Program Analysis and Verification

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http://www.cs.tau.ac.il/~maon/teaching/2013-2014/paav/paav1314b.html

Noam Rinetzky

Lecture 9: Abstract Interpretation I

Slides credit: Roman Manevich, Mooly Sagiv, Eran Yahav

We begin …

- Mobiles
- Scribe
- Home assignment – next lesson

Previously

- Operational Semantics
  - Large step (Natural)
  - Small step (SOS)

How?

- Denotational Semantics
  - aka mathematical semantics

What?

- Axiomatic Semantics
  - aka Hoare Logic
  - aka axiomatic (manual) verification

Why?

From verification to analysis

- Manual program verification
  - Verifier provides assertions
    - Loop invariants

- Automatic program verification (P analysis)
  - Tool automatically synthesize assertions
    - Finds loop invariants

Manual proof for `max`

```
nums : array
N : int

x := 0

res := nums[0]


while x < N

    if nums[x] > res then

        res := nums[x]


    x := x + 1
```

Manual proof for `max`

```
nums : array
N : int
{ N≥0 }
x := 0
{ N≥0 ∧ x=0 }
res := nums[0]
{ x=0 }
Inv = { x≤N }
while x < N
    { x=k ∧ k<N }
    if nums[x] > res then
        { x=k ∧ k<N }
        res := nums[x]
        { x=k ∧ k<N }
    { x=k ∧ k<N }
    x := x + 1
    { x=k+1 ∧ k≤N }
{ x≤N ∧ x≥N }
{ x=N }
```

Can we find this proof automatically?

```
nums : array
N : int
{ N≥0 }
x := 0
{ N≥0 ∧ x=0 }
res := nums[0]
{ x=0 }
Inv = { x≤N }
while x < N
    { x=k ∧ k<N }
    if nums[x] > res then
        { x=k ∧ k<N }
        res := nums[x]
        { x=k ∧ k<N }
    { x=k ∧ k<N }
    x := x + 1
    { x=k+1 ∧ k≤N }
{ x≤N ∧ x≥N }
{ x=N }
```

Observation: predicates in proof have the general form

$$\bigwedge \text{constraint}$$

where constraint has the form

$X - Y \leq c$

or

$\pm X \leq c$

Zone Abstract Domain (Analysis)

- Developed by Antoine Mine in his Ph.D. thesis
- Uses constraints of the form $X - Y \leq c$ and $\pm X \leq c$
- Built on top of Difference Bound Matrices (DBM) and shortest-path algorithms
  - $O(n^3)$ time
  - $O(n^2)$ space

Analysis with Zone abstract domain

```
nums : array
N : int
{ N≥0 }
x := 0
{ N≥0 ∧ x=0 }
res := nums[0]
{ N≥0 ∧ x=0 }
Inv = { N≥0 ∧ 0≤x≤N }
while x < N
    { N≥0 ∧ 0≤x<N }
    if nums[x] > res then
        { N≥0 ∧ 0≤x<N }
        res := nums[x]
        { N≥0 ∧ 0≤x<N }
    { N≥0 ∧ 0≤x<N }
    x := x + 1
    { N≥0 ∧ 0<x≤N }
{N≥0 ∧ 0≤x ∧ x=N }
```

Static Analysis with Zone Abstraction

```
nums : array
N : int
{ N≥0 }
x := 0
{ N≥0 ∧ x=0 }
res := nums[0]
{ x=0 }
Inv = { x≤N }
while x < N
    { x=k ∧ k≤N }
    if nums[x] > res then
        { x=k ∧ k<N }
        res := nums[x]
        { x=k ∧ k<N }
    { x=k ∧ k<N }
    x := x + 1
    { x=k+1 ∧ k≤N }
{ x≤N ∧ x≥N }
{ x=N }
```

Manual Proof

Abstract Interpretation [Cousot'77]

- Mathematical foundation of static analysis

Abstract Interpretation [Cousot'77]

- Mathematical foundation of static analysis

  - Abstract (semantic) domains ("abstract states")
  - Transformer functions ("abstract steps")
  - Chaotic iteration ("abstract computation")

Abstract Interpretation [CC77]

- A very general mathematical framework for approximating semantics
  - Generalizes Hoare Logic
  - Generalizes weakest precondition calculus
- Allows designing sound static analysis algorithms
  - Usually compute by iterating to a fixed-point
  - *Not specific to any programming language style*
- Results of an abstract interpretation are (loop) invariants
  - Can be interpreted as axiomatic verification assertions and used for verification

Abstract Interpretation by Example

Motivating Application: Optimization

- A compiler optimization is defined by a program transformation:

  T : Prog $\rightarrow$ Prog

- The transformation is semantics-preserving:

  $\forall s \in \text{State}.\ S_{\text{sos}} [\![C]\!]\, s = S_{\text{sos}} [\![\text{T}(C)]\!]\, s$

- The transformation is applied to the program only if an *enabling condition* is met
- We use static analysis for inferring enabling conditions

Common Subexpression Elimination

- If we have two variable assignments

  x := a op b
  …
  y := a op b

  and the values of x, a, and b have not changed between the assignments, rewrite the code as

  x = a op b
  …
  y := x

op ∈ {+, -, *, ==, <=}

- Eliminates useless recalculation
- Paves the way for more optimizations
  - e.g., dead code elimination

What do we need to prove?

```
{ true }
C1
x := a op b
C2
{ x = a op b }
y := a op b
C3
```

CSE →

```
{ true }
C1
x := a op b
C2
{ x = a op b }
y := x
C3
```

Available Expressions Analysis

- A static analysis that infers for every program point a set of facts of the form
  AV = { $x = y$ | $x, y \in$ Var } $\cup$
  { $x = -y$ | $x, y \in$ Var } $\cup$
  { $x = y \; op \; z$ | $y, z \in$ Var, $op \in \{+, -, *, <=\}$ }

- For every program with n=|Var| variables number of possible facts is finite: |AV|=$O(n^3)$
  – Yields a trivial algorithm … but, is it efficient?

Which proof is more desirable?

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ x=a+b }
y := a + b
...
```

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ z=a+c }
y := a + b
...
```

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ x=a+b ∧ z=a+c }
y := a + b
...
```

Which proof is more desirable?

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ x=a+b }
y := a + b
...
```

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ z=a+c }
y := a + b
...
```

More detailed predicate =
more optimization opportunities

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ x=a+b ∧ z=a+c }
y := a + b
...
```

x=a+b ∧ z=a+c ⇒ x=a+b

x=a+b ∧ z=a+c ⇒ z=a+c

Implication formalizes “more detailed” relation between predicates

Developing a theory of approximation

- Formulae are suitable for many analysis-based proofs but we may want to represent predicates in other ways:
  - Sets of "facts"
  - Automata
  - Linear (in)equalities
  - … ad-hoc representation

- Wanted: a uniform theory to represent semantic values and approximations

A Blast from the Past

- Recall denotational semantics
  - Pre-orders
  - Partial orders
    - Complete partial orders CPO,
    - Pointed CPO
    - Lattices
    - Least upper bounds
    - Chains
  - Monotonic functions
  - Fixpoints

Abstract Domains

- Mathematical foundations

Preorder

- We say that a binary order relation $\sqsubseteq$ over a set D is a preorder if the following conditions hold for every d, d', d'' $\in$ D
  - Reflexive: $d \sqsubseteq d$
  - Transitive: $d \sqsubseteq d'$ and $d' \sqsubseteq d''$ implies $d \sqsubseteq d''$

- There may exist d, d' such that

  $d \sqsubseteq d'$ and $d' \sqsubseteq d$ yet $d \neq d'$

Preorder example

- **S**imple **A**vailable **E**xpressions
- Define SAV = $\{ x = y \mid x, y \in \text{Var} \} \cup$
  $\{ x = y + z \mid y, z \in \text{Var} \}$
- For D=2^{SAV} (sets of available expressions) define (for two subsets $A_1, A_2 \in$ D)

  $A_1 \sqsubseteq^{\text{imp}} A_2$ if and only if $\bigwedge A_1 \Rightarrow \bigwedge A_2$

Can we decide $A_1 \sqsubseteq^{\text{imp}} A_2$?

- A_1 is "more detailed" if it implies all facts of A_2

- Compare {x=y $\wedge$ x=a+b} with {x=y $\wedge$ y=a+b}
  - Which one should we choose?

The meaning of implication

- A predicate P represents the set of states
  $models(P) = \{ s \mid s \models P \}$
- $P \Rightarrow Q$ means
  $models(P) \subseteq models(Q)$

Partially ordered sets

- A partially ordered set (poset) is a pair (D , $\sqsubseteq$)
  - D is a set of elements – a (semantic) domain
  - $\sqsubseteq$ is a partial order between pairs of elements from D. That is $\sqsubseteq$: $D \times D$ with the following properties, for all d, d', d'' in D
    - Reflexive: $d \sqsubseteq d$
    - Transitive: $d \sqsubseteq d'$ and $d' \sqsubseteq d''$ implies $d \sqsubseteq d''$
    - **Anti-symmetric: $d \sqsubseteq d'$ and $d' \sqsubseteq d$ implies $d = d'$**

Unique "most detailed" element

- Notation: if $d \sqsubseteq d'$ and $d \neq d'$ we write $d \sqsubset d'$

From preorders to partial orders

- We can transform a preorder into a poset by
  1. Coarsening the ordering

  2. Switching to a canonical form by choosing a representative for the set of equivalent elements
     d^* for $\{ d' \mid d \sqsubseteq d' \text{ and } d' \sqsubseteq d \}$

Coarsening for SAV

- For $D=2^{SAV}$ (sets of available expressions) define (for two subsets $A_1, A_2 \in D$)
  $A_1 \sqsubseteq^{coarse} A_2$ if and only if $A_1 \supseteq A_2$
- Notice that if $A_1 \supseteq A_2$ then $\bigwedge A_1 \Rightarrow \bigwedge A_2$
- Compare {x=y $\wedge$ x=a+b} with {x=y $\wedge$ y=a+b}
- How about {x=y $\wedge$ x=a+b $\wedge$ y=a+b} ?

