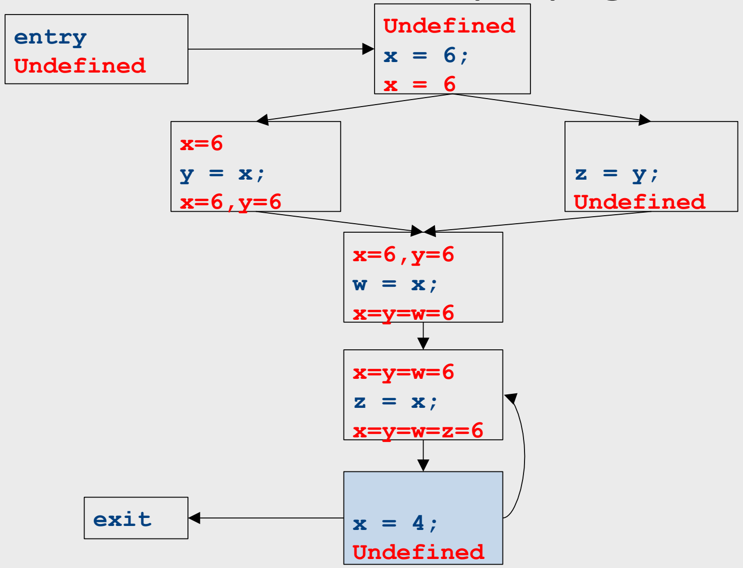

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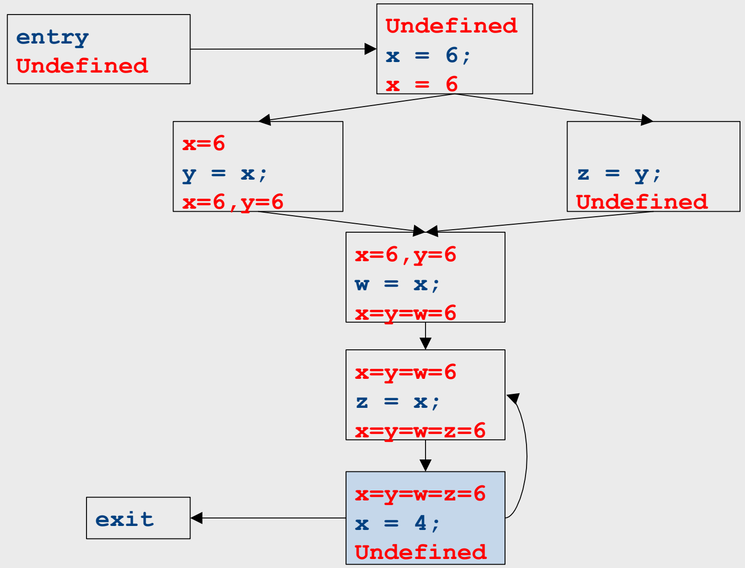

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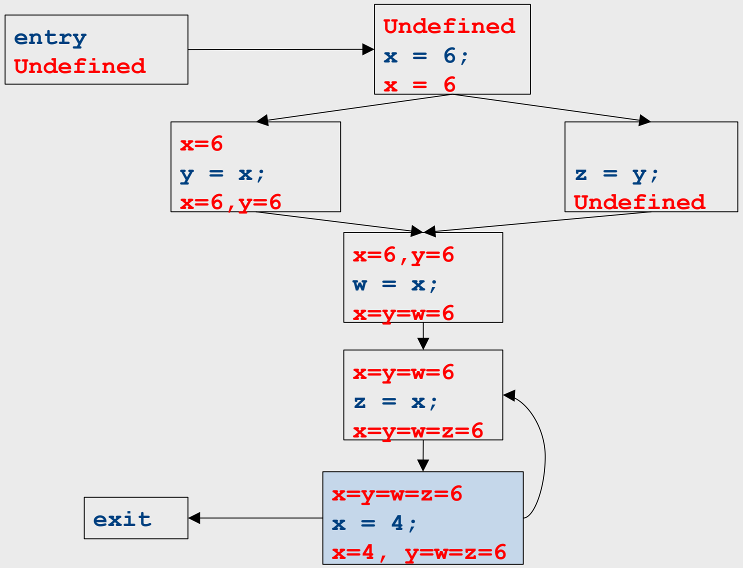

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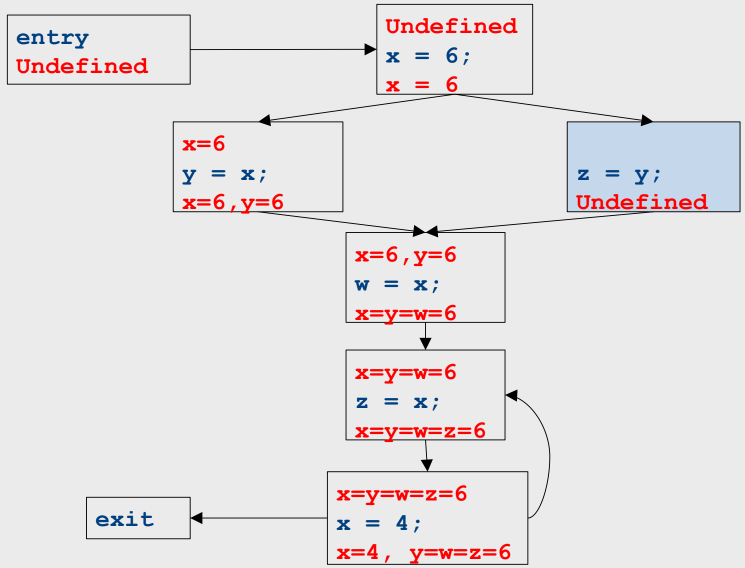

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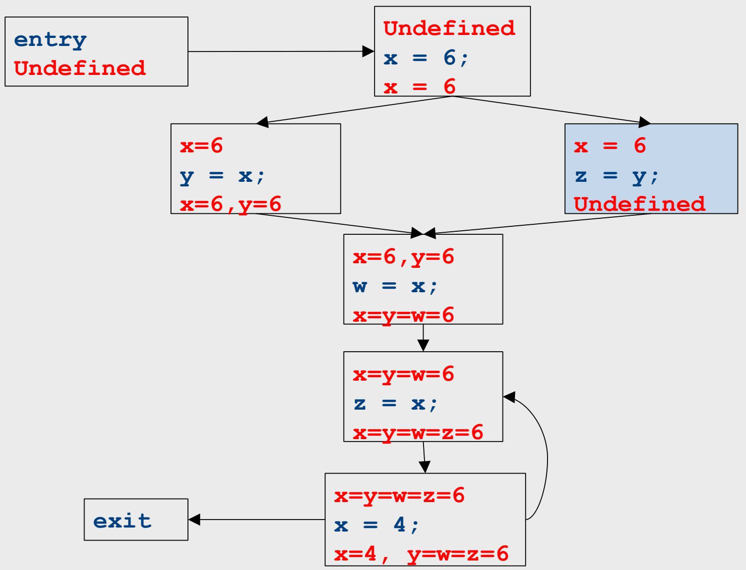
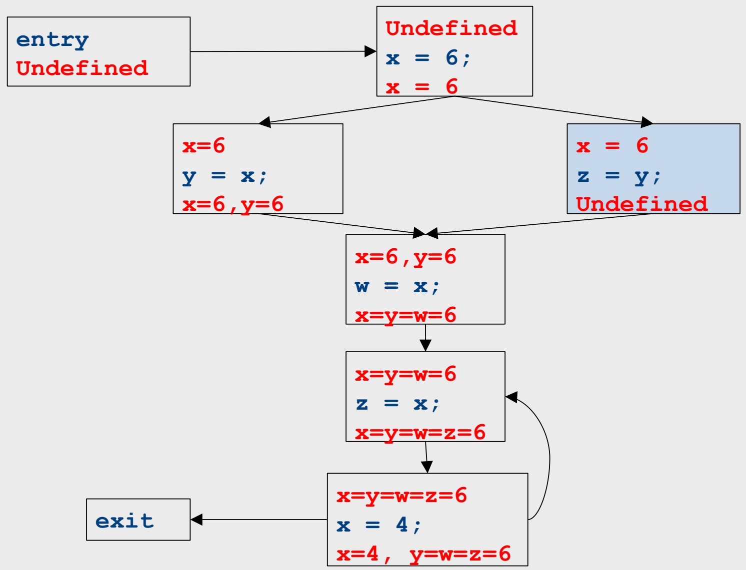


