Program Analysis and Verification

0368-4479

http://www.cs.tau.ac.il/~maon/teaching/2013-2014/paav/paav1314b.html

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Lecture 4: Denotational Semantics

Slides credit: Roman Manevich, Mooly Sagiv, Eran Yahav

Good manners

- Mobiles

Admin

- Grades
  - First home assignment will be published on Tuesday
    - (contents according to progress today)
    - Due lesson 6

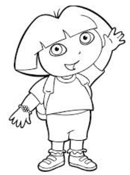
✓ Scribes (this week)

? Scribes (next week)

- From now on – in singles

What do we mean?

$$[\![\; P \;]\!] =$$



P: x→7

syntax

semantics

Why formal semantics?

- Implementation-independent definition of a programming language
- Automatically generating interpreters (and some day maybe full fledged compilers)
- **Verification and debugging**
  - if you don't know what it does, how do you know its incorrect?

Programming Languages

- Syntax
    - “how do I write a program?”
  - BNF
  - “Parsing”

- Semantics
    - “What does my program mean?”
  - …

Program semantics

- Operational: State-transformer
- Denotational: Mathematical objects
- Axiomatic: Predicate-transformer

Denotational semantics

- Giving mathematical models of programming languages
  - Meanings for program phrases (statements) defined abstractly as elements of some suitable mathematical structure.

- *It is not necessary for the semantics to determine an implementation, but it should provide criteria for showing that an implementation is correct*
  - *Dana Scott 1980*

Syntax: **While**

Abstract syntax:

$$a ::= n \mid x \mid a_1 + a_2 \mid a_1 \star a_2 \mid a_1 - a_2$$

$$b ::= \texttt{true} \mid \texttt{false}$$

$$\mid a_1 = a_2 \mid a_1 \leq a_2 \mid \neg b \mid b_1 \wedge b_2$$

$$S ::= x \texttt{:=} a \mid \texttt{skip} \mid S_1 \texttt{;}\ S_2$$

$$\mid \texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2$$

$$\mid \texttt{while}\ b\ \texttt{do}\ S$$

Syntactic categories

$n \in$ **Num**	numerals
$x \in$ **Var**	program variables
$a \in$ **Aexp**	arithmetic expressions
$b \in$ **Bexp**	boolean expressions
$S \in$ **Stm**	statements

Denotational semantics

- **A:** Aexp $\rightarrow (\Sigma \rightarrow N)$
- **B**: Bexp $\rightarrow (\Sigma \rightarrow T)$
- **S:** Stm $\rightarrow (\Sigma \rightharpoonup \Sigma)$
- Defined by structural induction

$\mathcal{A}[\![a]\!]$, $\mathcal{B}[\![b]\!]$, $S_{ns}[\![S]\!]$, $S_{sos}[\![S]\!]$

Semantic categories

Z	Integers {0, 1, -1, 2, -2, …}
T	Truth values {**ff**, **tt**}
State	**Var** $\rightarrow$ **Z**

Example state:	s=[x↦5, y↦7, z↦0]
Lookup:	s x = 5
Update:	s[x↦6] = [x↦6, y↦7, z↦0]

Denotational Semantics

- A “mathematical” semantics
  - $[\![S]\!]$ is a mathematical object
  - A fair amount of mathematics is involved
- Compositional
  - $[\![\texttt{while}\ b\ \texttt{do}\ S]\!] = F([\![b]\!], [\![S]\!])$
    - Recall:

$$\frac{\langle S, s\rangle \rightarrow s',\ \langle \texttt{while}\ \ b\ \mathrm{do}\ S, s'\rangle \rightarrow s''}{\langle \texttt{while}\ \ b\ \mathrm{do}\ S, s\rangle \rightarrow s''} \quad \text{if } \mathcal{B}[\![b]\!]\, s = \mathbf{tt}$$

- More abstract and canonical than Op. Sem.
  - No notion of “execution”
    - Merely definitions
  - No small step vs. big step
- Concurrency is an issue

Denotational Semantics

- Denotational semantics is also called
  - Fixed point semantics
  - Mathematical semantics
  - Scott-Strachey semantics

- The mathematical objects are called denotations
  - **Denotation**: meaning; especially, a direct specific meaning as distinct from an implied or associated idea
    - Though we still maintain a **computational intuition**

Important features

- **Syntax independence**: The denotations of programs should not involve the syntax of the source language.
- **Soundness**: All observably distinct programs have distinct denotations;
- **Full abstraction**: Two programs have the same denotations precisely when they are observationally equivalent.
- **Compositionality**