Canonical form for SAV

- For an available expressions element A define *Explicate*(A) = minimal set B such that:
  1. $A \subseteq B$
  2. x=y $\in B$ implies y=x $\in B$
  3. x=y $\in B$ and y=z $\in B$ implies x=z $\in B$
  4. x=y+z $\in B$ implies x=z+y $\in B$
  5. x=y $\in B$ and x=z+w $\in B$ implies y=z+w $\in B$
  6. x=y $\in B$ and z=x+w $\in B$ implies z=y+w $\in B$
  7. x=z+w $\in B$ and y=z+w $\in B$ implies x=y $\in B$
- Makes all implicit facts explicit

- Define $A^* = Explicate(A)$
- Define (for two subsets $A_1, A_2 \in$ D)
  $A_1 \sqsubseteq^{exp} A_2$ if and only if $A_1^* \supseteq A_2^*$
- **Lemma:** $A_1 \sqsubseteq^{exp} A_2$ if and only $A_1 \sqsubseteq^{imp} A_2$

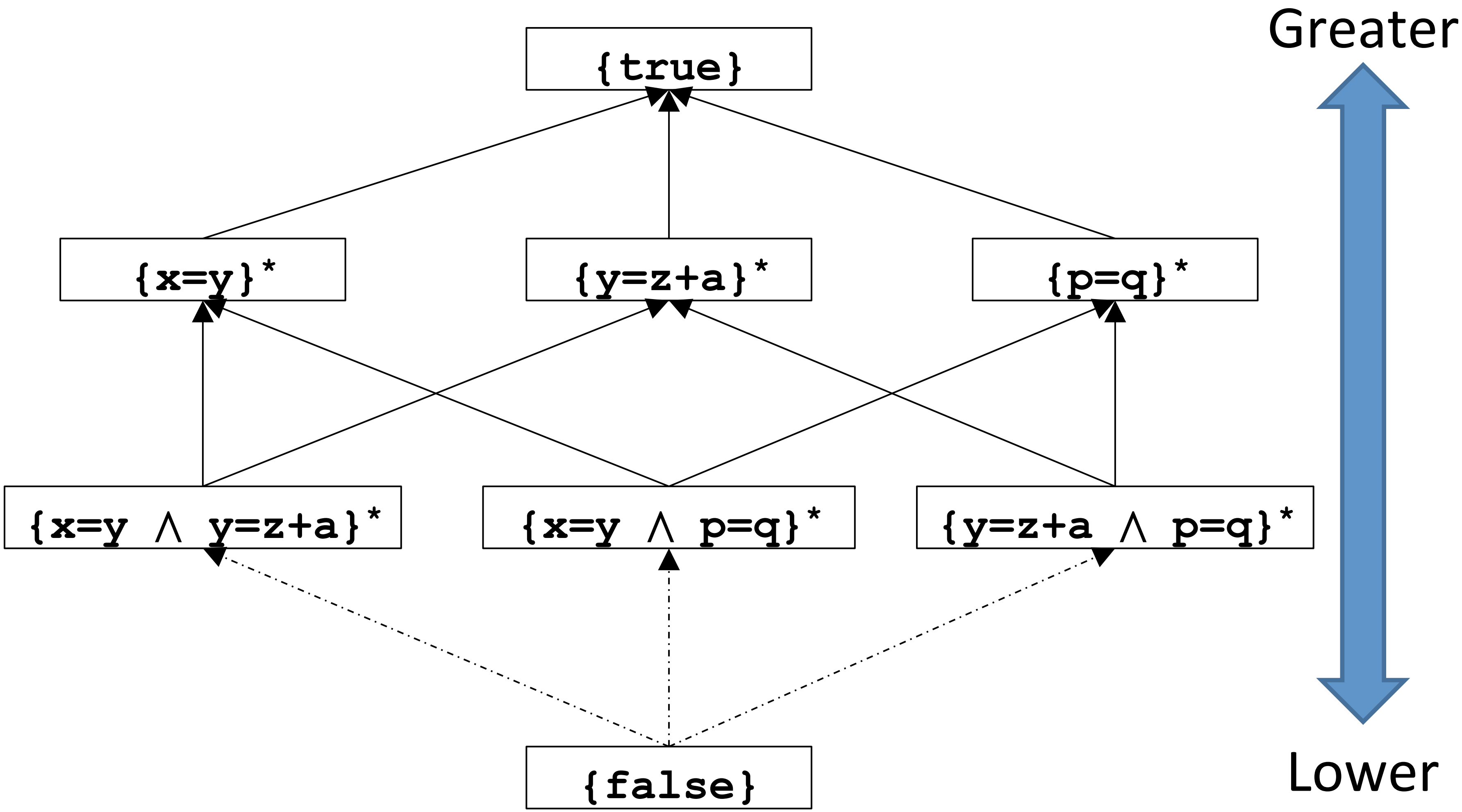
Therefore
$A_1 \sqsubseteq^{imp} A_2$ is decidable

Some posets-related terminology

- If $x \sqsubseteq y$ we can say
  - x is *lower* than y
  - x is *more precise* than y
  - x is *more concrete* than y
  - x *under-approximates* y

  - y is *greater* than x
  - y is *less precise* than x
  - y is *more abstract* than x
  - y *over-approximates* x

Visualizing ordering for SAV



D={x=y, y=x, p=q, q=p, y=z+a, y=a+z, z=y+z, x=z+a}

Pointed poset

- A poset (D, ⊑) with a least element ⊥ is called a pointed poset
  - For all d∈D we have that ⊥ ⊑ d
- The pointed poset is denoted by (D , ⊑, ⊥)
- We can always transform a poset (D, ⊑) into a pointed poset by adding a special bottom element

  (D ∪ {⊥}, ⊑ ∪ {⊥⊑d | d∈D}, ⊥)
- Greatest element for SAV = {**true** = ?}
- Least element for SAV = {**false**= ?}

Annotating conditions

[if$_p$]

$$\frac{\{\, b \wedge P \,\}\, S_1 \,\{\, Q \,\},\quad \{\, \neg b \wedge P \,\}\, S_2 \,\{\, Q \,\}}{\{\, P \,\}\ \texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2\ \{\, Q \,\}}$$

$\{\, P \,\}$

`if` b `then`

  $\{\, b \wedge P \,\}$

  S_1

  $\{\, Q_1 \,\}$

`else`

  S_2

  $\{\, Q_2 \,\}$

$\{\, Q \,\}$

Q approximates Q_1 and Q_2

We need a general way to approximate a set of semantic elements by a single semantic element

Join operator

- Assume a poset (D, $\sqsubseteq$)
- Let X $\subseteq$ D be a subset of D (finite/infinite)
- The join of X is defined as
  - $\sqcup$X = the least upper bound (LUB) of all elements in X *if it exists*
  - $\sqcup$X = $\min_{\sqsubseteq}$\{ b | forall x$\in$X we have that x$\sqsubseteq$b\}
  - The supremum of the elements in X
  - A kind of abstract union (disjunction) operator
- Properties of a join operator
  - Commutative: x $\sqcup$ y = y $\sqcup$ x
  - Associative: (x $\sqcup$ y) $\sqcup$ z = x $\sqcup$ (y $\sqcup$ z)
  - Idempotent: x $\sqcup$ x = x

Meet operator

- Assume a poset $(D, \sqsubseteq)$
- Let $X \subseteq D$ be a subset of D (finite/infinite)
- The meet of X is defined as
  - $\sqcap X$ = the greatest lower bound (GLB) of all elements in X *if it exists*
  - $\sqcap X = \max_{\sqsubseteq}\{ b \mid \text{forall } x \in X \text{ we have that } b \sqsubseteq x\}$
  - The infimum of the elements in X
  - A kind of abstract intersection (conjunction) operator
- Properties of a join operator
  - Commutative: $x \sqcap y = y \sqcap x$
  - Associative: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
  - Idempotent: $x \sqcap x = x$

Complete lattices

- A complete lattice $(D, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ is
- A set of elements D
- A partial order $x \sqsubseteq y$
- A join operator $\sqcup$
- A meet operator $\sqcap$
- A bottom element
  $\bot$ = ?
- A top element
  $\top$ = ?

Complete lattices

- A complete lattice $(D, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ is
- A set of elements D
- A partial order $x \sqsubseteq y$
- A join operator $\sqcup$
- A meet operator $\sqcap$
- A bottom element
  $\bot = \sqcup \emptyset$
- A top element
  $\top = \sqcup D$

Transfer Functions

- Mathematical foundations

Towards an automatic proof

- **Goal:** automatically compute an annotated program proving as many facts of the form
  $x = y + z$ as possible
- **Decision 1:** develop a forward-going proof
- **Decision 2:** draw predicates from a finite set D
  - "looking under the light of the lamp"
  - A compromise that simplifies problem by focusing attention – possibly miss some facts that hold
- **Challenge 1:** handle straight-line code
- **Challenge 2:** handle conditions
- **Challenge 3:** handle loops

Domain for SAV

- Define *atomic facts* (for SAV) as
  $\theta = \{ x = y \mid x, y \in \text{Var} \} \cup \{ x = y + z \mid x, y, z \in \text{Var} \}$
  - For $n = |\text{Var}|$ number of atomic facts is $O(n^3)$
- Define *sav-predicates* as $\Pi = 2^\theta$
- For $D \subseteq \theta$, Conj(D) $= \wedge D$
  - $\text{Conj}(\{a=b, c=b+d, b=c\}) = (a=b) \wedge (c=b+d) \wedge (b=c)$
- Note:
  - $\text{Conj}(D_1 \cup D_2) = \text{Conj}(D_1) \wedge \text{Conj}(D_1)$
  - $\text{Conj}(\{\}) \Leftrightarrow \text{true}$

Challenge 2: handling straight-line code

handling straight-line code: Goal

- Given a program of the form

  $x_1 := a_1; \ldots x_n := a_n$

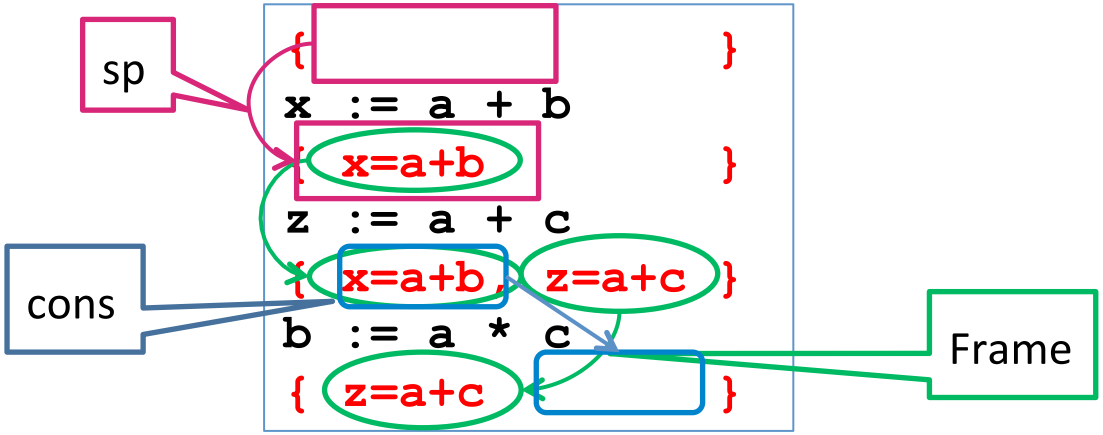
- Find predicates $P_0, \ldots, P_n$ such that
  1. $\{P_0\}\ x_1 := a_1\ \{P_1\} \ldots \{P_{n\text{-}1}\}\ x_n := a_n\ \{P_n\}$ is a proof
     - **sp**($x_i := a_\text{i}, P_{i\text{-}1}$) $\Rightarrow P_i$
  2. $P_\text{i} = \text{Conj}(D_\text{i})$
     - D_i is a set of simple (SAV) facts

Example

```
{                 }
x := a + b
{                 }
z := a + c
{                 }
b := a * c
{                 }
```

- Find a proof that satisfies both conditions

Example



- Can we make this into an algorithm?