Global constant propagation



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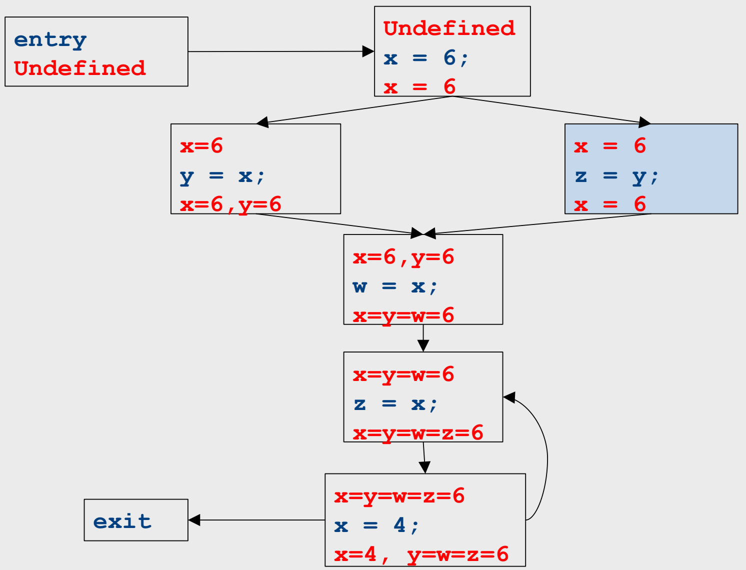
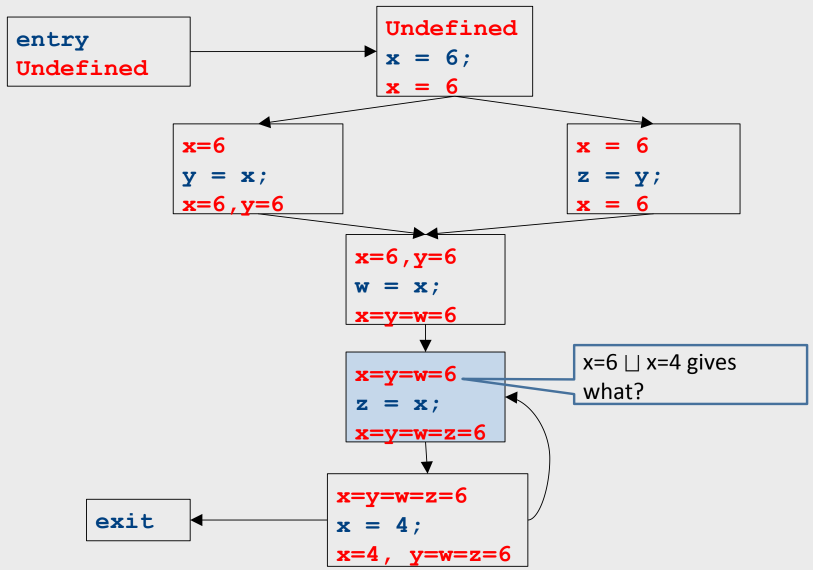
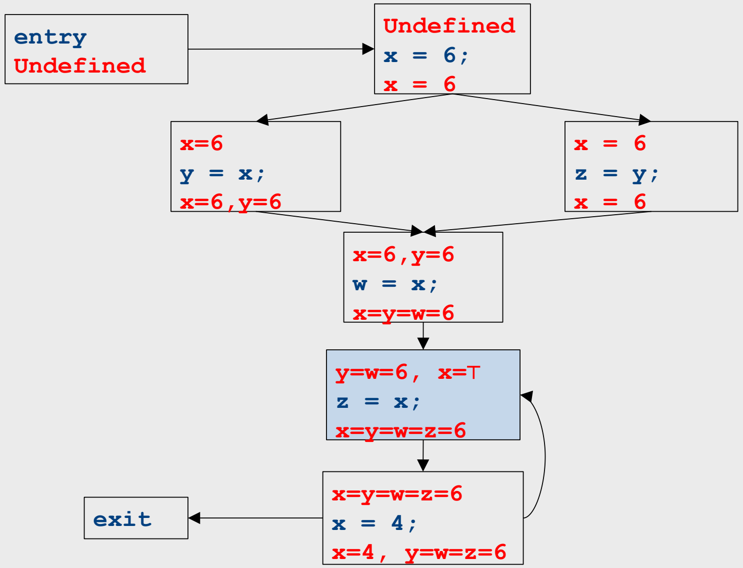


Global constant propagation



Global constant propagation



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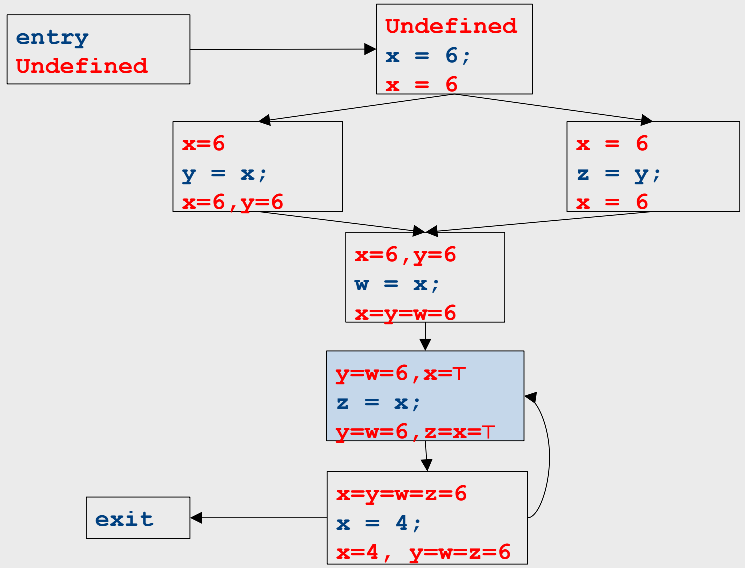
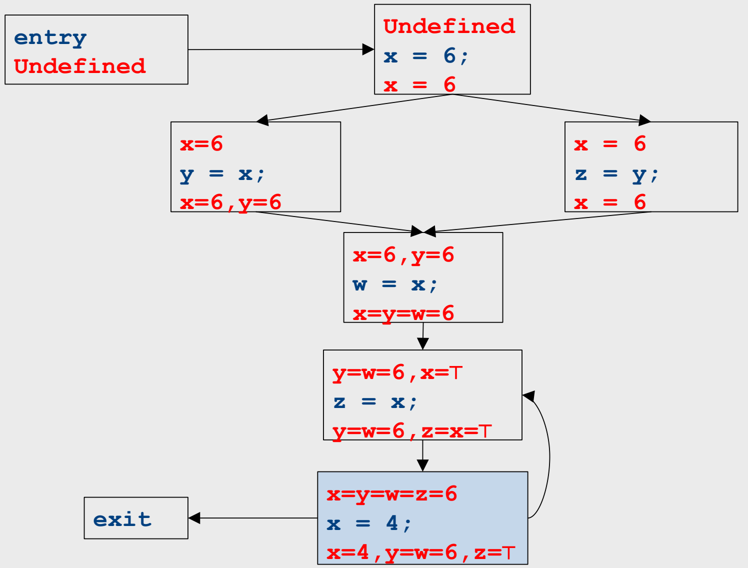
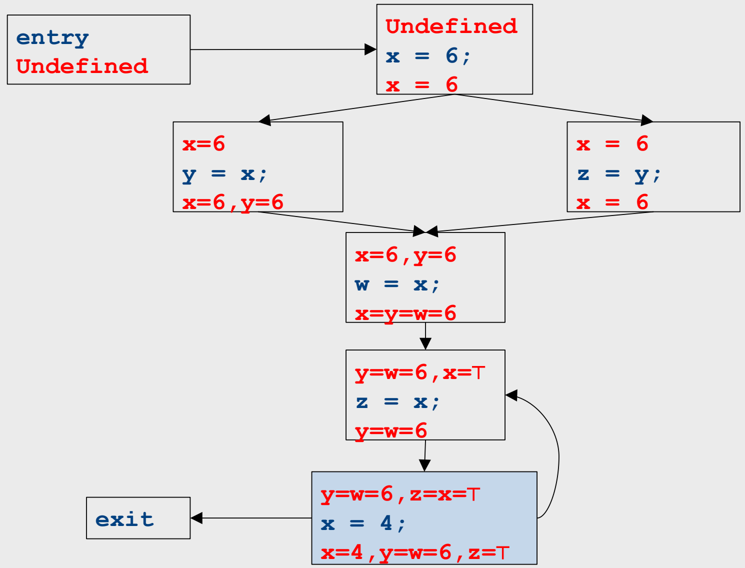