Plan

- Denotational semantics of While (1st attempt)
- Math
  - Complete partial orders
  - Monotonicity
  - Continuity
- Denotational semantics of **`while`**

Denotational semantics

- **A:** Aexp $\rightarrow$ ($\Sigma \rightarrow$N)
- **B**: Bexp $\rightarrow$($\Sigma \rightarrow$T)
- **S:** Stm $\rightarrow$($\Sigma \rightharpoonup \Sigma$)
- Defined by structural induction
  - Compositional definition

Denotational semantics

- **A:** Aexp → (Σ→N)
- **B**: Bexp →(Σ→T)
- **S:** Stm →(Σ⇀Σ)
- Defined by structural induction
  - Compositional definition

$\mathcal{A}[\![a]\!]$, $\mathcal{B}[\![b]\!]$, $S_{ns}[\![S]\!]$, $S_{sos}[\![S]\!]$

Denotational semantics of Aexp

- **A:** Aexp $\rightarrow$ ($\Sigma \rightarrow$N)
- **A** $[\![n]\!] = \{(\sigma, n) \mid \sigma \in \Sigma\}$
- **A** $[\![X]\!] = \{(\sigma, \sigma X) \mid \sigma \in \Sigma\}$
- **A** $[\![a_0+a_1]\!] = \{(\sigma, n_0+n_1) \mid (\sigma, n_0) \in \mathbf{A}[\![a_0]\!], (\sigma, n_1) \in \mathbf{A}[\![a_1]\!]\}$
- **A** $[\![a_0-a_1]\!] = \{(\sigma, n_0-n_1) \mid (\sigma, n_0) \in \mathbf{A}[\![a_0]\!], (\sigma, n_1) \in \mathbf{A}[\![a_1]\!]\}$
- **A** $[\![a_0 \times a_1]\!] = \{(\sigma, n_0 \times n_1) \mid (\sigma, n_0) \in \mathbf{A}[\![a_0]\!], (\sigma, n_1) \in \mathbf{A}[\![a_1]\!]\}$

Functions
represented as
sets of pairs

Lemma: A $[\![a]\!]$ is a function

Denotational semantics of Aexp with λ

- $\mathbf{A}$: Aexp $\rightarrow$ $(\Sigma \rightarrow N)$
- $\mathbf{A}[\![n]\!] = \lambda\sigma\in\Sigma.n$
- $\mathbf{A}[\![X]\!] = \lambda\sigma\in\Sigma.\sigma(X)$
- $\mathbf{A}[\![a_0+a_1]\!] = \lambda\sigma\in\Sigma.(\mathbf{A}[\![a_0]\!]\sigma+\mathbf{A}[\![a_1]\!]\sigma)$
- $\mathbf{A}[\![a_0-a_1]\!] = \lambda\sigma\in\Sigma.(\mathbf{A}[\![a_0]\!]\sigma-\mathbf{A}[\![a_1]\!]\sigma)$
- $\mathbf{A}[\![a_0\times a_1]\!] = \lambda\sigma\in\Sigma.(\mathbf{A}[\![a_0]\!]\sigma\times\mathbf{A}[\![a_1]\!]\sigma)$

Functions represented as lambda expressions

Denotational semantics of Bexp

- $\mathbf{B}$: Bexp $\rightarrow (\Sigma \rightarrow T)$
- $\mathbf{B}[\![\text{true}]\!] = \{(\sigma, \text{true}) \mid \sigma \in \Sigma\}$
- $\mathbf{B}[\![\text{false}]\!] = \{(\sigma, \text{false}) \mid \sigma \in \Sigma\}$
- $\mathbf{B}[\![a_0 = a_1]\!] = \{(\sigma, \text{true}) \mid \sigma \in \Sigma \;\&\; \mathbf{A}[\![a_0]\!]\sigma = \mathbf{A}[\![a_1]\!]\sigma\} \cup$
  $\{(\sigma, \text{false}) \mid \sigma \in \Sigma \;\&\; \mathbf{A}[\![a_0]\!]\sigma \neq \mathbf{A}[\![a_1]\!]\sigma\}$
- $\mathbf{B}[\![a_0 \leq a_1]\!] = \{(\sigma, \text{true}) \mid \sigma \in \Sigma \;\&\; \mathbf{A}[\![a_0]\!]\sigma \leq \mathbf{A}[\![a_1]\!]\sigma\} \cup$
  $\{(\sigma, \text{false}) \mid \sigma \in \Sigma \;\&\; \mathbf{A}[\![a_0]\!]\sigma \not\leq \mathbf{A}[\![a_1]\!]\sigma\}$
- $\mathbf{B}[\![\neg b]\!] = \{(\sigma, \neg_T\, t) \mid \sigma \in \Sigma, (\sigma, t) \in \mathbf{B}[\![b]\!]\}$
- $\mathbf{B}[\![b_0 \wedge b_1]\!] = \{(\sigma, t_0 \wedge_T t_1) \mid \sigma \in \Sigma, (\sigma, t_0) \in \mathbf{B}[\![b_0]\!], (\sigma, t_1) \in \mathbf{B}[\![b_1]\!]\}$
- $\mathbf{B}[\![b_0 \vee b_1]\!] = \{(\sigma, t_0 \vee_T t_1) \mid \sigma \in \Sigma, (\sigma, t_0) \in \mathbf{B}[\![b_0]\!], (\sigma, t_1) \in \mathbf{B}[\![b_1]\!]\}$