Algorithm for straight-line code

- **Goal:** find predicates $P_0, \ldots, P_n$ such that
  1. $\{P_0\}\ x_1 := a_1\ \{P_1\} \ldots \{P_{n-1}\}\ x_n := a_n\ \{P_n\}$ is a proof
     That is: $\mathbf{sp}(x_i := a_i, P_{i-1}) \Rightarrow P_i$
  2. Each P_i has the form Conj(D_i) where D_i is a set of simple (SAV) facts
- **Idea:** define a function $F^{SAV}[x:=a] : \Pi \rightarrow \Pi$ s.t.
  if $F^{SAV}[x:=a](D) = D'$
  then $\mathbf{sp}(x := a, \mathrm{Conj}(D)) \Rightarrow \mathrm{Conj}(D')$
  – We call F the abstract transformer for $x:=a$
- Initialize $D_0=\{\}$
- For each i: compute $D_{i+1} = \mathrm{Conj}(F^{SAV}[x_i := a_i]\ D_i)$
- Finally $P_i = \mathrm{Conj}(D_i)$

Defining an SAV abstract transformer

- **Goal:** define a function $F^{SAV}[x:=a] : \Pi \rightarrow \Pi$ s.t.
  if $F^{SAV}[x:=a](D) = D'$
  then **sp**($x := a$, Conj(D)) $\Rightarrow$ Conj(D')

Defining an SAV abstract transformer

- **Goal:** define a function $F^{SAV}[x{:=}a] : \Pi \rightarrow \Pi$ s.t.
  if $F^{SAV}[x{:=}a](D) = D'$
  then **sp**($x := a$, Conj(D)) $\Rightarrow$ Conj(D')
- **Idea:** define rules for individual facts
  and generalize to sets of facts by the
  conjunction rule

Defining an SAV abstract transformer

- **Goal:** define a function $F^{SAV}[x{:=}a] : \Pi \rightarrow \Pi$ s.t.
  if $F^{SAV}[x{:=}a](D) = D'$
  then **sp**($x := a$, Conj(D)) $\Rightarrow$ Conj(D')
- **Idea:** define rules for individual facts and generalize to sets of facts by the conjunction rule

[kill-lhs] $\{ x{=}\omega \}\ x{:=}a\ \{ \}$

[kill-rhs-1] $\{ y{=}x{+}w \}\ x{:=}a\ \{ \}$

[kill-rhs-2] $\{ y{=}w{+}x \}\ x{:=}a\ \{ \}$

[gen] $\{ \}\ x{:=}\ \omega\ \{ x{=}\omega \}$

[preserve] $\{ y{=}z{+}w \}\ x{:=}a\ \{ y{=}z{+}w \}$

ω Is either a variable v or an addition expression $v{+}w$

SAV abstract transformer example

```
{                  }
x := a + b
{ x=a+b            }
z := a + c
{ x=a+b, z=a+c }
b := a * c
{ z=a+c            }
```

[kill-lhs] { $x=\omega$ } $x:=a$ { }

[kill-rhs-1] { $y=x+w$ } $x:=a$ { }

[kill-rhs-2] { $y=w+x$ } $x:=a$ { }

[gen] { } $x:= \omega$ { $x=\omega$ }

[preserve] { $y=z+w$ } $x:=a$ { $y=z+w$ }

ω Is either a variable v or an addition expression $v+w$

Problem 1: large expressions

```
{              }
x := a + b + c
{              }
y := a + b + c
{              }
```

Missed CSE opportunity

- Large expressions on the right hand sides of assignments are problematic
  - Can miss optimization opportunities
  - Require complex transformers
- Solution: transform code to normal form where right-hand sides have bounded size

Solution: Simplify Prog. Lang.

```
{                }
x := a + b + c
{                }
y := a + b + c
{                }
```

➡

```
{                }
i1 := a + b
{                }
x := i1 + c
{                    }
i2 := a + b
{                                }
y := i2 + c
{                                                }
```

- Main idea: simplify expressions by storing intermediate results in new temporary variables
  - Three-address code
- Number of variables in simplified statements ≤ 3

Solution: Simplify Prog. Lang.

```
{                }
x := a + b + c
{                }
y := a + b + c
{                }
```

```
{                }
i1 := a + b
{ i1=a+b         }
x := i1 + c
{ i1=a+b, x=i1+c }
i2 := a + b
{ i1=a+b, x=i1+c, i2=a+b }
y := i2 + c
{ i1=a+b, x=i1+c, i2=a+b, y=i2+c }
```

Need to infer `i1=i2`

- Main idea: simplify expressions by storing intermediate results in new temporary variables
  - Three-address code
- Number of variables in simplified statements ≤ 3

Problem 2: Transformer Precision

```
{                  }
i1 := a + b
{ i1=a+b           }
x := i1 + c
{ i1=a+b, x=i1+c }
i2 := a + b
{ i1=a+b, x=i1+c, i2=a+b }
y := i2 + c
{ i1=a+b, x=i1+c, i2=a+b, y=i2+c }
```

Need to infer `i1=i2`

- Our transformer only infers syntactically available expressions – ones that appear in the code explicitly
- We want a transformer that looks deeper into the semantics of the predicates
  - Takes equalities into account

Solution: Use Canonical Form

- **Idea:** make as many implicit facts explicit by
  - Using symmetry and transitivity of equality
  - Commutativity of addition
  - Meaning of equality – can substitute equal variables
- For *P*=Conj(*D*) let *Explicate*(*D*) = minimal set D^* such that:
  1. $D \subseteq D^*$
  2. x=y $\in D^*$ implies y=x $\in D^*$
  3. x=y $\in D^*$ y=z $\in D^*$ implies x=z $\in D^*$
  4. x=y+z $\in D^*$ implies x=z+y $\in D^*$
  5. x=y $\in D^*$ and x=z+w $\in D^*$ implies y=z+w $\in D^*$
  6. x=y $\in D^*$ and z=x+w $\in D^*$ implies z=y+w $\in D^*$
  7. x=z+w $\in D^*$ and y=z+w $\in D^*$ implies x=y $\in D^*$
- Notice that *Explicate*(*D*) $\Leftrightarrow$ D
  - *Explicate* is a special case of a reduction operator

Sharpening the transformer

- **Define:** $F^*[x{:=}a] = \text{Explicate} \circ F^{SAV}[x{:=}a]$

```
{                }
i1 := a + b
{ i1=a+b, i1=b+a        }
x := i1 + c
{ i1=a+b, i1=b+a, x=i1+c, x=c+i1 }
i2 := a + b
{ i1=a+b, i1=b+a, x=i1+c, x=c+i1, i2=a+b,
  i2=b+a, i1=i2, i2=i1, x=i2+c, x=c+i2, }
y := i2 + c
{ ... }
```

Since sets of facts and their conjunction are isomorphic we will use them interchangeably

An algorithm for annotating SLP

- Annotate(P, x:=a) = {P} x:=a $F^*[x:=a](P)$

- Annotate(P, S_1; S_2) = {P} S_1; {Q_1} S_2 {Q_2
  - Annotate(P, S_1) = {P} S_1 {Q_1}
  - Annotate(Q_1, S_2) = {Q_1} S_2 {Q_2}

Challenge 2: handling conditions

handling conditions: Goal

[if$_p$] $$\frac{\{\, b \wedge P \,\}\, S_1\, \{\, Q \,\},\quad \{\, \neg b \wedge P \,\}\, S_2\, \{\, Q \,\}}{\{\, P \,\}\ \texttt{if}\ \ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2\ \{\, Q \,\}}$$

- Annotate a program
  `if` b `then` S_1 `else` S_2
  with predicates from Π

$\{ P \}$
`if` b `then`
    $\{ b \wedge P \}$
    S_1
    $\{ Q_1 \}$
`else`
    $\{ \neg b \wedge P \}$
    S_2
    $\{ Q_2 \}$
$\{ Q \}$

handling conditions: Goal

$$[\text{if}_p] \quad \frac{\{\, b \land P \,\}\, S_1 \,\{\, Q \,\},\quad \{\, \neg b \land P \,\}\, S_2 \,\{\, Q \,\}}{\{\, P \,\}\ \texttt{if}\ \ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2\ \{\, Q \,\}}$$

- Annotate a program
  `if` b `then` S_1 `else` S_2
  with predicates from Π
- **Assumption 1:** *P* is given (otherwise use true)
- **Assumption 2:** *b* is a simple binary expression
  e.g., x=y, x≠y, x<y (why?)

$\{\, P \,\}$
`if` b `then`
    $\{\, b \land P \,\}$
    S_1
    $\{\, Q_1 \,\}$
`else`
    $\{\, \neg b \land P \,\}$
    S_2
    $\{\, Q_2 \,\}$
$\{\, Q \,\}$

Annotating conditions

$$[\text{if}_p]\quad \frac{\{\, b \wedge P \,\}\, S_1 \,\{\, Q \,\},\quad \{\, \neg b \wedge P \,\}\, S_2 \,\{\, Q \,\}}{\{\, P \,\}\ \texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2\ \{\, Q \,\}}$$

1. Start with P or $\{b \wedge P\}$ and annotate S_1 (yielding Q_1)
2. Start with P or $\{\neg b \wedge P\}$ and annotate S_2 (yielding Q_2)
3. How do we infer a Q such that $Q_1 \Rightarrow Q$ and $Q_2 \Rightarrow Q$?

$\{\, P \,\}$
`if` b `then`
    $\{\, b \wedge P \,\}$
    S_1
    $\{\, Q_1 \,\}$
`else`
    $\{\, \neg b \wedge P \,\}$
    S_2
    $\{\, Q_2 \,\}$
$\{\, Q \,\}$

Possibly an SAV-fact

Possibly an SAV-fact

Joining predicates

[if$_p$]
$$\frac{\{ b \land P \}\, S_1\, \{ Q \},\quad \{ \neg b \land P \}\, S_2\, \{ Q \}}{\{ P \}\ \texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2\ \{ Q \}}$$

1. Start with P or $\{b \land P\}$ and annotate S_1 (yielding Q_1)
2. Start with P or $\{\neg b \land P\}$ and annotate S_2 (yielding Q_2)
3. How do we infer a Q such that $Q_1 \Rightarrow Q$ and $Q_2 \Rightarrow Q$?