Global constant propagation



Global constant propagation



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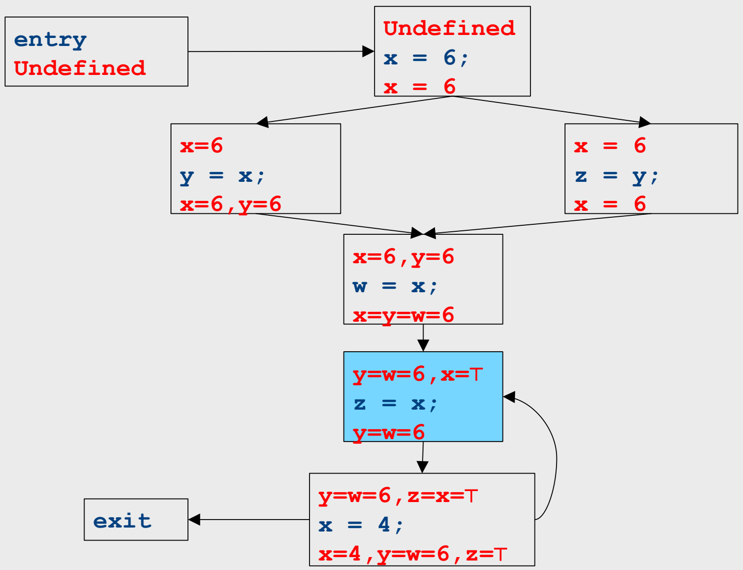
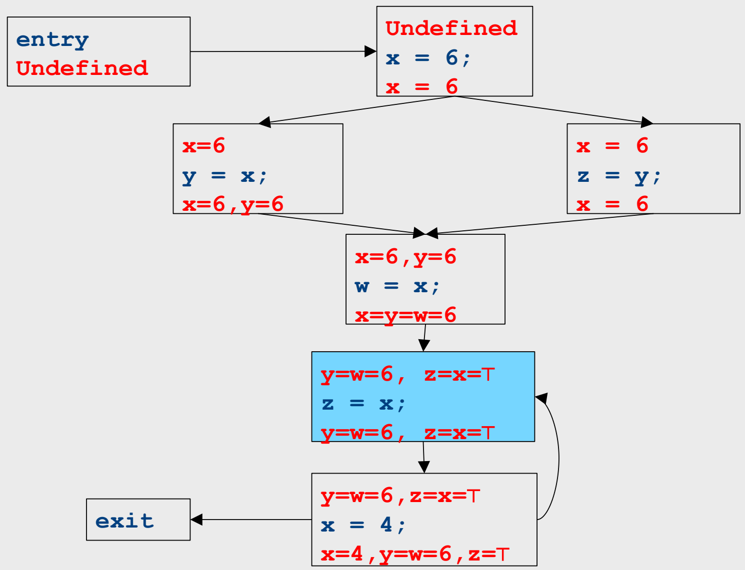


Global constant propagation



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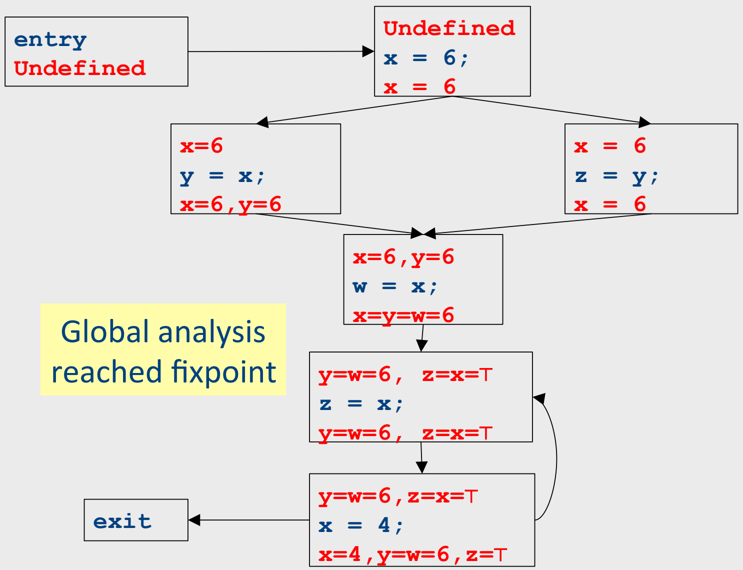

Dataflow for constant propagation

- Direction: **Forward**
- Semilattice: Vars⟶ {Undefined, 0, 1, -1, 2, -2, …, Not-a-Constant}
  - Join mapping for variables point-wise
    {x↦1,y↦1,z↦1} ⊔ {x↦1,y↦2,z↦Not-a-Constant} =
    {x↦1,y↦Not-a-Constant,z↦Not-a-Constant}
- Transfer functions:
  - $f_{\mathbf{x=k}}(V) = V|_{x\mapsto k}$ *(update V by mapping x to k)*
  - $f_{\mathbf{x=a+b}}(V) = V|_{x\mapsto ?}$
- Initial value: **x is Undefined**
  - (When might we use some other value?)

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  - (When might we use some other value?)

Proving termination

- Our algorithm for running these analyses continuously loops until no changes are detected
- Given this, how do we know the analyses will eventually terminate?
  - In general, **we don't**

Terminates?

Liveness Analysis

- A variable is live at a point in a program if later in the program its value will be read before it is written to again

Join semilattice definition

- A join semilattice is a pair $(V, \sqcup)$, where
- V is a domain of elements
- $\sqcup$ is a join operator that is
  - commutative: $x \sqcup y = y \sqcup x$
  - associative: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
  - idempotent: $x \sqcup x = x$
- If $x \sqcup y = z$, we say that z is the join or (**L**east **U**pper **B**ound) of x and y
- Every join semilattice has a bottom element denoted $\bot$ such that $\bot \sqcup x = x$ for all x

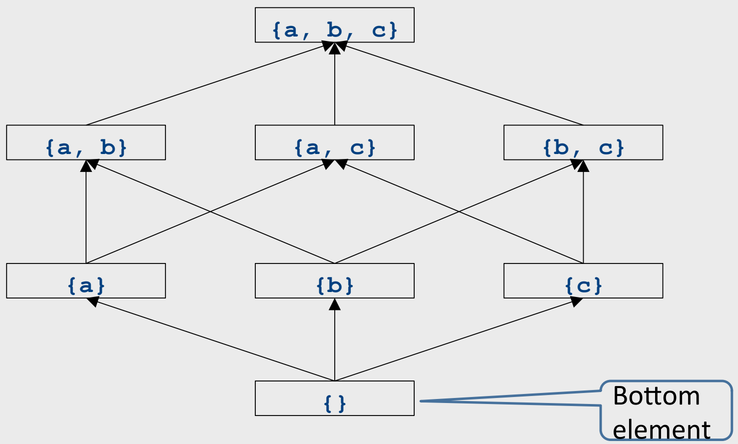
Partial ordering induced by join

- Every join semilattice $(V, \sqcup)$ induces an ordering relationship $\sqsubseteq$ over its elements
- Define $x \sqsubseteq y$ iff $x \sqcup y = y$
- Need to prove
  - Reflexivity: $x \sqsubseteq x$
  - Antisymmetry: If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$
  - Transitivity: If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$

A join semilattice for liveness

- Sets of live variables and the set union operation
- Idempotent:
  - $x \cup x = x$
- Commutative:
  - $x \cup y = y \cup x$
- Associative:
  - $(x \cup y) \cup z = x \cup (y \cup z)$
- Bottom element:
  - The empty set: $\emptyset \cup x = x$
- Ordering over elements = subset relation

Join semilattice example for liveness



Dataflow framework

- A global analysis is a tuple (**D, V, ⊔, F, I**), where
  - **D** is a direction (forward or backward)
    - The order to visit statements within a basic block, **NOT** the order in which to visit the basic blocks
  - **V** is a set of values (sometimes called domain)
  - ⊔ is a join operator over those values
  - **F** is a set of transfer functions $f_s : \mathbf{V} \to \mathbf{V}$ (for every statement s)
  - **I** is an initial value

Running global analyses

- Assume that (**D, V, ⊔, F, I**) is a forward analysis
- For every statement s maintain values before - IN[s] - and after - OUT[s]
- Set OUT[**s**] = ⊥ for all statements **s**
- Set OUT[**entry**] = **I**
- Repeat until no values change:
  - For each statement **s** with predecessors PRED[s]={$\mathbf{p_1}$, $\mathbf{p_2}$, … , $\mathbf{p_n}$}
    - Set IN[**s**] = OUT[$\mathbf{p_1}$] ⊔ OUT[$\mathbf{p_2}$] ⊔ … ⊔ OUT[$\mathbf{p_n}$]
    - Set OUT[**s**] = f_s(IN[**s**])
- The order of this iteration does not matter
  - Chaotic iteration

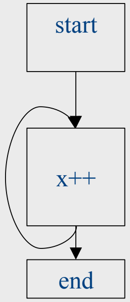
Proving termination

- Our algorithm for running these analyses continuously loops until no changes are detected
- Problem: how do we know the analyses will eventually terminate?