Lemma: $B[\![b]\!]$ is a function

Denotational semantics of statements?

- Intuition:
  - Running a statement s starting from a state σ yields another state σ'
- Can we define **S** $[\![s]\!]$ as a function that maps σ to σ' ?
  - **S** $[\![.]\!]$: Stm $\rightarrow (\Sigma \rightarrow \Sigma)$

Denotational semantics of commands?

- Problem: running a statement might not yield anything if the statement does not terminate
- Solution: a special element $\perp$ to denote a special outcome that stands for non-termination
  - For any set X, we write $X_\perp$ for $X \cup \{\perp\}$

- Convention:
  - whenever $f \in X \to X_\perp$ we extend f to $X_\perp \to X_\perp$ "strictly" so that $f(\perp) = \perp$

Denotational semantics of statements?

- We try:
  - $S[\![.]\!] : Stm \rightarrow (\Sigma_\perp \rightarrow \Sigma_\perp)$
- $S[\![skip]\!]\sigma = \sigma$
- $S[\![s_0 ; s_1]\!]\sigma = S[\![s_1]\!](S[\![s_0]\!]\sigma)$
- $S[\![\text{if } b \text{ then } s_0 \text{ else } s_1]\!]\sigma =$
  $\text{if } B[\![b]\!]\sigma \text{ then } S[\![s_0]\!]\sigma \text{ else } S[\![s_1]\!]\sigma$

Examples

- $S [\![X{:=}\ 2;\ X{:=}1]\!] \sigma = \sigma[X \mapsto 1]$
- $S [\![\text{if true then } X{:=}2;\ X{:=}1 \text{ else } \ldots]\!] \sigma = \sigma[X \mapsto 1]$

  – The semantics does not care about intermediate states

  – So far, we did not explicitly need $\perp$

Denotational semantics of loops?

- $S \llbracket \text{while b do s} \rrbracket \sigma = ?$

Denotational semantics of loops?

- Goal: Find a function from states to states such which defines the meaning of W

- Intuition:
  - while b do s

    ~
  - if b then (s; while b do s) else skip

Denotational semantics of loops?

- Goal: Find a function from states to states such which defines the meaning of W

- Intuition:
  - S⟦while b do s⟧

    =
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Denotational semantics of loops?

- Goal: Find a function from states to states such which defines the meaning of W

- Intuition:
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    =
  - S⟦if b then (s; while b do s) else skip⟧

Denotational semantics of loops?

- Abbreviation W=S $[\![\text{while b do s}]\!]$
- Solution 1:
  - W(σ) = if B$[\![b]\!]\sigma$ then W(S$[\![s]\!]$ σ) else σ
- Unacceptable solution
  - Defines W in terms of itself
  - It not evident that a suitable W exists
  - It may not describe W uniquely (e.g., for while true do skip)

Denotational semantics of loops?

- Goal: Find a function from states to states such which defines the meaning of W

- Approach: Solve domain equation
  - S⟦while b do s⟧

    =
  - S⟦if b then (s; while b do s) else skip⟧

Introduction to Domain Theory

- We will solve the unwinding equation through a general theory of recursive equations
- Think of programs as processors of streams of bits (streams of 0's and 1's, possibly terminated by $) What properties can we expect?

Motivation

- Let “isone” be a function that must return “1$” when the input string has at least a 1 and “0$” otherwise
  - isone(00…0$) = 0$
  - isone(xx…1…$) =1$
  - isone(0…0) =?
- **Monotonicity**: in terms of information
  - Output is never retracted
    - More information about the input is reflected in more information about the output
  - How do we express monotonicity precisely?