   Q_1=Conj(D_1), Q_2=Conj(D_2)
   Define: $Q = Q_1 \sqcup Q_2$
   $= \text{Conj}(D_1 \cap D_2)$

The join operator for SAV

```
{ P }
if b then
      { b ∧ P }
      S1
      { Q1 }
else
      { ¬b ∧ P }
      S2
      { Q2 }
{ Q }
```

Joining predicates

- Q_1=Conj(D_1), Q_2=Conj(D_2)
- We want to soundly approximate $Q_1 \vee Q_2$ in Π
- Define: $Q = Q_1 \sqcup Q_2$
  $= \text{Conj}(D_1 \cap D_2)$
- Notice that $Q_1 \Rightarrow Q$ and $Q_2 \Rightarrow Q$
  meaning $Q_1 \vee Q_2 \Rightarrow Q$

Handling conditional expressions

- Let *D* be a set of facts and *b* be an expression
- Goal: Elements in Π that soundly approximate
  - $D \wedge bexpr$
  - $D \wedge \neg bexpr$
- Technique: Add statement `assume` *bexpr*

  $\langle \texttt{assume}\ bexpr, s\rangle \Rightarrow^{sos} s$ if $\mathcal{B}[\![bexpr]\!]\, s = \mathbf{tt}$

- Find a function F[`assume` *bexpr*] : $\Pi \to \Pi$

  $\text{Conj}(D) \wedge bexpr \Rightarrow \text{Conj}(\text{F}[\texttt{assume}\ bexpr])$

Handling conditional expressions

- F[`assume` ***bexpr***] : $\Pi \rightarrow \Pi$ such that
  Conj(*D*) $\wedge$ ***bexpr*** $\Rightarrow$ Conj(F[`assume` ***bexpr***])

- β(***bexpr***) = if ***bexpr*** is an SAV-fact then {***bexpr***} else {}
  - Notice ***bexpr*** $\Rightarrow$ β(***bexpr***)
  - Examples
    - β(y=z) = {y=z}
    - β(y<z) = {}

- F[`assume` ***bexpr***](*D*) = *D* $\cup$ β(***bexpr***)

Example

```
{               }
if (x = y)
    {           }
    a := b + c
    {                           }
    d := b - c
    {                           }
else
    {               }
    a := b + c
    {               }
    d := b + c
    {                                         }
{               }
```

Example

```
{                  }
if (x = y)
     { x=y, y=x }
     a := b + c
     { x=y, y=x, a=b+c, a=c+b }
     d := b - c
     { x=y, y=x, a=b+c, a=c+b }
else
     {                  }
     a := b + c
     { a=b+c, a=c+b }
     d := b + c
     { a=b+c, a=c+b, d=b+c, d=c+b, a=d, d=a }
{ a=b+c, a=c+b }
```

Recap

- We now have an **algorithm** for soundly annotating loop-free code
- Generates forward-going proofs
- Algorithm operates on abstract syntax tree of code
  - Handles straight-line code by applying F^*
  - Handles conditions by recursively annotating true and false branches and then intersecting their postconditions

An algorithm for conditions

- Annotate(P, `if` *bexpr* `then` S_1 `else` S_2) =
  {P}
  `if` *bexpr* `then` S_1 `else` S_2
  $Q_1 \sqcup Q_2$
  - Annotate($P \cup \beta(bexpr)$, S_1) =
    F[`assume` *bexpr*](P) S_1 {Q_1}
  - Annotate($P \cup \beta(\neg bexpr)$, S_2)=
    F[`assume` $\neg$*bexpr*](P) S_2 {Q_2}

Example

```
{                 }
if (x = y)
    { x=y, y=x }
    a := b + c
    { x=y, y=x, a=b+c, a=c+b }
    d := b - c
    { x=y, y=x, a=b+c, a=c+b }
else
    {                 }
    a := b + c
    { a=b+c, a=c+b }
    d := b + c
    { a=b+c, a=c+b, d=b+c, d=c+b, a=d, d=a }
{ a=b+c, a=c+b }
```

Challenge 2: handling loops

Challenge 2: handling loops

handling loops: Goal

[while$_p$]

$$\frac{\{bexpr \wedge P\}\ S\ \{P\}}{\{P\}\ \texttt{while}\ \ b\ \texttt{do}\ S\ \{\neg bexpr \wedge P\}}$$

$\{P\}$

Inv = $\{N\}$

`while` *bexpr* `do`

$\{bexpr \wedge N\}$

S

$\{Q\}$

$\{\neg bexpr \wedge N\}$

handling loops: Goal

$$[\text{while}_p] \quad \frac{\{ bexpr \land P \}\ S\ \{ P \}}{\{ P \}\ \texttt{while}\ \ b\ \texttt{do}\ S\ \{\neg bexpr \land P \}}$$

- Annotate a program `while` *bexpr* `do` *S* with predicates from Π
  - s.t. $P \Rightarrow N$
- **Main challenge:** find *N*
- **Assumption 1:** *P* is given (otherwise use true)
- **Assumption 2:** *bexpr* is a simple binary expression

$\{ P \}$
Inv = $\{ N \}$
`while` *bexpr* `do`
    $\{ bexpr \land N \}$
    S
    $\{ Q \}$
$\{\neg bexpr \land N \}$

Example: annotate this program

```
{ y=x+a, y=a+x, w=d, d=w }
Inv = {                                          }
while (x ≠ z) do
    {                                            }
    x := x + 1
    {
                                        }
    y := x + a
    {                                            }
    d := x + a
    {                                            }
{                                                }
```

Example: annotate this program

```
{ y=x+a, y=a+x, w=d, d=w }
Inv = { y=x+a, y=a+x }
while (x ≠ z) do
    { y=x+a, y=a+x }
    x := x + 1
    { }
    y := x + a
    { y=x+a, y=a+x }
    d := x + a
    { y=x+a, y=a+x, d=x+a, d=a+x, y=d, d=y }
{ y=x+a, y=a+x, x=z, z=x }
```

handling loops: Idea

[while$_p$] $$\frac{\{bexpr \wedge P\}\, S\, \{P\}}{\{P\}\ \texttt{while}\ \ b\ \texttt{do}\ S\ \{\neg bexpr \wedge P\}}$$

- **Idea:** try to guess a loop invariant from a small number of loop unrollings
  - We know how to annotate S (by induction)

$\{P\}$
Inv = $\{N\}$
`while` *bexpr* `do`
    $\{bexpr \wedge N\}$
    S
    $\{Q\}$
$\{\neg bexpr \wedge N\}$

k-loop unrolling

```
{ P }
Inv = { N }
while (x ≠ z) do
     x := x + 1
     y := x + a
     d := x + a
```

```
{ y=x+a, y=a+x, w=d, d=w }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q1 = {        }
```

```
{ P }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q1 = {        }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q2 = {        }
```

…

k-loop unrolling

```
{ P }
Inv = { N }
while (x ≠ z) do
      x := x + 1
      y := x + a
      d := x + a
```

⟺

```
{ y=x+a, y=a+x, w=d, d=w }
if (x ≠ z)
      x := x + 1
      y := x + a
      d := x + a
Q1 = { y=x+a, y=a+x }
```

```
{ P }
if (x ≠ z)
      x := x + 1
      y := x + a
      d := x + a
Q1 = { y=x+a, y=a+x }
if (x ≠ z)
      x := x + 1
      y := x + a
      d := x + a
Q2 = { y=x+a, y=a+x }
```

…

k-loop unrolling

```
{ P }
Inv = { N }
while (x ≠ z) do
     x := x + 1
     y := x + a
     d := x + a
```

The following must hold:

$P \Rightarrow N$

$Q_1 \Rightarrow N$

$Q_2 \Rightarrow N$

…

$Q_k \Rightarrow N$

```
{ y=x+a, y=a+x, w=d, d=w }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q1 = { y=x+a, y=a+x }
```

```
{ P }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q1 = { y=x+a, y=a+x }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q2 = { y=x+a, y=a+x }
```

…

k-loop unrolling

```
{ P }
Inv = { N }
while (x ≠ z) do
     x := x + 1
     y := x + a
     d := x + a
```

```
{ y=x+a, y=a+x, w=d, d=w }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q1 = { y=x+a, y=a+x }
```

```
{ P }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q1 = { y=x+a, y=a+x }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q2 = { y=x+a, y=a+x }
```

...

The following must hold:

$P \Rightarrow N$

$Q_1 \Rightarrow N$

$Q_2 \Rightarrow N$

…

$Q_k \Rightarrow N$

…

Observation 1: No need to explicitly unroll loop – we can reuse postcondition from unrolling k-1 for k

We can compute the following sequence:

$N_0 = P$

$N_1 = N_1 \sqcup Q_1$

$N_2 = N_1 \sqcup Q_2$

…

$N_k = N_{k-1} \sqcup Q_k$

k-loop unrolling

```
{ P }
Inv = { N }
while (x ≠ z) do
     x := x + 1
     y := x + a
     d := x + a
```

```
{ y=x+a, y=a+x, w=d, d=w }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q1 = { y=x+a, y=a+x }
```

The following must hold:

$P \Rightarrow N$
$Q_1 \Rightarrow N$
$Q_2 \Rightarrow N$
…
$Q_k \Rightarrow N$
…

Observation 2: N_k monotonically decreases set of facts.
Question: does it stabilizes for some k?