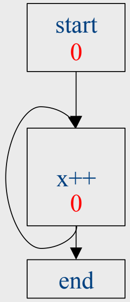
A non-terminating analysis

- The following analysis will loop infinitely on any CFG containing a loop:
- Direction: Forward
- Domain: ℕ
- Join operator: **max**
- Transfer function: $f(n) = n + 1$
- Initial value: 0

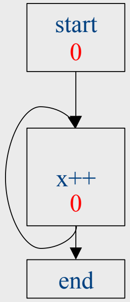
A non-terminating analysis



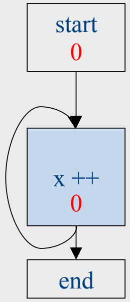
Initialization



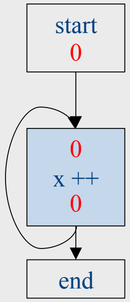
Fixed-point iteration



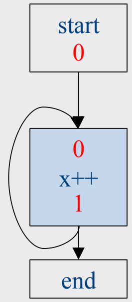
Choose a block



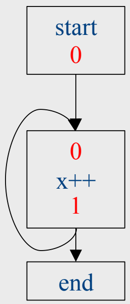
Iteration 1



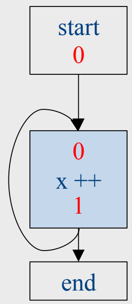
Iteration 1



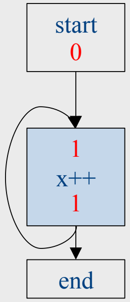
Choose a block



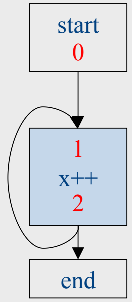
Iteration 2



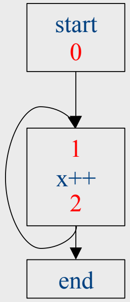
Iteration 2



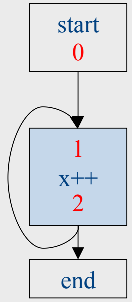
Iteration 2



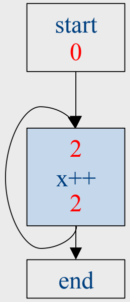
Choose a block



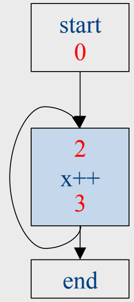
Iteration 3



Iteration 3

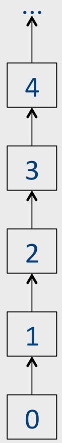


Iteration 3



Why doesn't this terminate?

- Values can increase without bound
- Note that "increase" refers to the lattice ordering, not the ordering on the natural numbers
- The height of a semilattice is the length of the longest increasing sequence in that semilattice
- The dataflow framework is not guaranteed to terminate for semilattices of infinite height
- Note that a semilattice can be infinitely large but have finite height
  - e.g. constant propagation



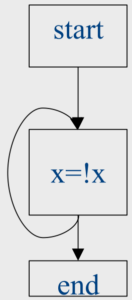
Height of a lattice

- An increasing chain is a sequence of elements $\bot \sqsubset a_1 \sqsubset a_2 \sqsubset \ldots \sqsubset a_k$
  - The length of such a chain is k
- The height of a lattice is the length of the maximal increasing chain
- For liveness with *n* program variables:
  - $\{\} \subset \{v_1\} \subset \{v_1,v_2\} \subset \ldots \subset \{v_1,\ldots,v_n\}$
- For available expressions it is the number of expressions of the form a=b op c
  - For *n* program variables and *m* operator types: $m \cdot n^3$

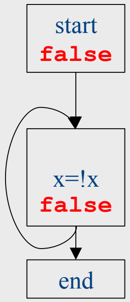
Another non-terminating analysis

- This analysis works on a finite-height semilattice, but will not terminate on certain CFGs:
- Direction: Forward
- Domain: Boolean values **`true`** and **`false`**
- Join operator: Logical OR
- Transfer function: Logical NOT
- Initial value: **`false`**

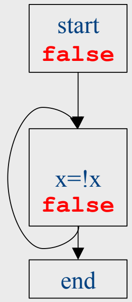
A non-terminating analysis



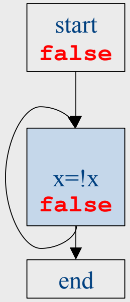
Initialization



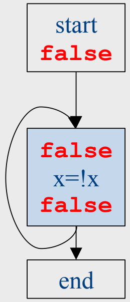
Fixed-point iteration



Choose a block



Iteration 1



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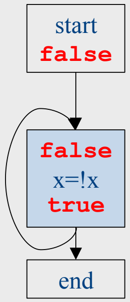
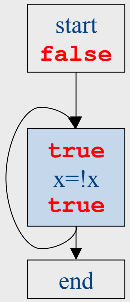
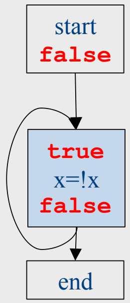


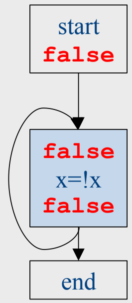
Iteration 2



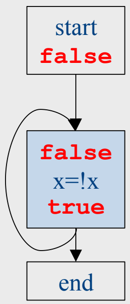
Iteration 2



Iteration 3

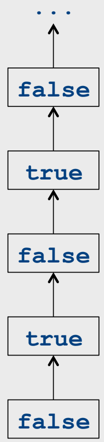


Iteration 3



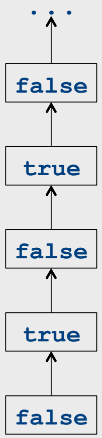
Why doesn't it terminate?

- Values can loop indefinitely
- Intuitively, the join operator keeps pulling values up
- If the transfer function can keep pushing values back down again, then the values might cycle forever



Why doesn't it terminate?

- Values can loop indefinitely
- Intuitively, the join operator keeps pulling values up
- If the transfer function can keep pushing values back down again, then the values might cycle forever
- How can we fix this?



Monotone transfer functions

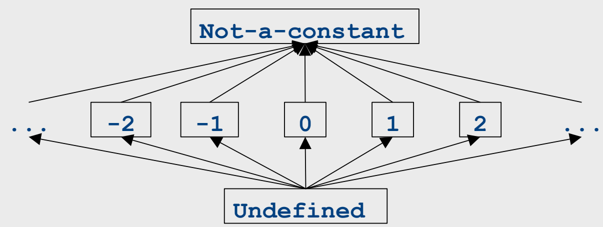
- A transfer function f is monotone iff
  if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$
- Intuitively, if you know less information about a program point, you can't "gain back" more information about that program point
- Many transfer functions are monotone, including those for liveness and constant propagation
- Note: Monotonicity does **not** mean that $x \sqsubseteq f(x)$
  - (This is a different property called extensivity)

Liveness and monotonicity

- A transfer function f is monotone iff
  if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$
- Recall our transfer function for **a = b + c** is
  - $f_{a = b + c}(V) = (V - \{a\}) \cup \{b, c\}$
- Recall that our join operator is set union and induces an ordering relationship
  $X \sqsubseteq Y$ iff $X \subseteq Y$
- Is this monotone?

Is constant propagation monotone?

- A transfer function f is monotone iff
  if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$
- Recall our transfer functions
  - $f_{\mathbf{x=k}}(V) = V|_{x \mapsto k}$ *(update V by mapping x to k)*
  - $f_{\mathbf{x=a+b}}(V) = V|_{x \mapsto \text{Not-a-Constant}}$ *(assign Not-a-Constant)*
- Is this monotone?



The grand result

- Theorem: A dataflow analysis with a **finite-height semilattice** and family of **monotone transfer functions** always terminates
- Proof sketch:
  - The join operator can only bring values up
  - Transfer functions can never lower values back down below where they were in the past (monotonicity)
  - Values cannot increase indefinitely (finite height)

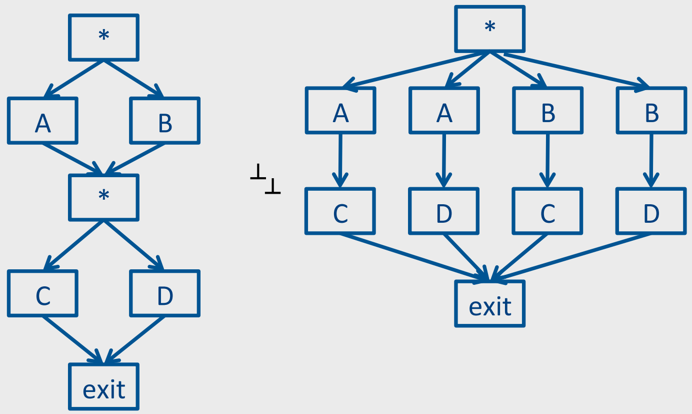
An "optimality" result

- A transfer function *f* is distributive if
  $f(a \sqcup b) = f(a) \sqcup f(b)$
  for every domain elements *a* and *b*
- If all transfer functions are distributive then the fixed-point solution is the solution that would be computed by joining results from all (potentially infinite) control-flow paths
  - Join over all paths
- Optimal if we ignore program conditions

An "optimality" result

If(*)
A
else
B;

If(*)
C
else
D;



$$f_C(f_A(\bot) \sqcup f_B(\bot)) \sqcup f_D(f_A(\bot) \sqcup f_B(\bot)) \overset{?}{=} f_C(f_A(\bot)) \sqcup f_D(f_A(\bot)) \sqcup f_C(f_B(\bot)) \sqcup f_D(f_B(\bot))$$

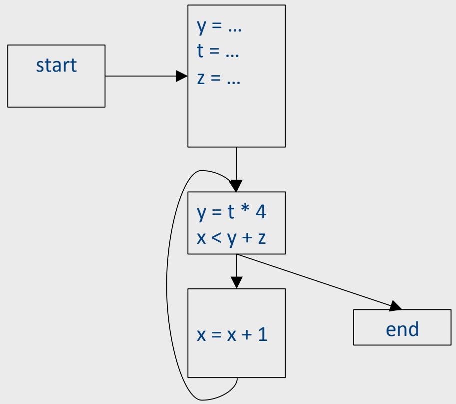
An "optimality" result

- A transfer function *f* is distributive if
  $f(a \sqcup b) = f(a) \sqcup f(b)$
  for every domain elements *a* and *b*
- If all transfer functions are distributive then the fixed-point solution is equal to the solution computed by joining results from all (potentially infinite) control-flow paths
  - Join over all paths
- Optimal if we pretend all control-flow paths can be executed by the program
- Which analyses use distributive functions?