Montonicity

- Define a partial order
  $x \sqsubseteq y$
  - A partial order is reflexive, transitive, and anti-symmetric
  - y is a refinement of x
    - "more precise"
- For streams of bits $x \sqsubseteq y$ when x is a prefix of y
- For programs, a typical order is:
  - No output (yet) $\sqsubseteq$ some output

Montonicity

- A set equipped with a partial order is a poset
- Definition:
  - D and E are postes
  - A function f: D →E is monotonic if
    $\forall x, y \in D: x \sqsubseteq_D y \Rightarrow f(x) \sqsubseteq_E f(y)$
  - The semantics of the program ought to be a monotonic function
    - More information about the input leads to more information about the output

Montonicity Example

- Consider our "isone" function with the prefix ordering
- Notation:
  - 0^k is the stream with k consecutive 0's
  - 0^∞ is the infinite stream with only 0's
- Question (revisited): what is isone(0^k)?
  - By definition, isone(0^k\$) = 0\$ and isone(0^k1\$) = 1\$
  - But $0^k \sqsubseteq 0^k\$$ and $0^k \sqsubseteq 0^k1\$$
  - "isone" must be monotone, so:
    - isone(0^k) $\sqsubseteq$ isone(0^k\$) = 0\$
    - isone(0^k) $\sqsubseteq$ isone(0^k1\$) = 1\$
  - Therefore, monotonicity requires that isone(0^k) is a common prefix of 0\$ and 1\$, namely ε

Motivation

- Are there other constraints on "isone"?
- Define "isone" to satisfy the equations
  - isone(ε)=ε
  - isone(1s)=1$
  - isone(0s)=isone(s)
  - isone($)=0$
- What about 0^{∞}?

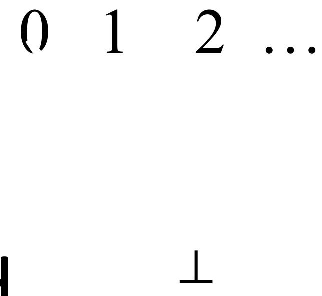
- **Continuity**: finite output depends only on finite input (no infinite lookahead)
  - Intuition: A program that can produce observable results can do it in a finite time

Chains

- A chain is a countable increasing sequence $\langle x_i \rangle = \{x_i \in X \mid x_0 \sqsubseteq x_1 \sqsubseteq \ldots\}$
- An upper bound of a set if an element "bigger" than all elements in the set
- The least upper bound is the "smallest" among upper bounds:
  - $x_i \sqsubseteq \sqcup\langle x_i \rangle$ for all $i \in N$
  - $\sqcup\langle x_i \rangle \sqsubseteq y$ for all upper bounds y of $\langle x_i \rangle$ and it is unique if it exists

Complete Partial Orders

- Not every poset has an upper bound
  - with ⊥ ⊑ n and n⊑n for all n ∈N
  - {1, 2} does not have an upper bound
- Sometimes chains have no upper bound



The chain

0 ≤1≤2≤…

does not have an upper bound

Complete Partial Orders

- It is convenient to work with posets where every chain (not necessarily every set) has a least upper bound
- A partial order P is complete if every chain in P has a least upper bound also in P
- We say that P is a complete partial order (cpo)
- A cpo with a least ("bottom") element $\perp$ is a pointed cpo (pcpo)

Examples of cpo's

- Any set P with the order
  $x \sqsubseteq y$ if and only if $x = y$ is a cpo
  It is discrete or flat
- If we add $\perp$ so that $\perp \sqsubseteq x$ for all $x \in P$, we get a flat pointed cpo
- The set N with $\leq$ is a poset with a bottom, but not a complete one
- The set $N \cup \{ \infty \}$ with $n \leq \infty$ is a pointed cpo
- The set N with $\geq$ is a cpo without bottom
- Let S be a set and P(S) denotes the set of all subsets of S ordered by set inclusion
  - P(S) is a pointed cpo

Constructing cpos

- If D and E are pointed cpos, then so is
  $D \times E$
  $(x, y) \sqsubseteq_{D \times E} (x', y')$ iff $x \sqsubseteq_D x'$ and $y \sqsubseteq_E y'$
  $\bot_{D \times E} = (\bot_D, \bot_E)$
  $\bigsqcup (x_i, y_i) = (\bigsqcup_D x_i, \bigsqcup_E y_i)$