We can compute the following sequence:

$N_0 = P$
$N_1 = N_1 \sqcup Q_1$
$N_2 = N_1 \sqcup Q_2$
…
$N_k = N_{k-1} \sqcup Q_k$

```
{ P }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q1 = { y=x+a, y=a+x }
if (x ≠ z)
     x := x + 1
     y := x + a
     d := x + a
Q2 = { y=x+a, y=a+x }
```

…

Algorithm for annotating a loop

Annotate(P, `while` *bexpr* `do S`) =

Initialize $N' := N_c := P$

repeat

let Annotate(P, **`if b then S else skip`**) be

$\{N_c\}$ **`if`** *bexpr* **`then`** *S* **`else skip`** $\{N'\}$

$N_c := N_c \sqcup N'$

until $N' = N_c$

return {P}

INV= N'

`while` *bexpr* `do`

F[`assume` *bexpr*](*N*)

Annotate(F[`assume` *bexpr*](*N*), S)

F[`assume` ¬*bexpr*](*N*)

A technical issue

- Unrolling loops is quite inconvenient and inefficient (but we can avoid it as we just saw)
- How do we handle more complex control-flow constructs, e.g., `goto` , `break`, exceptions…?
- **Solution:** model control-flow by labels and `goto` statements
- Would like a dedicated data structure to explicitly encode control flow in support of the analysis
- **Solution:** control-flow graphs (CFGs)

Intermediate language example

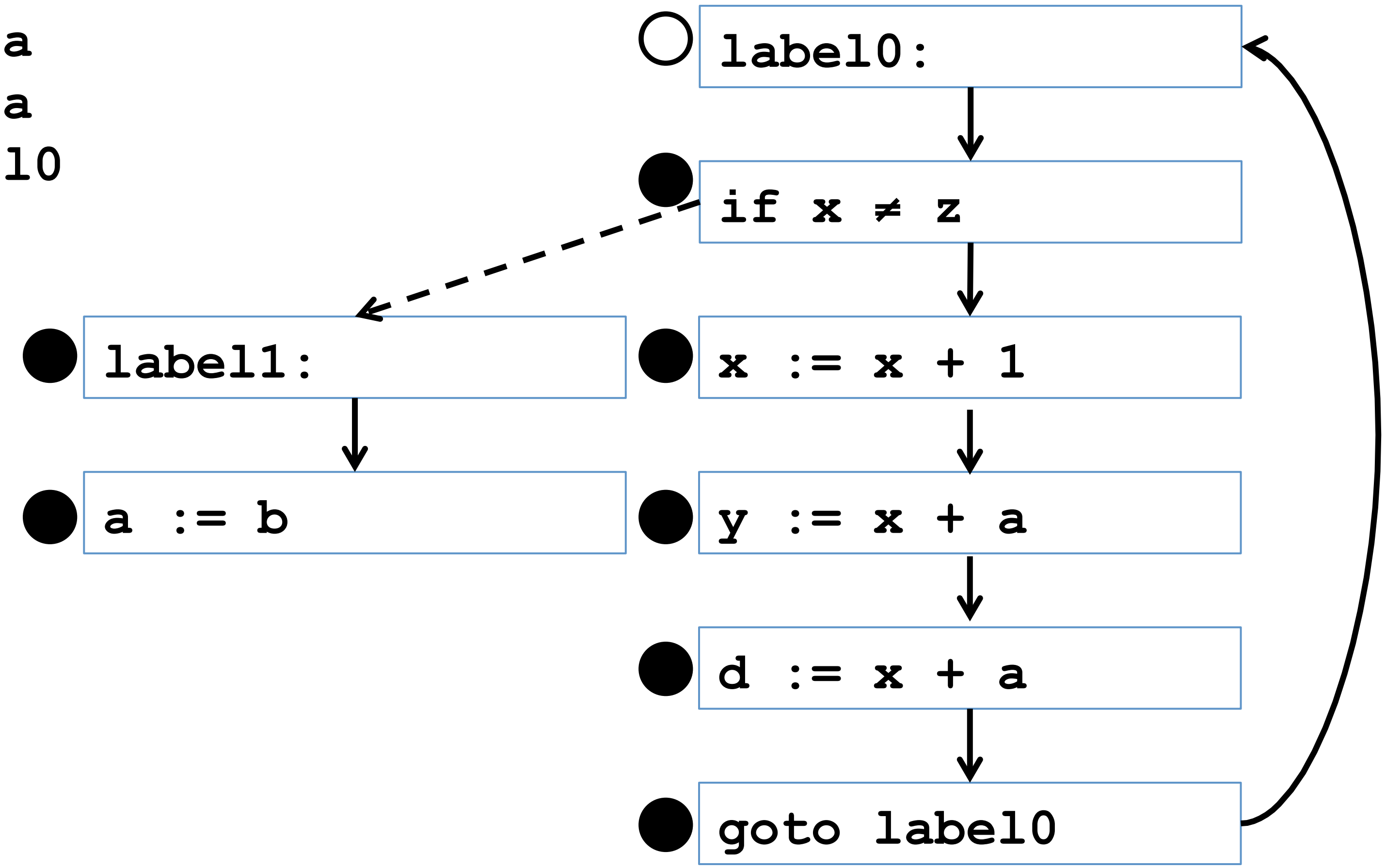
```
while (x ≠ z) do
    x := x + 1
    y := x + a
    d := x + a
a := b
```

```
label0:
   if x ≠ z goto label1
   x := x + 1
   y := x + a
   d := x + a
   goto label0

label1:
   a := b
```

Control-flow graph example

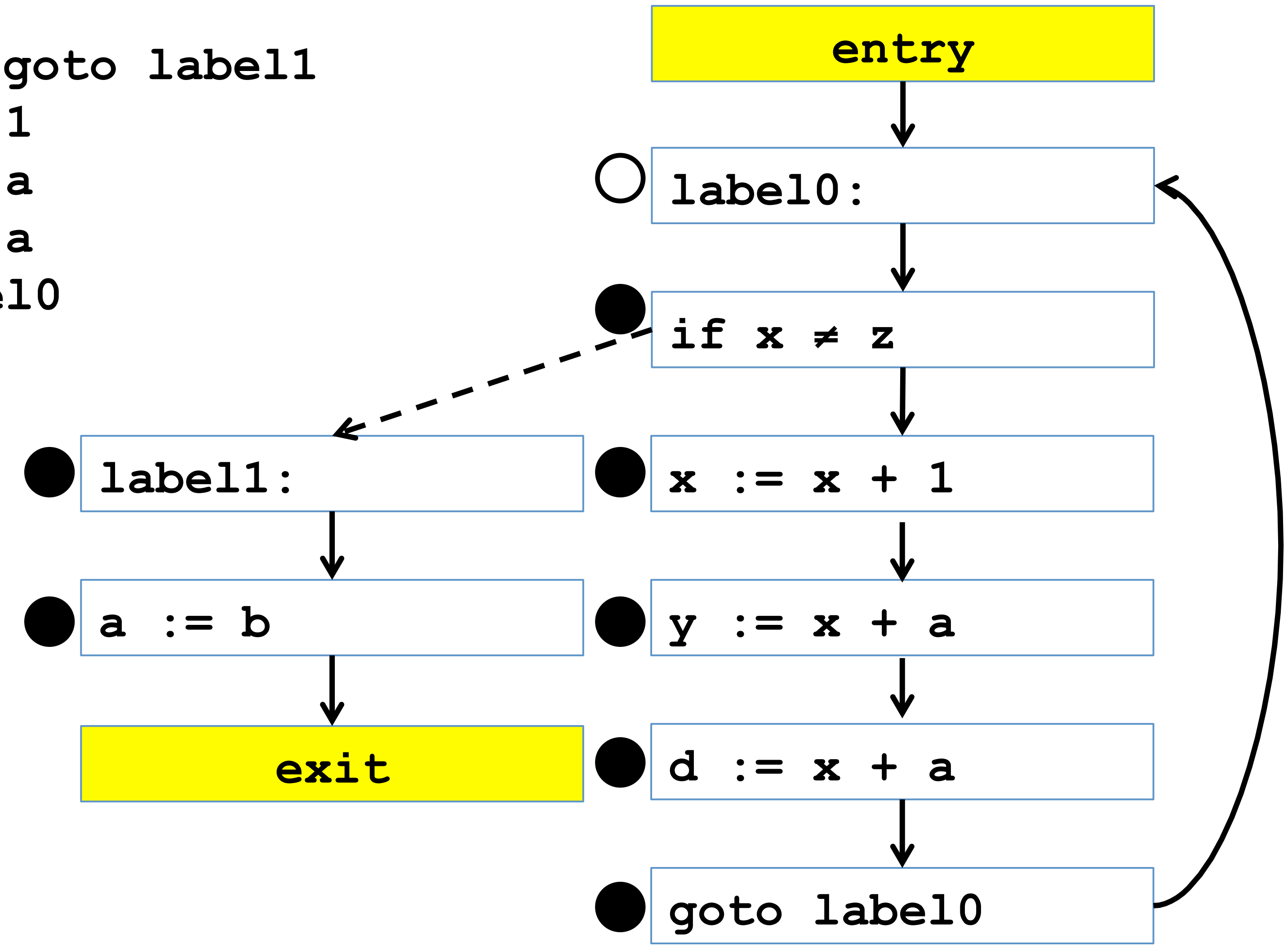
line number

```
label0:
   if x ≠ z goto label1
   x := x + 1
   y := x + a
   d := x + a
   goto label0

label1:
   a := b
```



Control-flow graph example

```
○ label0:
●    if x ≠ z goto label1
●    x := x + 1
●    y := x + a
●    d := x + a
●    goto label0

● label1:
●    a := b
```

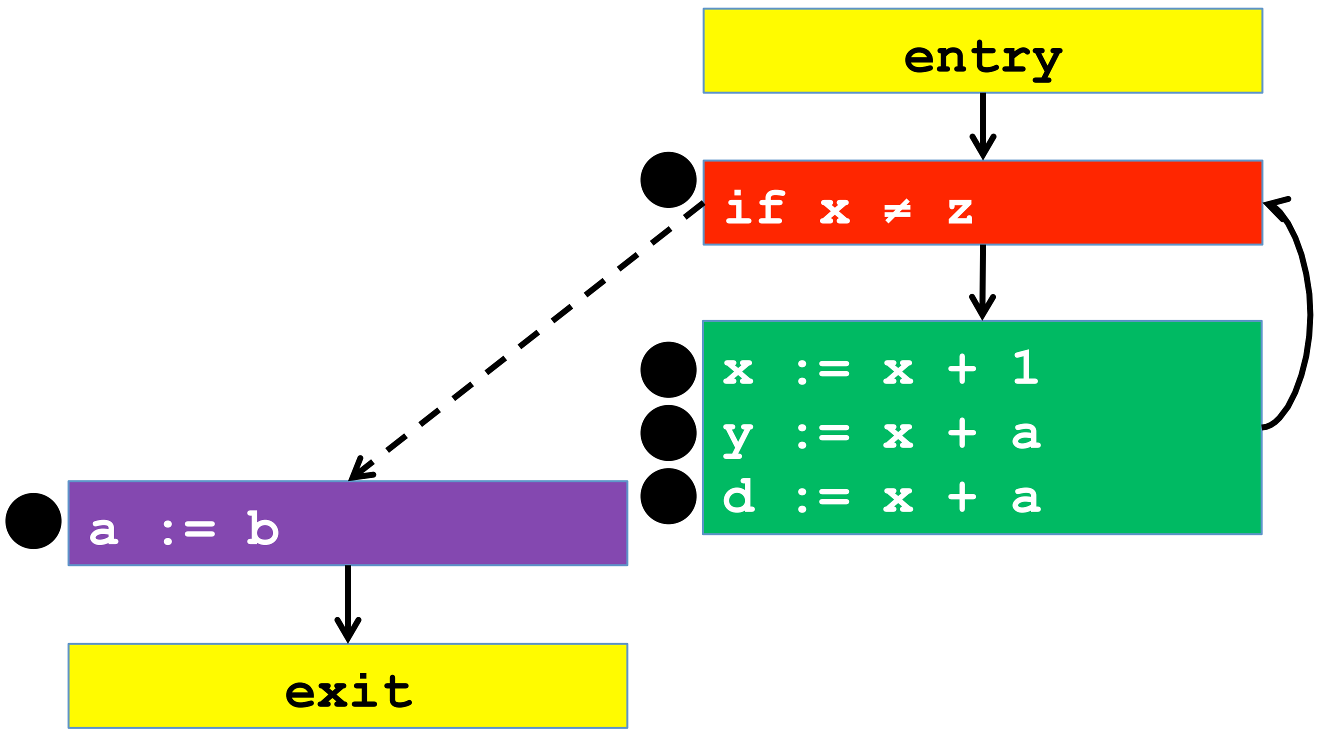


Control-flow graph

- Node are statements or labels
- Special nodes for entry/exit
- A edge from node *v* to node *w* means that after executing the statement of *v* control passes to *w*
  - Conditions represented by splits and join node
  - Loops create cycles
- Can be generated from abstract syntax tree in linear time
  - Automatically taken care of by the front-end
- Usage: store analysis results in CFG nodes