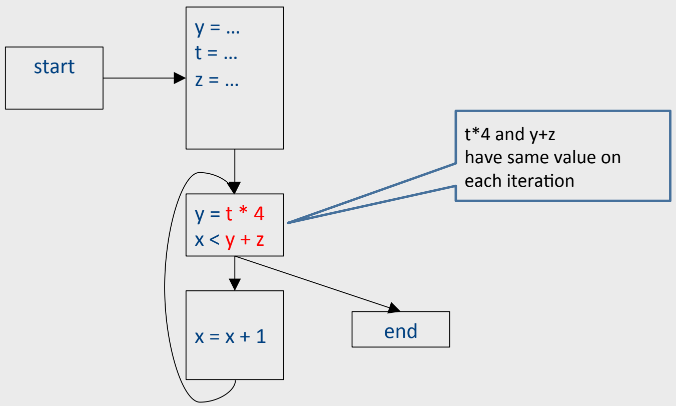
Loop optimizations

- Most of a program's computations are done inside loops
  - Focus optimizations effort on loops
- The optimizations we've seen so far are independent of the control structure
- Some optimizations are specialized to loops
  - Loop-invariant code motion
  - (Strength reduction via induction variables)
- Require another type of analysis to find out where expressions get their values from
  - Reaching definitions
    - (Also useful for improving register allocation)

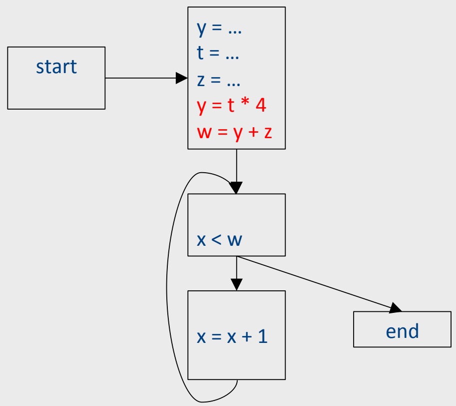
Loop invariant computation



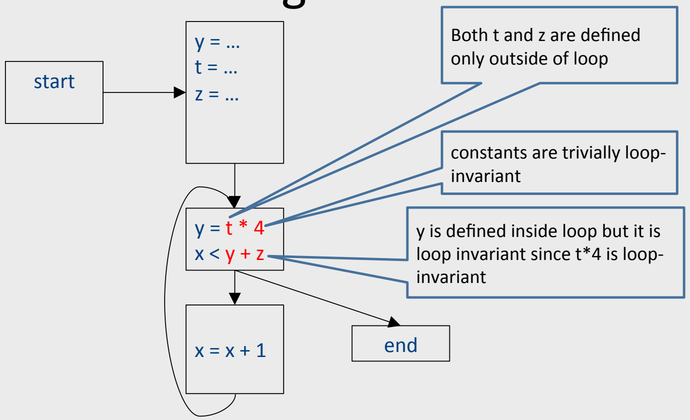
Loop invariant computation



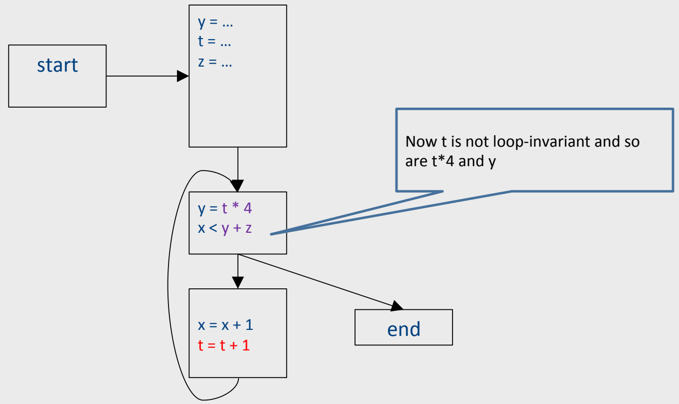
Code hoisting



What reasoning did we use?



What about now?



Loop-invariant code motion

- *d*: t = a_1 op a_2
  - *d* is a program location
- a_1 op a_2 loop-invariant (for a loop *L*) if computes the same value in each iteration
  - Hard to know in general
- Conservative approximation
  - Each a_i is a constant, or
  - All definitions of a_i that reach *d* are outside *L*, or
  - Only one definition of of a_i reaches *d*, and is loop-invariant itself
- Transformation: hoist the loop-invariant code outside of the loop

Reaching definitions analysis

- A definition *d*: t = … reaches a program location if there is a path from the definition to the program location, along which the defined variable is never redefined

Reaching definitions analysis

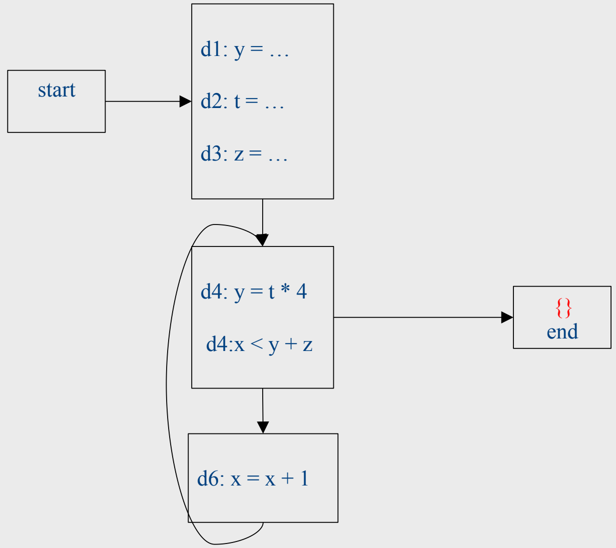
- A definition *d*: t = … reaches a program location if there is a path from the definition to the program location, along which the defined variable is never redefined
- Direction: Forward
- Domain: sets of program locations that are definitions `
- Join operator: union
- Transfer function:

  $f_{d:\ a=b\ op\ c}(\mathrm{RD}) = (\mathrm{RD} - \mathit{defs}(a)) \cup \{d\}$

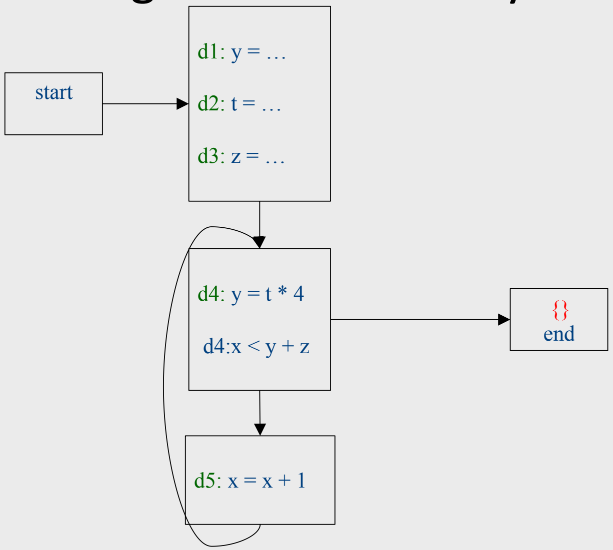
  $f_{d:\ not\text{-}a\text{-}def}(\mathrm{RD}) = \mathrm{RD}$

  – Where *defs*(*a*) is the set of locations defining *a* (statements of the form *a*=...)
- Initial value: {}

