Program Analysis and Verification

0368-4479

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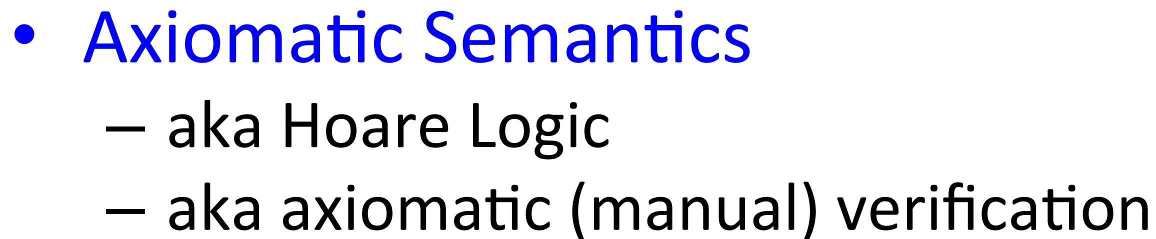
Lecture 5: Abstract Interpretation

Slides credit: Roman Manevich, Mooly Sagiv, Eran Yahav

Previously

- Operational Semantics
  - Large step (Natural)
  - Small step (SOS)

How?



- Axiomatic Semantics
  - aka Hoare Logic
  - aka axiomatic (manual) verification

Why?

From verification to analysis

- Manual **program verification**
  - Verifier provides assertions
    - Loop invariants
- Automatic **program verification** (Program analysis)
  - Tool automatically synthesize assertions
    - Finds loop invariants

Manual proof for `max`

```
nums : array
N : int

x := 0

res := nums[0]


while x < N

    if nums[x] > res then

        res := nums[x]


    x := x + 1
```

Manual proof for `max`

```
nums : array
N : unsigned int
{ N≥0 }
x := 0
{ N≥0 ∧ x=0 }
res := nums[0]
{ x=0 }
Inv = { x≤N }
while x < N
    { x=k ∧ k<N }
    if nums[x] > res then
        { x=k ∧ k<N }
        res := nums[x]
        { x=k ∧ k<N }
    { x=k ∧ k<N }
    x := x + 1
    { x=k+1 ∧ k≤N }
{ x≤N ∧ x≥N }
{ x=N }
```

We only prove no buffer (array) overflow

Can we find this proof automatically?

```
nums : array
N : unsigned int
{ N≥0 }
x := 0
{ N≥0 ∧ x=0 }
res := nums[0]
{ x=0 }
Inv = { x≤N }
while x < N
    { x=k ∧ k<N }
    if nums[x] > res then
        { x=k ∧ k<N }
        res := nums[x]
        { x=k ∧ k<N }
    { x=k ∧ k<N }
    x := x + 1
    { x=k+1 ∧ k≤N }
{ x≤N ∧ x≥N }
{ x=N }
```

Observation: predicates in proof have the general form

$\bigwedge$ constraint

where constraint has the form

$X - Y \leq c$

or

$\pm X \leq c$

Zone Abstract Domain (Analysis)

- Developed by Antoine Mine in his Ph.D. thesis
- Uses constraints of the form $X - Y \leq c$ and $\pm X \leq c$
- Built on top of Difference Bound Matrices (DBM) and shortest-path algorithms
  - $O(n^3)$ time
  - $O(n^2)$ space

Analysis with Zone abstract domain

```
nums : array
N : unsigned int
{ N≥0 }
x := 0
{ N≥0 ∧ x=0 }
res := nums[0]
{ N≥0 ∧ x=0 }
Inv = { N≥0 ∧ 0≤x≤N }
while x < N
    { N≥0 ∧ 0≤x<N }
    if nums[x] > res then
        { N≥0 ∧ 0≤x<N }
        res := nums[x]
        { N≥0 ∧ 0≤x<N }
    { N≥0 ∧ 0≤x<N }
    x := x + 1
    { N≥0 ∧ 0<x≤N }
{N≥0 ∧ 0≤x ∧ x=N }
```