CFG with Basic Blocks

- Stores basic blocks in a single node
- Extended blocks – maximal connected loop-free subgraphs



Next lecture:

abstract interpretation 4

Previously

- Available Expressions analysis
  - Abstract transformer for assignments
  - Processing serial composition
  - Processing conditions
  - Processing loop
- Control-flow graphs

CSE optimization

```
{ x = a + b }
  y := a + b
```

CSE →

```
y := x
```

Semantic domain

- Define *factoids*:
  $\theta = \{ x = y \mid x, y \in \text{Var} \} \cup$
  $\{ x = y + z \mid x, y, z \in \text{Var} \}$
- Define *predicates* as $\Pi = 2^{\theta}$
- Treat conjunctive formulas as sets of factoids
  {a=b, c=b+d, b=c} ~ (a=b) $\wedge$ (c=b+d) $\wedge$ (b=c)

Defining an SAV abstract transformer

- **Goal:** define a function $F^{SAV}[x:=aexpr] : \Pi \rightarrow \Pi$ s.t.
  if $F^{SAV}[x:=aexpr](D) = D'$
  then **sp**(x :=$aexpr$, D) $\Rightarrow$ D'
- **Idea:** define rules for individual facts and generalize to sets of facts by the conjunction rule

[kill-lhs] $\{ x=\omega \}$ x:= $aexpr$ $\{ \}$

ω Is either a variable v or an addition expression $v+w$

[kill-rhs-1] $\{ y=x+w \}$ x:= $aexpr$ $\{ \}$

[kill-rhs-2] $\{ y=w+x \}$ x:= $aexpr$ $\{ \}$

[gen] $\{ \}$ x:= ω $\{ x=\omega \}$

[preserve] $\{ y=z+w \}$ x:= $aexpr$ $\{ y=z+w \}$

Defining a reduction

- For an SAV-predicate D define *Explicate*(D) = minimal set D^* such that:
  1. $D \subseteq D^*$
  2. x=y $\in D^*$ implies y=x $\in D^*$
  3. x=y $\in D^*$ y=z $\in D^*$ implies x=z $\in D^*$
  4. x=y+z $\in D^*$ implies x=z+y $\in D^*$
  5. x=y $\in D^*$ and x=z+w $\in D^*$ implies y=z+w $\in D^*$
  6. x=y $\in D^*$ and z=x+w $\in D^*$ implies z=y+w $\in D^*$
  7. x=z+w $\in D^*$ and y=z+w $\in D^*$ implies x=y $\in D^*$
- **Define:** $F^*[x{:=}aexpr] = Explicate \circ F^{SAV}[x{:=}aexpr]$

Recap

An algorithm for annotating SLP

Annotate(P, S_1; S_2) =

let Annotate(P, S_1) be {P} A_1 {Q_1}

let Annotate(Q_1, S_2) be {Q_1} A_2 {Q_2}

return {P} A_1; {Q_1} A_2 {Q_2}

Handling conditional expressions

- We want to soundly approximate $D \land$ *bexpr* and $D \land \neg$*bexpr* in Π
- Define an artificial statement `assume` *bexpr*

  $\langle$`assume` ***bexpr*, *s***$\rangle \Rightarrow^{sos}$ ***s*** if $\mathcal{B}[\![bexpr]\!]\, s = \mathbf{tt}$
- Define β(*bexpr*) = if *bexpr* is factoid {*bexpr*} else {}
- Define F[`assume` *bexpr*](*D*) = *D* $\cup$ β(*bexpr*)
- Can sharpen

  F^*[`assume` *bexpr*] = *Explicate* ° F^{SAV}[`assume` *bexpr*]

An algorithm for annotating conditions

let P_t = F^*[`assume` *bexpr*] P
let P_f = F^*[`assume` ¬*bexpr*] P
let Annotate(P_t, S_1) be {P_t} A_1 {Q_1}
let Annotate(P_f, S_2) be {P_f} A_2 {Q_2}
return {P}

```
if bexpr then
    {Pt} A1 {Q1}
else
    {Pf} A2 {Q2}
{Q1 ⊔ Q2}
```

Algorithm for annotating loops

Annotate(P, `while` *bexpr* `do` S) =

N' := N_c := P // Initialize

repeat

let P_t = F[`assume` *bexpr*] N_c

let Annotate(P_t, S) be $\{N_c\}$ A_{body} $\{N'\}$

$N_c := N_c \sqcup N'$

until $N' = N_c$

return $\{P\}$

INV= $\{N'\}$

`while` *bexpr* `do`

$\{P_t\}$

A_{body}

{F[`assume` ¬*bexpr*](N)}

Algorithm for annotating a program

Annotate(P, S) =

case S is x:=*aexpr*

return {P} x:=*aexpr* {F^*[x:=*aexpr*] P}

case S **is** S_1; S_2

let Annotate(P, S_1) be {P} A_1 {Q_1}

let Annotate(Q_1, S_2) be {Q_1} A_2 {Q_2}

return {P} A_1; {Q_1} A_2 {Q_2}

case S **is** `if` *bexpr* `then` S_1 `else` S_2

let P_t = F[`assume` *bexpr*] P

let P_f = F[`assume` ¬*bexpr*] P

let Annotate(P_t, S_1) be {P_t} A_1 {Q_1}

let Annotate(P_f, S_2) be {P_f} A_2 {Q_2}

return {P} `if` *bexpr* `then` {P_t} A_1 {Q_1}
`else` {P_f} A_2 {Q_2}
{$Q_1 \sqcup Q_2$}

case S **is** `while` *bexpr* `do` S

N' := N_c := P // Initialize

repeat

let P_t = F[`assume` *bexpr*] N_c

let Annotate(P_t, S) be {N_c} A_{body} {N'}

$N_c := N_c \sqcup N'$

until $N' = Nc$

return {P} INV= {N'} `while` *bexpr* `do` {P_t} A_{body} {F[`assume` ¬*bexpr*](N)}

Exercise: apply algorithm

```
{               }
y := a+b
{               }
x := y
{                       }
while (x≠z) do
    {                                          }
    w := a+b
    {                                          }
    x := a+b
    {                                          }
    a := z
    {                                          }
```

Step 1/18

```
{}
y := a+b
{ y=a+b }*
x := y
while (x≠z) do
    w := a+b
    x := a+b
    a := z
```

Not all factoids are shown – apply *Explicate* to get all factoids

Step 2/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
while (x≠z) do
   w := a+b
   x := a+b
   a := z
```

Step 3/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv' = { y=a+b, x=y, x=a+b }*
while (x≠z) do
   w := a+b
   x := a+b
   a := z
```

Step 4/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv' = { y=a+b, x=y, x=a+b }*
while (x≠z) do
   { y=a+b, x=y, x=a+b }*
   w := a+b
   x := a+b
   a := z
```

Step 5/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv' = { y=a+b, x=y, x=a+b }*
while (x≠z) do
    { y=a+b, x=y, x=a+b }*
    w := a+b
    { y=a+b, x=y, x=a+b, w=a+b, w=x, w=y }*
    x := a+b
    a := z
```

Step 6/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv’ = { y=a+b, x=y, x=a+b }*
while (x≠z) do
    { y=a+b, x=y, x=a+b }*
    w := a+b
    { y=a+b, x=y, x=a+b, w=a+b, w=x, w=y }*
    x := a+b
    { y=a+b, w=a+b, w=y, x=a+b, w=x, x=y }*
    a := z
```

Step 7/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv' = { y=a+b, x=y, x=a+b }*
while (x≠z) do
   { y=a+b, x=y, x=a+b }*
   w := a+b
   { y=a+b, x=y, x=a+b, w=a+b, w=x, w=y }*
   x := a+b
   { y=a+b, w=a+b, w=y, x=a+b, w=x, x=y }*
   a := z
   { w=y, w=x, x=y, a=z }*
```

Step 8/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv'' = { x=y }*
while (x≠z) do
    { y=a+b, x=y, x=a+b }*
    w := a+b
    { y=a+b, x=y, x=a+b, w=a+b, w=x, w=y }*
    x := a+b
    { y=a+b, w=a+b, w=y, x=a+b, w=x, x=y }*
    a := z
    { w=y, w=x, x=y, a=z }*
```

Step 9/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv'' = { x=y }*
while (x≠z) do
    { x=y }*
    w := a+b
    { y=a+b, x=y, x=a+b, w=a+b, w=x, w=y }*
    x := a+b
    { y=a+b, w=a+b, w=y, x=a+b, w=x, x=y }*
    a := z
    { w=y, w=x, x=y, a=z }*
```

Step 10/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv'' = { x=y }*
while (x≠z) do
    { x=y }*
    w := a+b
    { x=y, w=a+b }*
    x := a+b
    { y=a+b, w=a+b, w=y, x=a+b, w=x, x=y }*
    a := z
    { w=y, w=x, x=y, a=z }*
```

Step 11/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv'' = { x=y }*
while (x≠z) do
    { x=y }*
    w := a+b
    { x=y, w=a+b }*
    x := a+b
    { x=a+b, w=a+b, w=x }*
    a := z
    { w=y, w=x, x=y, a=z }*
```

Step 12/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv'' = { x=y }*
while (x≠z) do
    { x=y }*
    w := a+b
    { x=y, w=a+b }*
    x := a+b
    { x=a+b, w=a+b, w=x }*
    a := z
    { w=x, a=z }*
```

Step 13/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv''' = { }
while (x≠z) do
    { x=y }*
    w := a+b
    { x=y, w=a+b }*
    x := a+b
    { x=a+b, w=a+b, w=x }*
    a := z
    { w=x, a=z }*
```