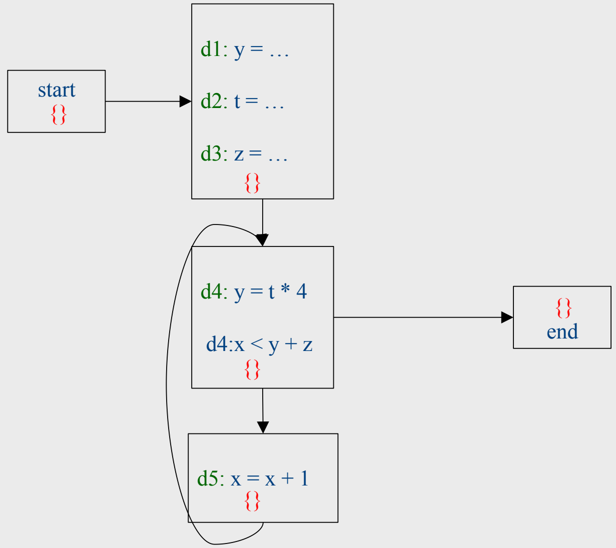
Reaching definitions analysis



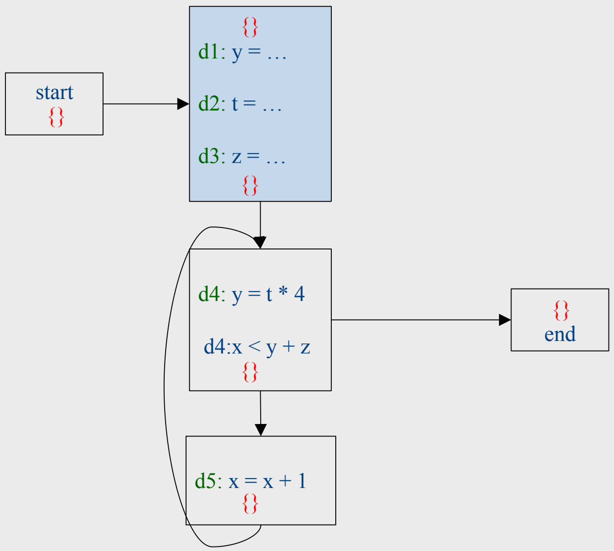
Reaching definitions analysis



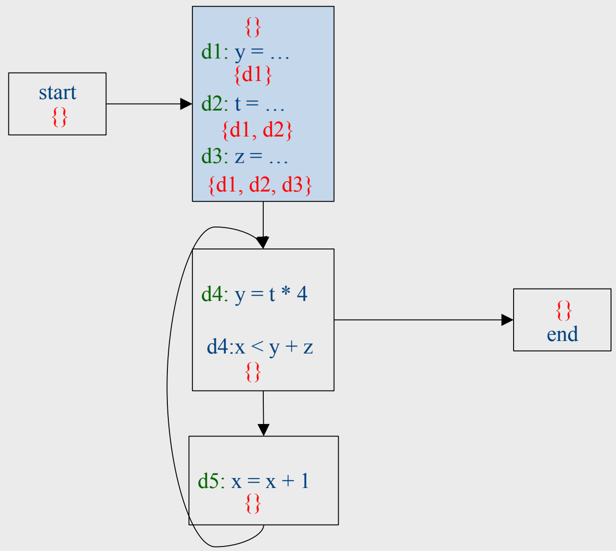
Initialization



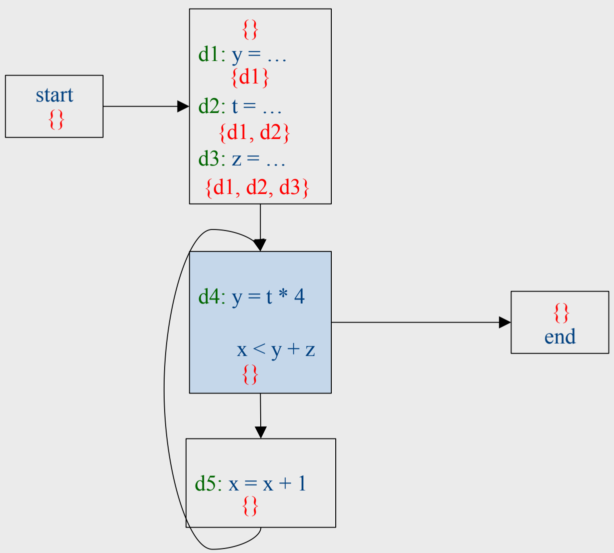
Iteration 1



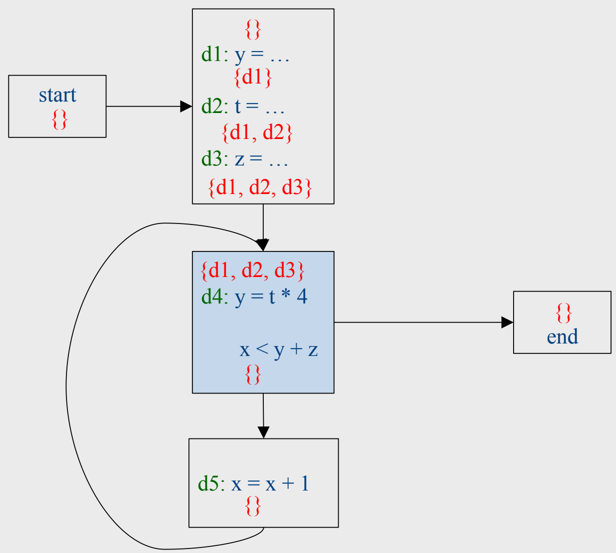
Iteration 1



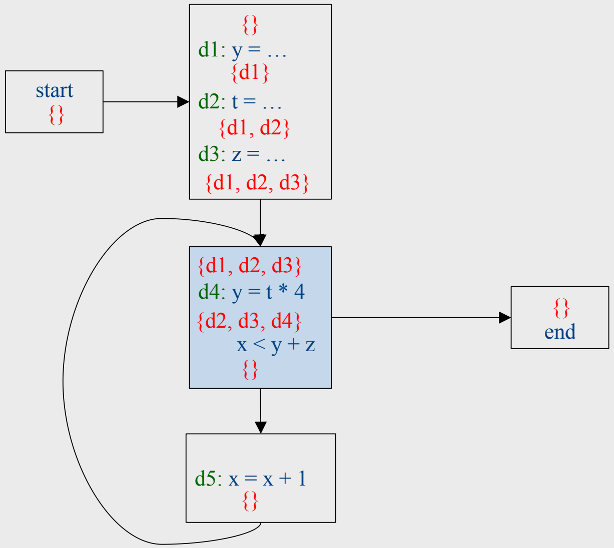
Iteration 2



Iteration 2



Iteration 2



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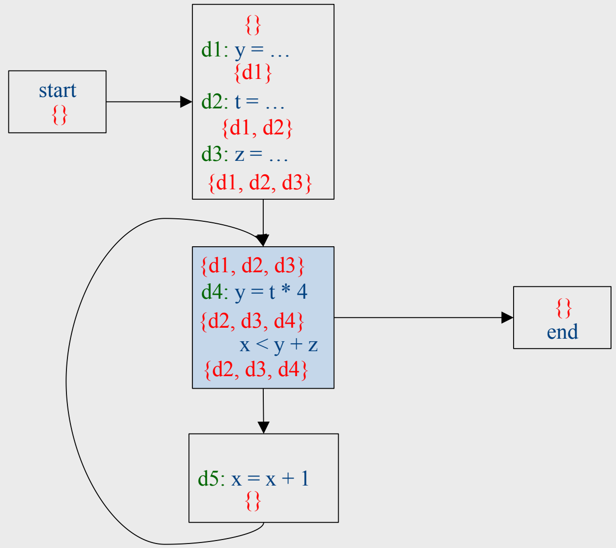
Iteration 2



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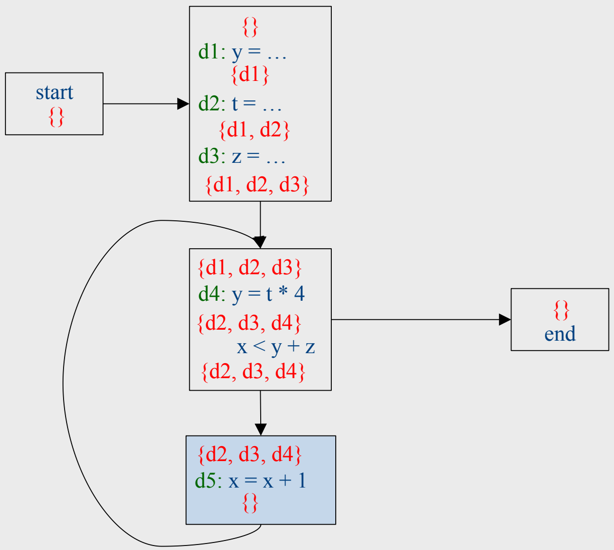
Iteration 3



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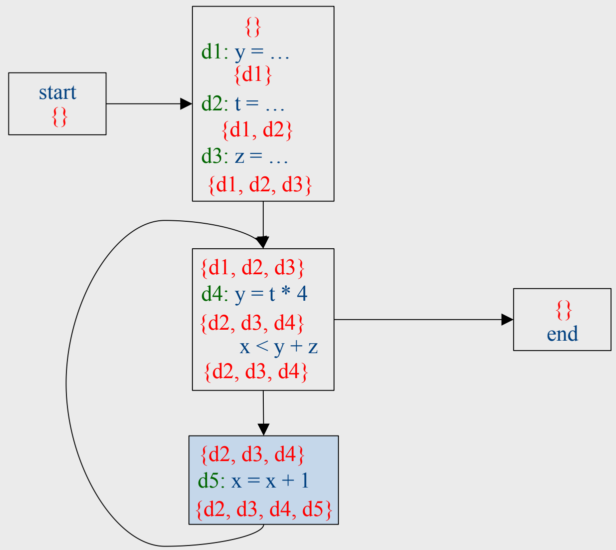
Iteration 3