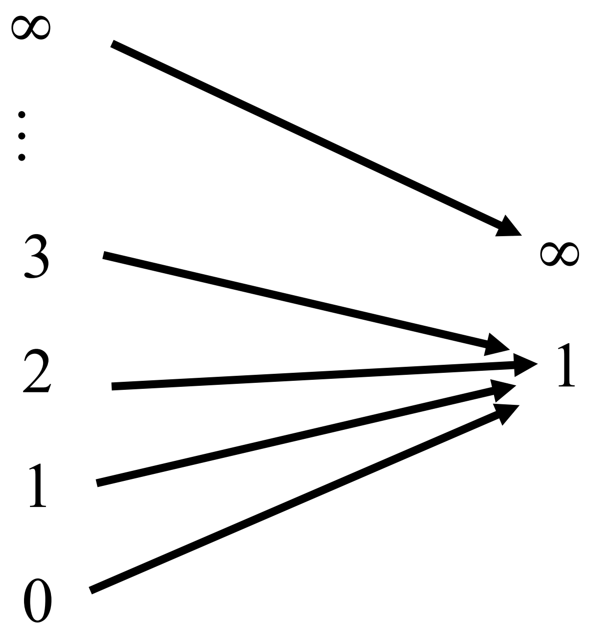
Constructing cpos (2)

- If S is a set of E is a pcpos, then so is
  $S \rightarrow E$
  $m \sqsubseteq m'$ iff $\forall s \in S$: $m(s) \sqsubseteq_E m'(s)$
  $\bot_{S \rightarrow E} = \lambda s.\ \bot_E$
  $\sqcup (m\ , m') = \lambda s. m(s) \sqcup_E m'(s)$

Continuity

- A monotonic function maps a chain of inputs into a chain of outputs:
  $x_0 \sqsubseteq x_1 \sqsubseteq \ldots \Rightarrow f(x_0) \sqsubseteq f(x_1) \sqsubseteq \ldots$
- It is always true that:
  $\sqcup_i \langle f(x_i)\rangle \sqsubseteq f(\sqcup_i \langle x_i \rangle)$
- But
  $f(\sqcup_i \langle x_i \rangle) \sqsubseteq \sqcup_i \langle f(x_i)\rangle$
  is not always true

A Discontinuity Example



$f(\sqcup_i \langle x_i \rangle) \neq \sqcup i \langle f(x_i) \rangle$

Continuity

- Each $f(x_i)$ uses a "finite" view of the input
- $f(\sqcup <x_i>)$ uses an "infinite" view of the input
- A function is **continuous** when
  $f(\sqcup <xi>) = \sqcup_i <f(x_i)>$
- The output generated using an infinite view of the input does not contain more information than all of the outputs based on finite inputs

Continuity

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- $f(\sqcup <x_i>)$ uses an "infinite" view of the input
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- **Scott's thesis**: The semantics of programs can be described by a continuous functions

Examples of Continuous Functions

- For the partial order (N $\cup${∞ }, $\leq$)
  - The identity function is continuous
    $id(\sqcup n_i) = \sqcup id(n_i)$
  - The constant function "five(n)=5" is continuous
    $five(\sqcup n_i) = \sqcup five(n_i)$
  - If isone(0^∞) =ε then isone is continuos

- For a flat cpo A, any monotonic function f: $A_\perp \rightarrow A_\perp$ such that f is strict is continuous

- Chapter 8 of the Wynskel textbook includes many more continuous functions

Fixed Points

- Solve equation: $$W(\sigma) = \begin{cases} W(S[\![s]\!]\ \sigma) & \text{if } B[\![b]\!](\sigma)=\text{true} \\ \sigma & \text{if } B[\![b]\!](\sigma)=\text{false} \\ \bot & \text{if } B[\![b]\!](\sigma)=\bot \end{cases}$$

  where $W:\Sigma_\bot \rightarrow \Sigma_\bot$; $W= S[\![\text{while be do s}]\!]$

- Alternatively, W = F(W) where:

$$F(W) = \lambda\sigma. \begin{cases} W(S[\![s]\!]\ \sigma) & \text{if } B[\![b]\!](\sigma)=\text{true} \\ \sigma & \text{if } B[\![b]\!](\sigma)=\text{false} \\ \bot & \text{if } B[\![b]\!](\sigma)=\bot \end{cases}$$

Fixed Point (cont)

- Thus we are looking for a solution for W = F(W)
  - a fixed point of F
- Typically there are many fixed points
- We may argue that W ought to be continuous $W \in [\Sigma_{\perp} \rightarrow \Sigma_{\perp}]$
- Cut the number of solutions
- We will see how to find the least fixed point for such an equation provided that F itself is continuous