Step 14/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv''' = { }
while (x≠z) do
    { }
    w := a+b
    { x=y, w=a+b }*
    x := a+b
    { x=a+b, w=a+b, w=x }*
    a := z
    { w=x, a=z }*
```

Step 15/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv''' = { }
while (x≠z) do
    { }
    w := a+b
    { w=a+b }*
    x := a+b
    { x=a+b, w=a+b, w=x }*
    a := z
    { w=x, a=z }*
```

Step 16/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv''' = { }
while (x≠z) do
    { }
    w := a+b
    { w=a+b }*
    x := a+b
    { x=a+b, w=a+b, w=x }*
    a := z
    { w=x, a=z }*
```

Step 17/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv''' = { }
while (x≠z) do
    { }
    w := a+b
    { w=a+b }*
    x := a+b
    { x=a+b, w=a+b, w=x }*
    a := z
    { w=x, a=z }*
```

Step 18/18

```
{}
y := a+b
{ y=a+b }*
x := y
{ y=a+b, x=y, x=a+b }*
Inv = { }
while (x≠z) do
   { }
   w := a+b
   { w=a+b }*
   x := a+b
   { x=a+b, w=a+b, w=x }*
   a := z
   { w=x, a=z }*
{ x=z }
```

Another Example

Constant Propagation

- **Optimization:** constant folding

simplifies constant expressions

`{ x=c }`
`y := aexpr` → **constant folding** → `y := eval(aexpr[c/x])`

  - Example: `x:=7; y:=x*9`
    transformed to: `x:=7; y:=7*9`
    and then to: `x:=7; y:=63`

- **Analysis:** constant propagation (CP)
  - Infers facts of the form *x*=*c*

CP semantic domain

?

CP semantic domain

- Define CP-*factoids*:
  $\theta = \{ x = c \mid x \in \text{Var}, c \in Z \}$
  - How many factoids are there?
- Define *predicates* as $\Pi = 2^\theta$
  - How many predicates are there?
  - Do all predicates make sense? $(x=5) \wedge (x=7)$
- Treat conjunctive formulas as sets of factoids

  $\{x=5, y=7\} \sim (x=5) \wedge (y=7)$

CP abstract transformer

- **Goal:** define a function
  $F^{CP}[x{:=}aexpr] : \Pi \rightarrow \Pi$ such that
  if $F^{CP}[x{:=}aexpr]\ P = P'$
  then **sp**(x:=$aexpr$, P) $\Rightarrow$ P'

?

CP abstract transformer

- **Goal:** define a function
  $F^{CP}[x{:=}aexpr] : \Pi \rightarrow \Pi$ such that
  if $F^{CP}[x{:=}aexpr]\ P = P'$
  then **sp**($x{:=}aexpr$, P) $\Rightarrow P'$

[kill]	$\{ x{=}c \}\ x{:=}aexpr\ \{ \}$
[gen-1]	$\{ \}\ x{:=}c\ \{ x{=}c \}$
[gen-2]	$\{ y{=}c, z{=}c' \}\ x{:=}y \text{ op } z\ \{ x{=}c \text{ op } c' \}$
[preserve]	$\{ y{=}c \}\ x{:=}aexpr\ \{ y{=}c \}$

Gen-kill formulation of transformers

- Suited for analysis propagating sets of factoids
  - Available expressions,
  - Constant propagation, etc.
- For each statement, define a set of killed factoids and a set of generated factoids
  F[S] P = (P \ kill(S)) ∪ gen(S)
  - F^{CP}[*x*:=*aexpr*] P = (P \ {*x*=*c*}) *aexpr* is not a constant
  - F^{CP}[*x*:=*k*] P = (P \ {*x*=*c*}) ∪ {*x*=*k*}
- Used in dataflow analysis – a special case of abstract interpretation

Does this still work?

Annotate(P, S_1; S_2) =

let Annotate(P, S_1) be {P} A_1 {Q_1}

let Annotate(Q_1, S_2) be {Q_1} A_2 {Q_2}

return {P} A_1; {Q_1} A_2 {Q_2}

Handling conditional expressions

- We want to soundly approximate $D \wedge$ *bexpr* and $D \wedge \neg$*bexpr* in Π
- Define an artificial statement `assume` *bexpr*

  $\langle$`assume` *bexpr*, *s*$\rangle \Rightarrow^{sos}$ *s* if $\mathcal{B}[\![bexpr]\!]\, s$ = **tt**
- Define β(*bexpr*) = if *bexpr* is CP-factoid {*bexpr*} else {}
- Define F[`assume` *bexpr*](*D*) = *D* $\cup$ β(*bexpr*)

Does this still work?

let P_t = F[`assume` *bexpr*] *P*
let P_f = F[`assume` ¬*bexpr*] *P*
let Annotate(P_t, S_1) be {P_t} A_1 {Q_1}
let Annotate(P_f, S_2) be {P_f} A_2 {Q_2}
return {*P*}

`if` *bexpr* `then`
{P_t} A_1 {Q_1}
`else`
{P_f} A_2 {Q_2}
{$Q_1 \sqcup Q_2$}

How do we define join for CP?

Join example

- $\{x=5, y=7\} \sqcup \{x=3, y=7, z=9\} =$

Does this still work?

Annotate(P, `while` *bexpr* `do` S) =
  N' := N_c := P // Initialize
  repeat
    let P_t = F[`assume` *bexpr*] N_c
    let Annotate(P_t, S) be {N_c} A_{body} {N'}
    N_c := $N_c \sqcup N'$
  until N' = Nc
  return {P} INV= {N'} `while` *bexpr* `do` {P_t} A_{body} {F[`assume` $\neg$*bexpr*](N)}

- What about correctness?
- What about termination?

Does this still work?

Annotate(P, `while` *bexpr* `do` S) =
  N' := N_c := P // Initialize
  repeat
    let P_t = F[`assume` *bexpr*] N_c
    let Annotate(P_t, S) be {N_c} A_{body} {N'}
    N_c := $N_c \sqcup N'$
  until N' = Nc
  return {P} INV= {N'} `while` *bexpr* `do` {P_t} A_{body} {F[`assume` $\neg$*bexpr*](N)}

- What about correctness?
  - If loop terminates then is N' a loop invariant?
- What about termination?

A termination principle

- $g : X \rightarrow X$ is a function
- How can we determine whether the sequence $x_0, x_1 = g(x_0), \ldots, x_{k+1}=g(x_k),\ldots$ stabilizes?
- Technique:
  1. Find ranking function rank : $X \rightarrow \mathbb{N}$
     (that is show that rank(x) $\geq$ 0 for all x)
  2. Show that if $x \neq g(x)$
     then rank($g(x)$) < rank(x)

Rank function for available expressions

- rank(P) = ?

Rank function for available expressions

Annotate(P, `while` *bexpr* `do` S) =
N' := N_c := P // Initialize
repeat
let P_t = F[`assume` *bexpr*] N_c
let Annotate(P_t, S) be {N_c} A_{body} {N'}
$N_c := N_c \sqcup N'$
until $N' = Nc$
return {P} INV= {N'} `while` *bexpr* `do` {P_t} A_{body} {F[`assume` ¬*bexpr*](N)}

- rank(P) = |P|
  number of factoids
- Prove that either $N_c = N_c \sqcup N'$
  or $\text{rank}(N_c \sqcup N') <^? \text{rank}(N_c)$

Rank function for constant propagation

Annotate(P, `while` *bexpr* `do` S) =
  N' := N_c := P // Initialize
  repeat
    let P_t = F[`assume` *bexpr*] N_c
    let Annotate(P_t, S) be $\{N_c\}$ A_{body} $\{N'\}$
    $N_c := N_c \sqcup N'$
  until $N' = Nc$
  return $\{P\}$ INV= $\{N'\}$ `while` *bexpr* `do` $\{P_t\}$ A_{body} {F[`assume` ¬*bexpr*](N)}

- rank(P) = ?

- Prove that either $N_c = N_c \sqcup N'$
  or rank(N_c) $>^?$ rank($N_c \sqcup N'$)

Rank function for constant propagation

Annotate(P, `while` *bexpr* `do` S) =
  N' := N_c := P // Initialize
  repeat
    let P_t = F[`assume` *bexpr*] N_c
    let Annotate(P_t, S) be $\{N_c\}$ A_{body} $\{N'\}$
    $N_c := N_c \sqcup N'$
  until $N' = Nc$
  return $\{P\}$ INV= $\{N'\}$ `while` *bexpr* `do` $\{P_t\}$ A_{body} {F[`assume` ¬*bexpr*](N)}

- rank(P) = |P|
  number of factoids
- Prove that either $N_c = N_c \sqcup N'$
  or $\text{rank}(N_c) >^? \text{rank}(N_c \sqcup N')$

What were the common elements?

- Two static analyses
  - Available Expressions (extended with equalities)
  - Constant Propagation
- Semantic domain
  - An approximation relation ⇒
    - A weaker one given by set inclusion
  - Join operator
- Abstract transformers for basic statements
  - Assignments
  - **`assume`** statements
- Initial precondition

Orders (Reminder)

Preorder

- Let D be a set of elements
- We say that a binary order relation $\sqsubseteq$ over D is a preorder if the following conditions hold for every d, d', d'' $\in D$
  - Reflexive: $d \sqsubseteq d$
  - Transitive: $d \sqsubseteq d'$ and $d' \sqsubseteq d''$ implies $d \sqsubseteq d''$
- There may exist d, d' such that