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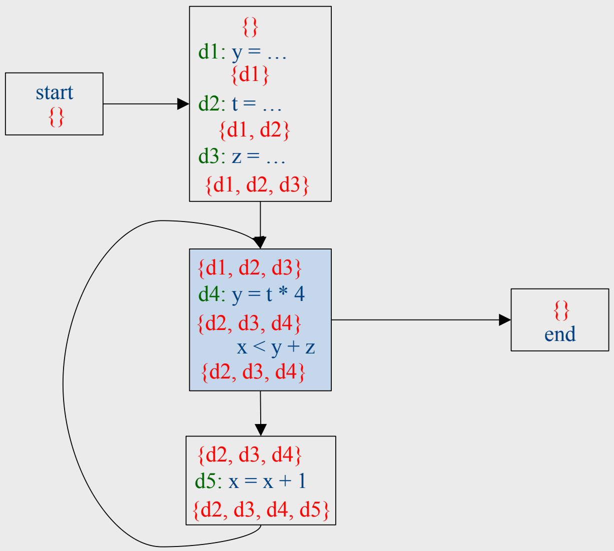
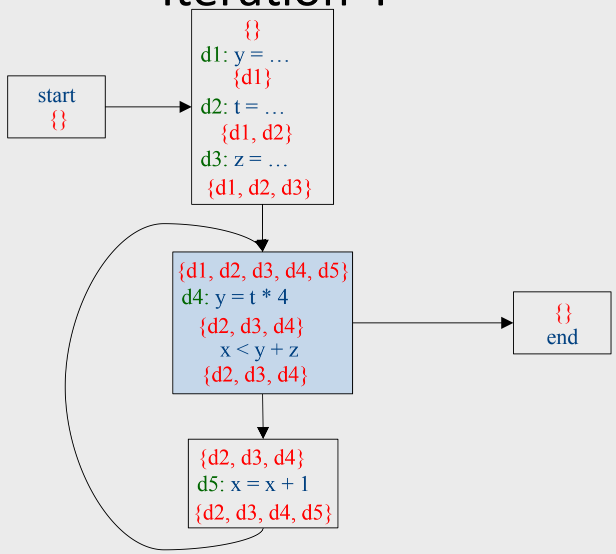
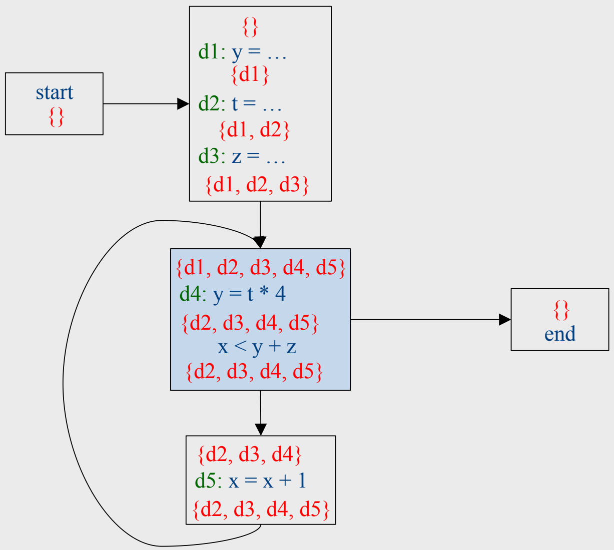
Iteration 4



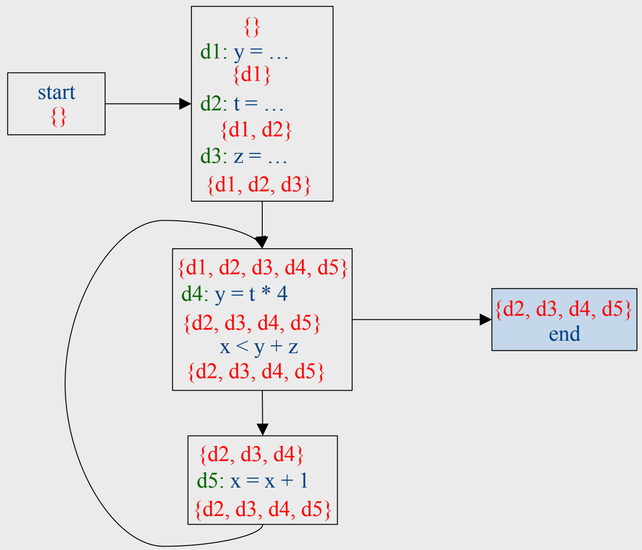
Iteration 4



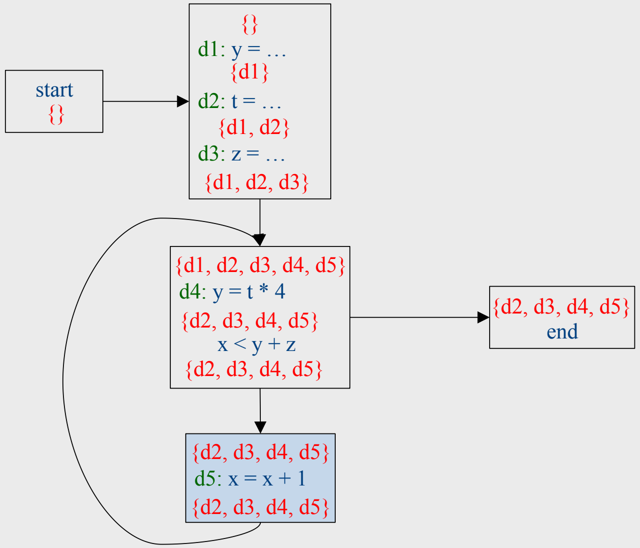
Iteration 4



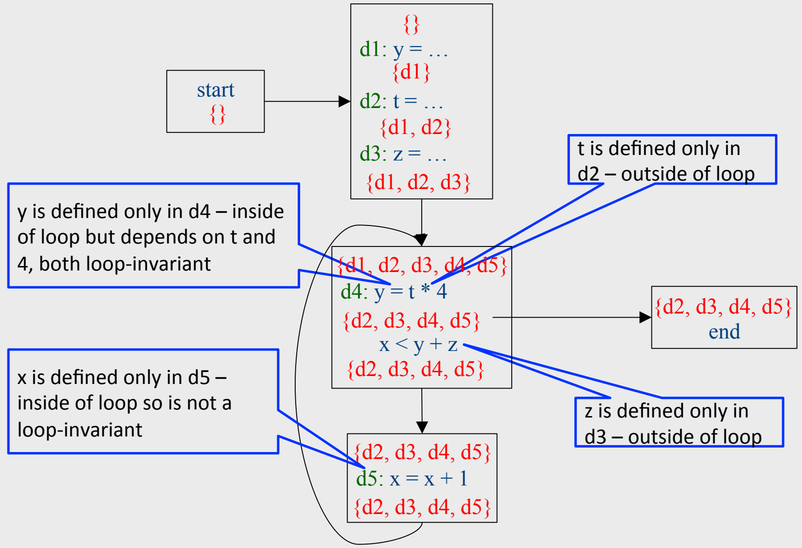
Iteration 5



Iteration 6



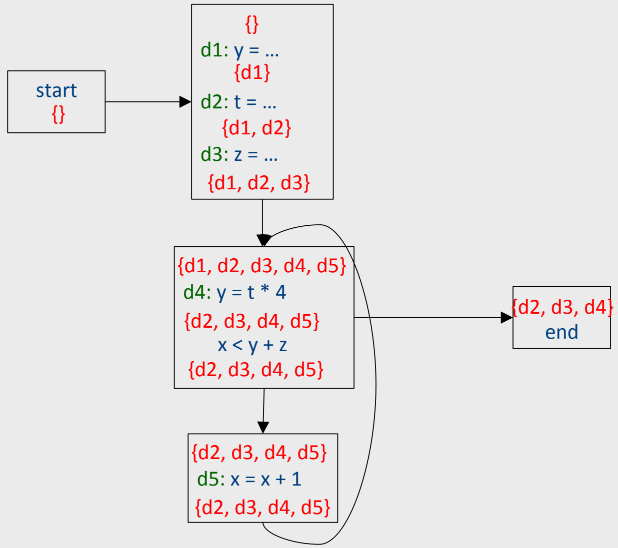
Which expressions are loop invariant?