Fixed Point Theorem

- Define $F^k = \lambda x. F(F(\ldots F(x)\ldots))$ (F composed k times)
- If D is a pointed cpo and $F : D \rightarrow D$ is continuous, then
  - for any fixed-point x of F and $k \in N$
    $F^k (\bot) \sqsubseteq x$
  - The least of all fixed points is
    $\sqcup_k F^k (\bot)$
- Proof:

i. By induction on k.
   - Base: $F^0 (\bot) = \bot \sqsubseteq x$
   - Induction step: $F^{k+1} (\bot) = F(F^k (\bot)) \sqsubseteq F(x) = x$

ii. It suffices to show that $\sqcup_k F^k (\bot)$ is a fixed-point
   - $F(\sqcup_k F^k (\bot)) = \sqcup_k F^{k+1} (\bot) = \sqcup_k F^k (\bot)$

Fixed-Points (notes)

- If F is continuous on a pointed cpo, we know how to find the least fixed point
- All other fixed points can be regarded as refinements of the least one
  - They contain more information, they are more precise
  - In general, they are also more arbitrary

Fixed-Points (notes)

- If F is continuous on a pointed cpo, we know how to find the least fixed point
- All other fixed points can be regarded as refinements of the least one
  - They contain more information, they are more precise
  - In general, they are also more arbitrary
  - They also make less sense for our purposes

Denotational Semantics of While

- $\Sigma_\perp$ is a flat pointed cpo
  - A state has more information on non-termination
  - Otherwise, the states must be equal to be comparable (information-wise)

- We want strict functions $\Sigma_\perp \to \Sigma_\perp$
  - therefore, continuous functions

- The partial order on $\Sigma_\perp \to \Sigma_\perp$
  $f \sqsubseteq g$ iff $f(x) = \perp$ or $f(x) = g(x)$ for all $x \in \Sigma_\perp$
  - g terminates with the same state whenever f terminates
  - g might terminate for more inputs

Denotational Semantics of While

- Recall that W is a fixed point of
  $F:[[\Sigma_{\bot} \rightarrow \Sigma_{\bot}] \rightarrow [\Sigma_{\bot} \rightarrow \Sigma_{\bot}]]$

- F is continuous $F(w) = \lambda\sigma. \begin{cases} w(S[\![s]\!](\sigma)) & \text{if } B[\![b]\!](\sigma)=\text{true} \\ \sigma & \text{if } B[\![b]\!](\sigma)=\text{false} \\ \bot & \text{if } B[\![b]\!](\sigma)=\bot \end{cases}$

- Thus, we set $S[\![\text{while } b \text{ do } c]\!] = \sqcup F^k(\bot)$
  - Least fixed point
    - Terminates least often of all fixed points

- Agrees on terminating states with all fixed point

Denotational Semantics of While

- $S [\![skip]\!] = \lambda\sigma.\sigma$
- $S [\![X := exp]\!] = \lambda\sigma.\sigma[X \mapsto A[\![exp]\!] \sigma]$
- $S [\![s_0 ; s_1]\!] = \lambda\sigma. S [\![s_1]\!] (S [\![s_0]\!] \sigma)$
- $S [\![\text{if } b \text{ then } s_0 \text{ else } s_1]\!] =$

  $\lambda\sigma. \text{ if } B[\![b]\!] \sigma \text{ then } S [\![s_0]\!] \sigma \text{ else } S [\![s_1]\!] \sigma$
- $S [\![\text{while } b \text{ do } s]\!] = \sqcup F^k(\bot)$
  - k=0, 1, …
  - $F = \lambda w. \lambda\sigma. \text{ if } B[\![b]\!](\sigma) = \text{true } w(S[\![s]\!](\sigma)) \text{ else } \sigma$

Example(1)

- while true do skip
- $F:[[\Sigma_\perp \rightarrow \Sigma_\perp] \rightarrow [\Sigma_\perp \rightarrow \Sigma_\perp]]$

$$F = \lambda w.\lambda\sigma. \begin{cases} w(S[\![s]\!](\sigma)) & \text{if } B[\![b]\!](\sigma)=\text{true} \\ \sigma & \text{if } B[\![b]\!](\sigma)=\text{false} \\ \perp & \text{if } B[\![b]\!](\sigma)=\perp \end{cases}$$

$B[\![\text{true}]\!]=\lambda\sigma.\text{true}$

$S[\![\text{skip}]\!]=\lambda\sigma.\sigma$

$F = \lambda\, w.\lambda\sigma.w(\sigma)$

$F^0(\perp)= \perp \sqcup F^1(\perp) = \perp \sqcup F^2(\perp) = \perp = \perp$

Example(2)