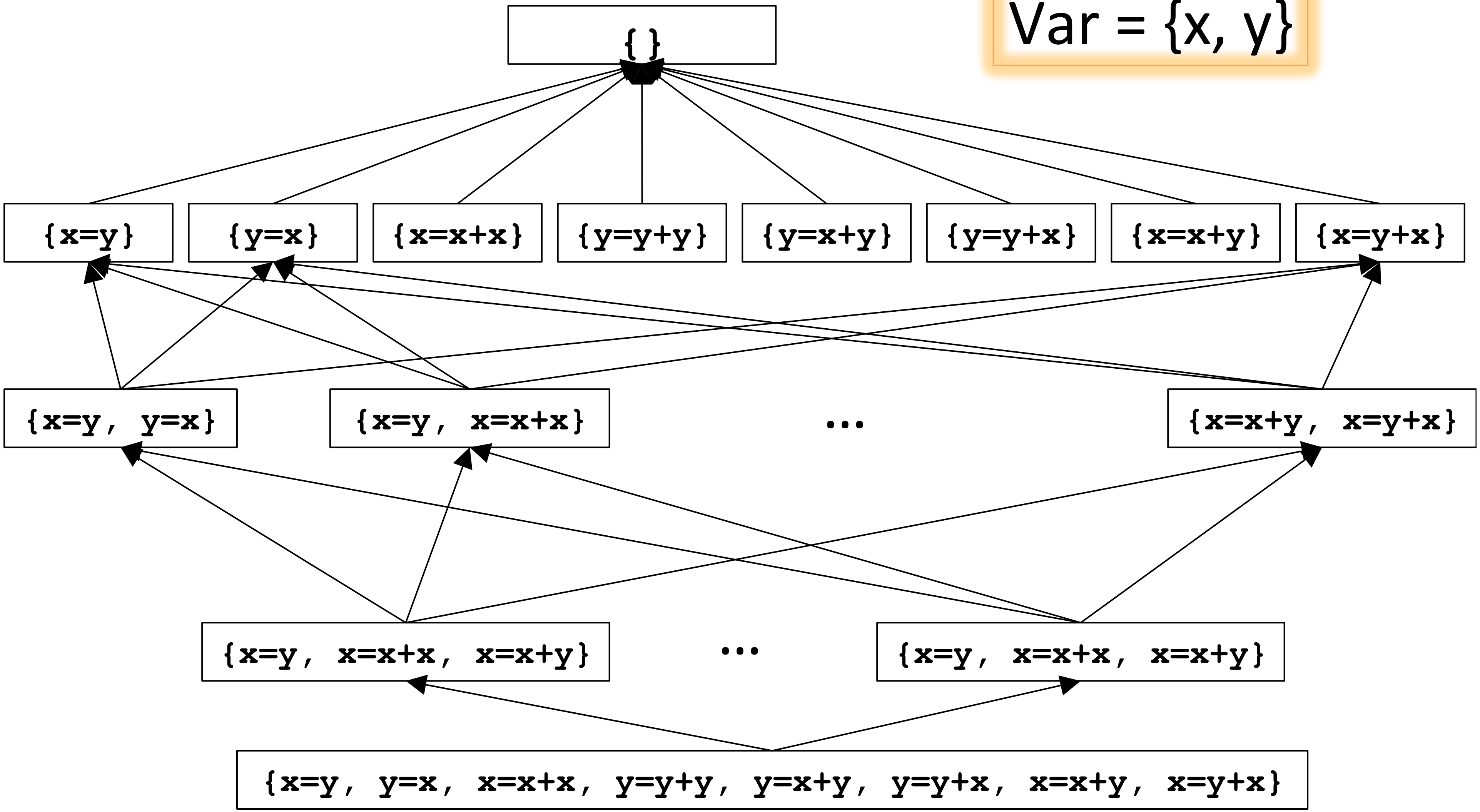
  $d \sqsubseteq d'$ and $d' \sqsubseteq d$ yet $d \neq d'$

Preorder examples

- SAV-predicates
  - SAV-factoids
    $\theta = \{ x = y \mid x, y \in \text{Var} \} \cup \{ x = y + z \mid x, y, z \in \text{Var} \}$
  - SAV-predicates $\Pi = 2^{\theta}$
  - Order relation 1: $P_1 \sqsubseteq^{set} P_2$ iff $P_1 \supseteq P_2$
  - Order relation 2: $P_1 \sqsubseteq^{imp} P_2$ iff $P_1 \Rightarrow P_2$
  - Which order relation is stronger (contains more pairs)?
  - Which order relation is easier to check?

SAV preorder 1: $P_1 \sqsubseteq^{set} P_2$ iff $P_1 \supseteq P_2$

Var = {x, y}



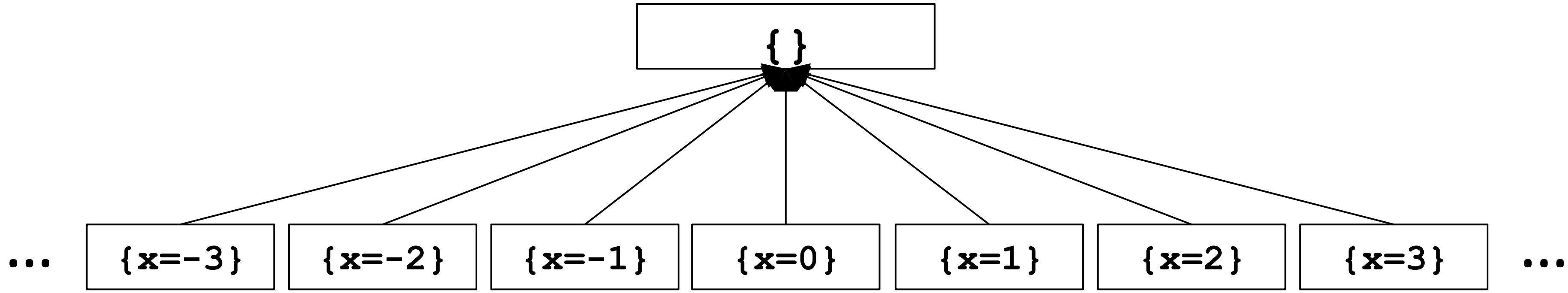
Preorder examples

- CP-predicates
  - CP-factoids
    $\theta = \{ x = c \mid x \in \text{Var}, c \in \text{Z} \}$
  - CP-predicates $\Pi = 2^{\theta}$
  - Order relation 1: $P_1 \sqsubseteq^{set} P_2$ iff $P_1 \supseteq P_2$
  - Order relation 2: $P_1 \sqsubseteq^{imp} P_2$ iff $P_1 \Rightarrow P_2$
  - Is there a difference?

Preorder examples

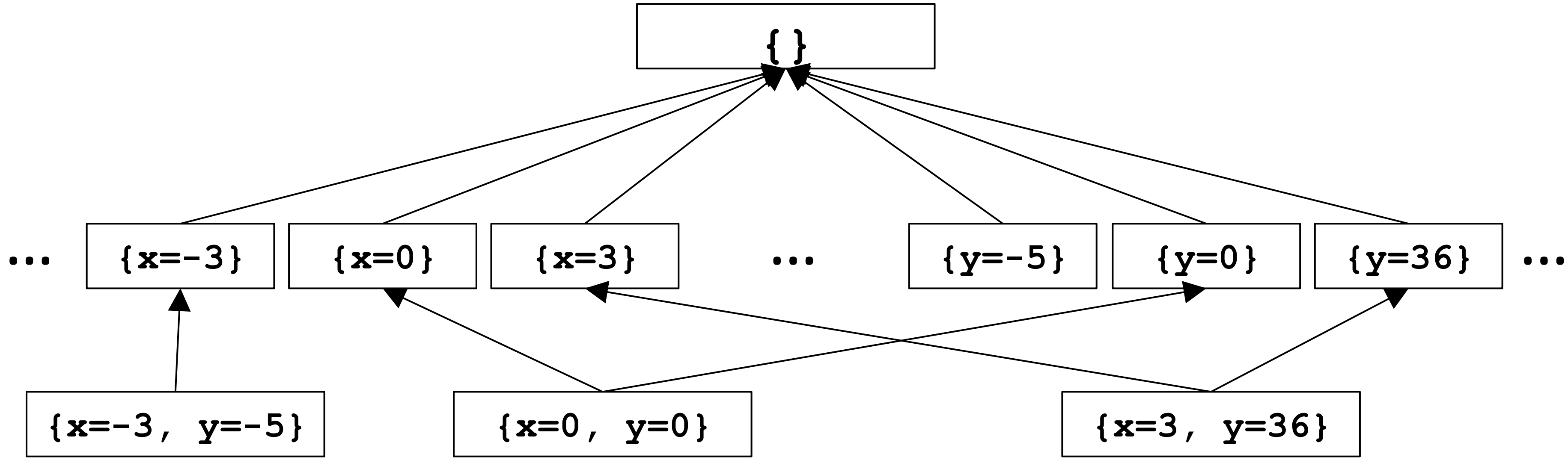
- CP-predicates
  - CP-factoids
    $\theta = \{ x = c \mid x \in \text{Var}, c \in Z \}$
  - CP-predicates $\Pi = 2^{\theta}$
  - Order relation 1: $P_1 \sqsubseteq^{set} P_2$ iff $P_1 \supseteq P_2$
  - Order relation 2: $P_1 \sqsubseteq^{imp} P_2$ iff $P_1 \Rightarrow P_2$
  - Is there a difference?
    - {x=5, x=7, x=9} $\supseteq$ {x=5, x=7}
    - {x=5, x=7, x=9} $\Rightarrow$ {x=5, x=7}
    - {x=5, x=7} $\Rightarrow$ {x=5, x=7, x=9}

CP preorder example



Var = {x}

CP preorder example



Var = {x, y}

The problem with preorders

- Equivalent elements have different representations
  - {x=y, x=a+b} *S*
  - {x=y, y=a+b} *S*
- Leads to unpredictability
- Which result should our static analysis give?

The problem with preorders

- Equivalent elements have different representations
  - {x=y, x=a+b} assert y==a+b
  - {x=y, y=a+b} assert y==a+b
- Leads to unpredictability
- Which result should our static analysis give?

The problem with preorders

- Equivalent elements have different representations
  - {x=y, x=a+b} assert x==a+b
  - {x=y, y=a+b} assert x==a+b
- Leads to unpredictability
- Which result should our static analysis give?

In practice many static analyses use preorders

Partially ordered sets (partial orders)

- A partially ordered set (Poset) is a pair (D , $\sqsubseteq$)
- D is a set of elements – a semantic domain
- $\sqsubseteq$ is a partial order between pairs of elements from D. That is $\sqsubseteq$: $D \times D$ with the following properties, for all d, d', d'' in D
  - Reflexive: d $\sqsubseteq$ d
  - Transitive: d $\sqsubseteq$ d' and d' $\sqsubseteq$ d'' implies d $\sqsubseteq$ d''
  - **Anti-symmetric: d $\sqsubseteq$ d' and d' $\sqsubseteq$ d implies d = d'**
- If d $\sqsubseteq$ d' and d $\neq$ d' we write d $\sqsubset$ d'

SAV partial order

- SAV-predicates
  - SAV-factoids
    $\theta = \{ x = y \mid x, y \in \text{Var} \} \cup \{ x = y + z \mid x, y, z \in \text{Var} \}$
  - SAV-predicates $\Pi = 2^{\theta}$
- Order relation 1: $P_1 \sqsubseteq^{set} P_2$ iff $P_1 \supseteq P_2$
  Is this a partial order?
- Order relation 2: $P_1 \sqsubseteq^{imp} P_2$ iff $P_1 \Rightarrow P_2$
  that is models(P_1) $\supseteq$ models(P_2)
  Is this a partial order?
- Order relation 3: $P_1 \sqsubseteq^{set*} P_2$ iff
  Explicate(P_1) $\sqsubseteq^{set}$ *Explicate*(P_2)
  Is this a partial order?