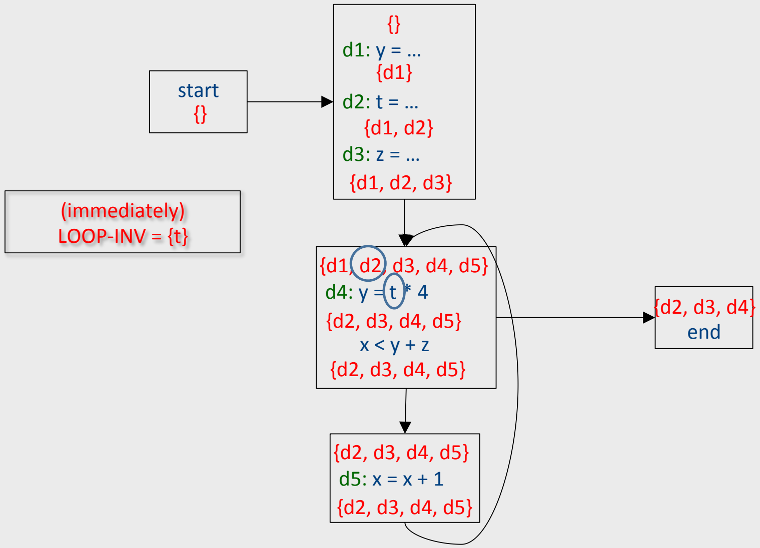
Inferring loop-invariant expressions

- For a statement s of the form t = a_1 op a_2
- A variable a_i is immediately loop-invariant if all reaching definitions IN[s]={d_1,…,d_k} for a_i are outside of the loop
- LOOP-INV = immediately loop-invariant variables and constants
  LOOP-INV = LOOP-INV ∪ {x | d: x = a_1 op a_2, d is in the loop, and both a_1 and a_2 are in LOOP-INV}
  - Iterate until fixed-point
- An expression is loop-invariant if all operands are loop-invariants

Computing LOOP-INV



Computing LOOP-INV



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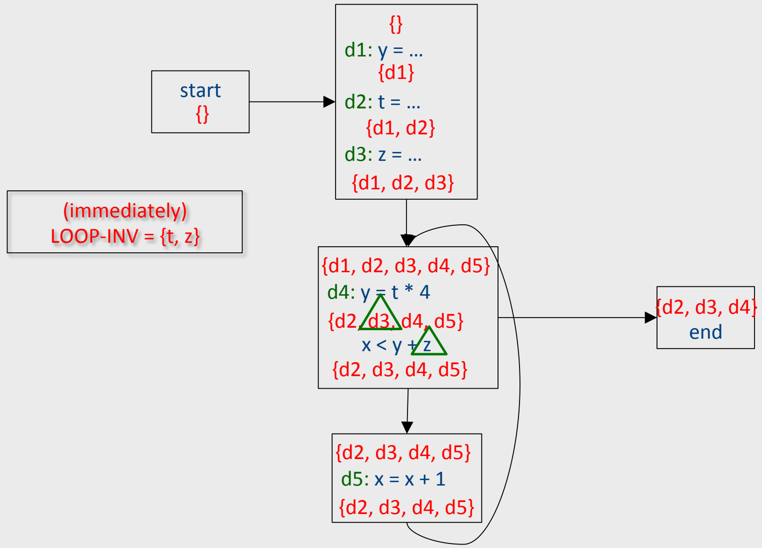

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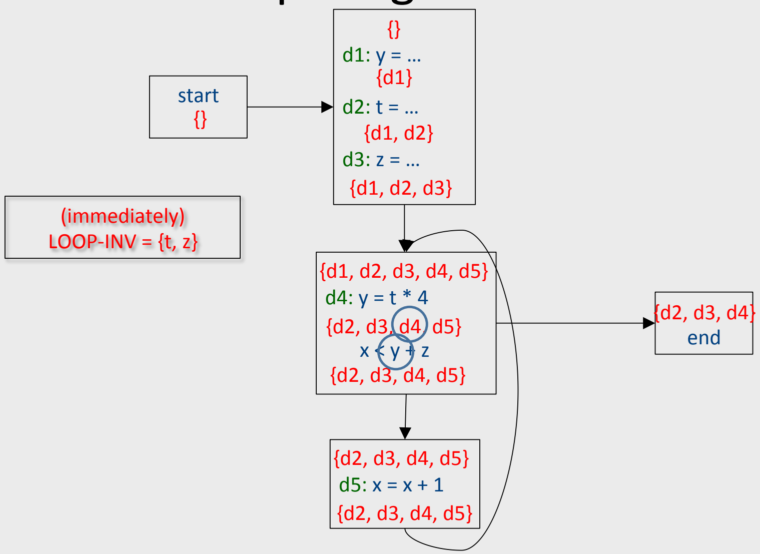

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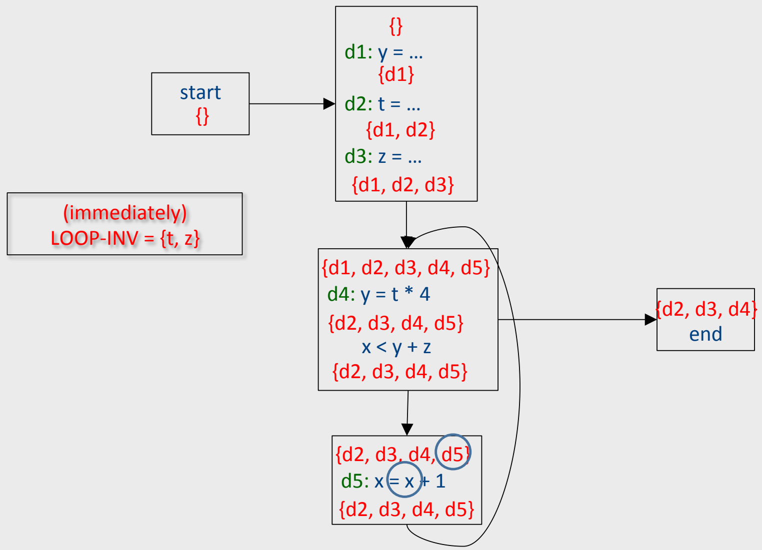

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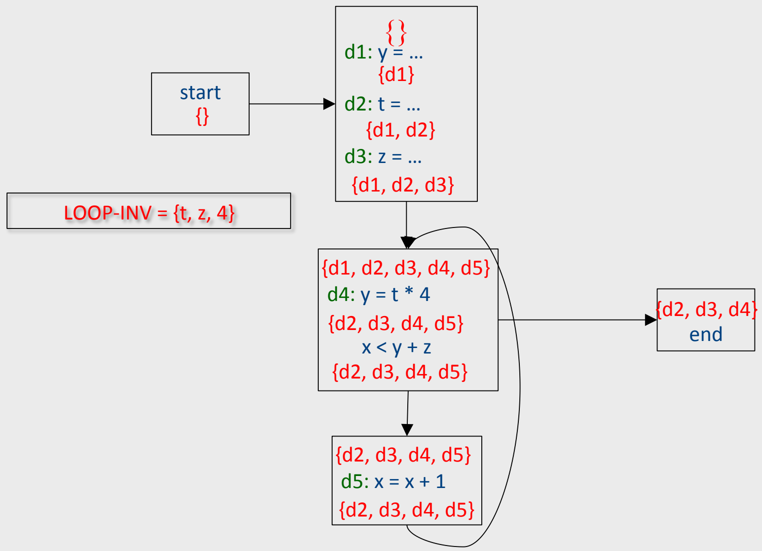
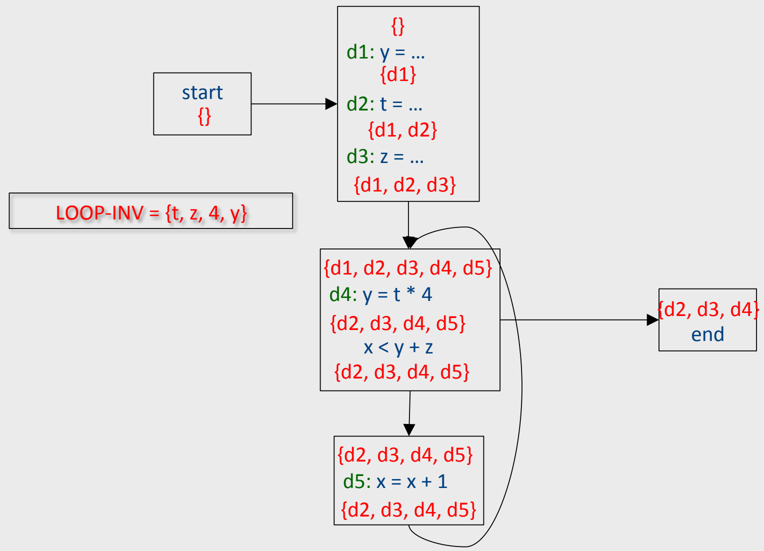


Computing LOOP-INV



Induction variables

```
while (i < x) {
    j = a + 4 * i
    a[j] = j
    i = i + 1
}
```

j is a linear function of the induction variable with multiplier 4

i is incremented by a loop-invariant expression on each iteration – this is called an induction variable

Strength-reduction

Prepare initial value

```
j = a + 4 * i
while (i < x) {
    j = j + 4
    a[j] = j
    i = i + 1
}
```

Increment by multiplier

Summary of optimizations

Analysis	Enabled Optimizations
Available Expressions	Common-subexpression elimination Copy Propagation
Constant Propagation	Constant folding
Live Variables	Dead code elimination Register allocation
Reaching Definitions	Loop-invariant code motion

Ad

- Advanced course on program analysis and verification
- Workshop on compile time techniques for detecting malicious JavaScripts
  - With Trusteer
    - Now IBM