- while false do s
- $F:[[\Sigma_\perp \rightarrow \Sigma_\perp] \rightarrow [\Sigma_\perp \rightarrow \Sigma_\perp]]$

$$F = \lambda w.\lambda\sigma.\begin{cases} w(S[\![s]\!](\sigma)) & \text{if } B[\![b]\!](\sigma)=\text{true} \\ \sigma & \text{if } B[\![b]\!](\sigma)=\text{false} \\ \perp & \text{if } B[\![b]\!](\sigma)=\perp \end{cases}$$

$B[\![\text{false}]\!]=\lambda\sigma.\text{false}$

$F = \lambda w.\lambda\sigma.\sigma$

$$F^0(\perp)=\perp \sqcup F^1(\perp) = \lambda\sigma.\sigma \sqcup F^2(\perp) = \lambda\sigma.\sigma = \lambda\sigma.\sigma$$

Example(3)

$[\![$ while $x \neq 3$ do $x = x - 1$ $]\!] = \sqcup F^k(\bot)$ k=0, 1, …

where

$F = \lambda w. \lambda\sigma.$ if $\sigma(x) \neq 3$ $w(\sigma[x \mapsto \sigma(x) - 1])$ else σ

$F^0(\bot)$	$\bot$
$F^1(\bot)$	if $\sigma(x) \neq 3$ $\bot(\sigma[x \mapsto \sigma(x) - 1])$ else σ
	if $\sigma(x) \neq 3$ then $\bot$ else σ
$F^2(\bot)$	if $\sigma(x) \neq 3$ then $F^1(\sigma[x \mapsto \sigma(x) - 1])$ else σ
	if $\sigma(x) \neq 3$ then (if $\sigma[x \mapsto \sigma(x) - 1]$ $x \neq 3$ then $\bot$ else $\sigma[x \mapsto \sigma(x) - 1]$) else σ
	if $\sigma(x) \neq 3$ (if $\sigma(x) \neq 4$ then $\bot$ else $\sigma[x \mapsto \sigma(x) - 1]$) else σ
	if $\sigma(x) \in \{3, 4\}$ then $\sigma[x \mapsto 3]$ else $\bot$
$F^k(\bot)$	if $\sigma(x) \in \{3, 4, \ldots k\}$ then $\sigma[x \mapsto 3]$ else $\bot$
lfp(F)	if $\sigma(x) \geq 3$ then $\sigma[x \mapsto 3]$ else $\bot$

Example 4 Nested Loops

```
P ==
Z := 0 ;
while X > 0 do (
    Y := X;
     while (Y>0) do
           Z := Z + Y ;
           Y: =  Y- 1; )
   X = X – 1
   )
```

Example 4 Nested Loops

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$$s[\![\text{inner-loop}]\!] = \begin{cases} [Y\mapsto 0][Z \mapsto \sigma(Z)+\sigma(Y) * (\sigma(Y)+1)/2] & \text{if } \sigma(Y)\geq 0 \\ \perp & \text{if } \sigma(Y)<0 \end{cases}$$

$$s[\![\text{outer-loop}]\!] = \begin{cases} [Y\mapsto 0] \; [X\mapsto 0] \; [Z \mapsto \sigma(X) \times (\sigma(X) + 1) \times (1 + (2\sigma(X) + 1)/3)/4] & \text{if } \sigma(X)\geq 0 \\ \perp & \text{if } \sigma(X)\geq 0 \end{cases}$$

$$s[\![S]\!] = \begin{cases} [Y\mapsto 0] \; [X\mapsto 0] \; [Z \mapsto \sigma(Z)+\sigma(X) \times (\sigma(X) + 1) \times (1 + (2\sigma(X) + 1)/3)/4] & \text{if } \sigma(X)<0 \\ \perp & \text{if } \sigma(X)<0 \end{cases}$$

Equivalence of Semantics

- $\forall \sigma, \sigma' \in \Sigma$:
  $\sigma' = S[\![s]\!]\sigma \Leftrightarrow \langle s, \sigma\rangle \rightarrow \sigma' \Leftrightarrow \langle s, \sigma\rangle \Rightarrow^* \sigma'$

Complete Partial Orders

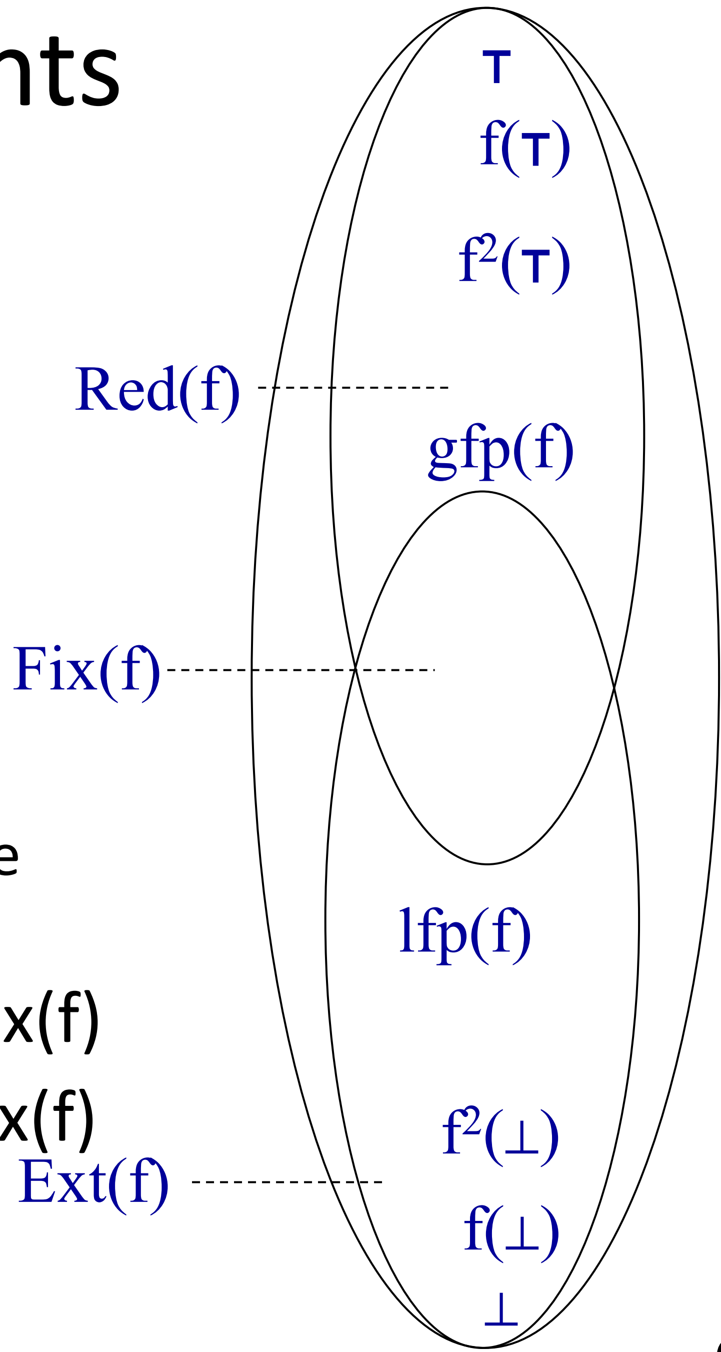
- Let (D, $\sqsubseteq$) be a partial order
  - D is a complete lattice if every subset has both greatest lower bounds and least upper bounds

Knaster-Tarski Theorem

- Let $f: L \rightarrow L$ be a monotonic function on a complete lattice L
- The least fixed point lfp(f) exists
  - $lfp(f) = \sqcap\{x \in L: f(x) \sqsubseteq x\}$

Fixed Points

- A monotone function f: L → L where $(L, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ is a complete lattice
- $Fix(f) = \{ l: l \in L, f(l) = l\}$
- $Red(f) = \{l: l \in L, f(l) \sqsubseteq l\}$
- $Ext(f) = \{l: l \in L, l \sqsubseteq f(l)\}$
  - $l_1 \sqsubseteq l_2 \Rightarrow f(l_1) \sqsubseteq f(l_2)$
- Tarski's Theorem 1955: if f is monotone then:
  - $lfp(f) = \sqcap Fix(f) = \sqcap Red(f) \in Fix(f)$
  - $gfp(f) = \sqcup Fix(f) = \sqcup Ext(f) \in Fix(f)$



Summary

- Denotational definitions are not necessarily better than operational semantics, and they usually require more mathematical work
- The mathematics may be done once and for all
- The mathematics may pay off:
- Some of its techniques are being transferred to operational semantics.
- It is trivial to prove that
  - If $B\llbracket b_1 \rrbracket = B\llbracket b_2 \rrbracket$ and $C \llbracket c_1 \rrbracket = C\llbracket c_2 \rrbracket$
  - Then $C\llbracket \text{while } b_1 \text{ do } c_1 \rrbracket = C\llbracket \text{while } b_2 \text{ do } c_2 \rrbracket$
    - compare with the operational semantics

Summary

- Denotational semantics provides a way to declare the meaning of programs in an abstract way
    - side-effects
    - loops
    - Recursion
    - Gotos
    - non-determinism
  - But not low level concurrency
- Fixed point theory provides a declarative way to specify computations
  - Many usages

The End