Static Analysis with Zone Abstraction

```
nums : array
N : unsigned int
{ N≥0 }
x := 0
{ N≥0 ∧ x=0 }
res := nums[0]
{ x=0 }
Inv = { x≤N }
while x < N
    { x=k ∧ k≤N }
    if nums[x] > res then
        { x=k ∧ k<N }
        res := nums[x]
        { x=k ∧ k<N }
    { x=k ∧ k<N }
    x := x + 1
    { x=k+1 ∧ k≤N }
{ x≤N ∧ x≥N }
{ x=N }
```

Manual Proof

Abstract Interpretation [Cousot'77]

- Mathematical foundation of static analysis

Abstract Interpretation [Cousot'77]

- Mathematical foundation of static analysis

  - Abstract (semantic) domains ("abstract states")
  - Transformer functions ("abstract steps")
  - Chaotic iteration ("abstract computation")

Abstract Interpretation [CC77]

- A very general mathematical framework for approximating semantics
  - Generalizes Hoare Logic
  - Generalizes weakest precondition calculus
- Allows designing sound static analysis algorithms
  - Usually compute by iterating to a fixed-point
  - *Not specific to any programming language style*
- Results of an abstract interpretation are (loop) invariants
  - Can be interpreted as axiomatic verification assertions and used for verification

Abstract Interpretation in 5 Slides

- Disclaimer
  - Do not worry if you feel that you do not understand the next 5 slides
    - You are not expected to …
  - This is just to give you a view of the land …

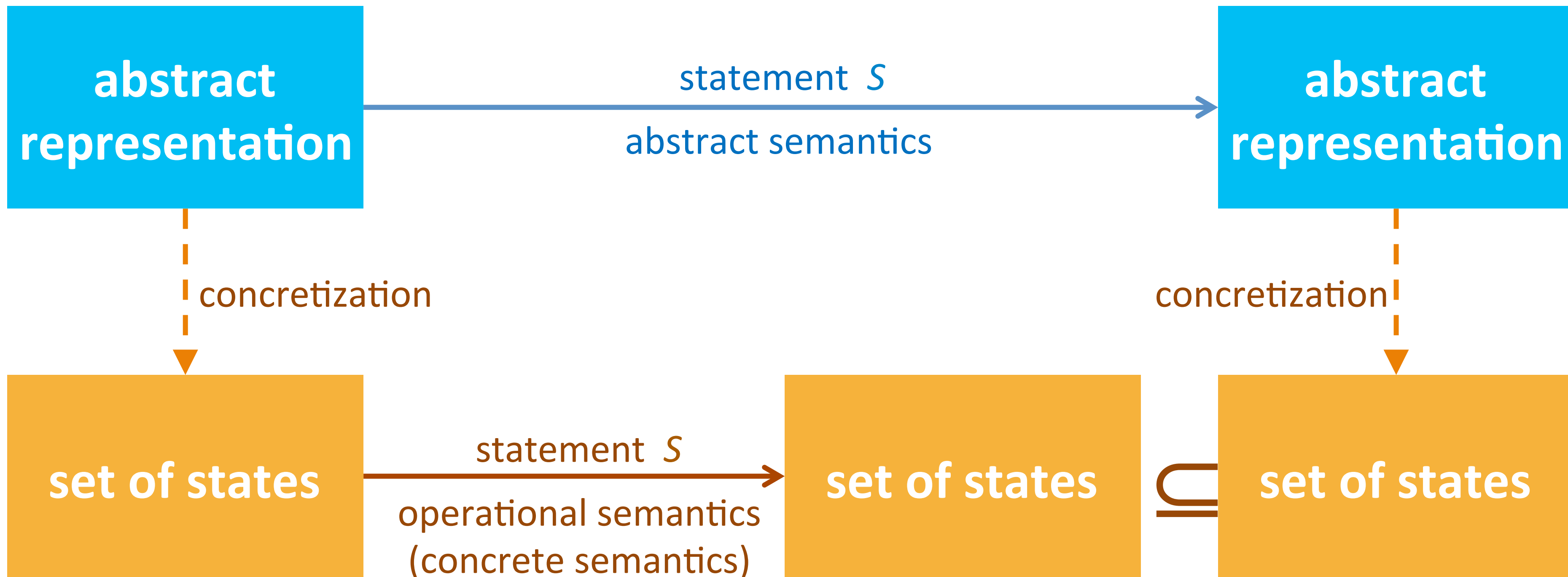
Collecting semantics

- For a set of program states **State**, we define the collecting lattice

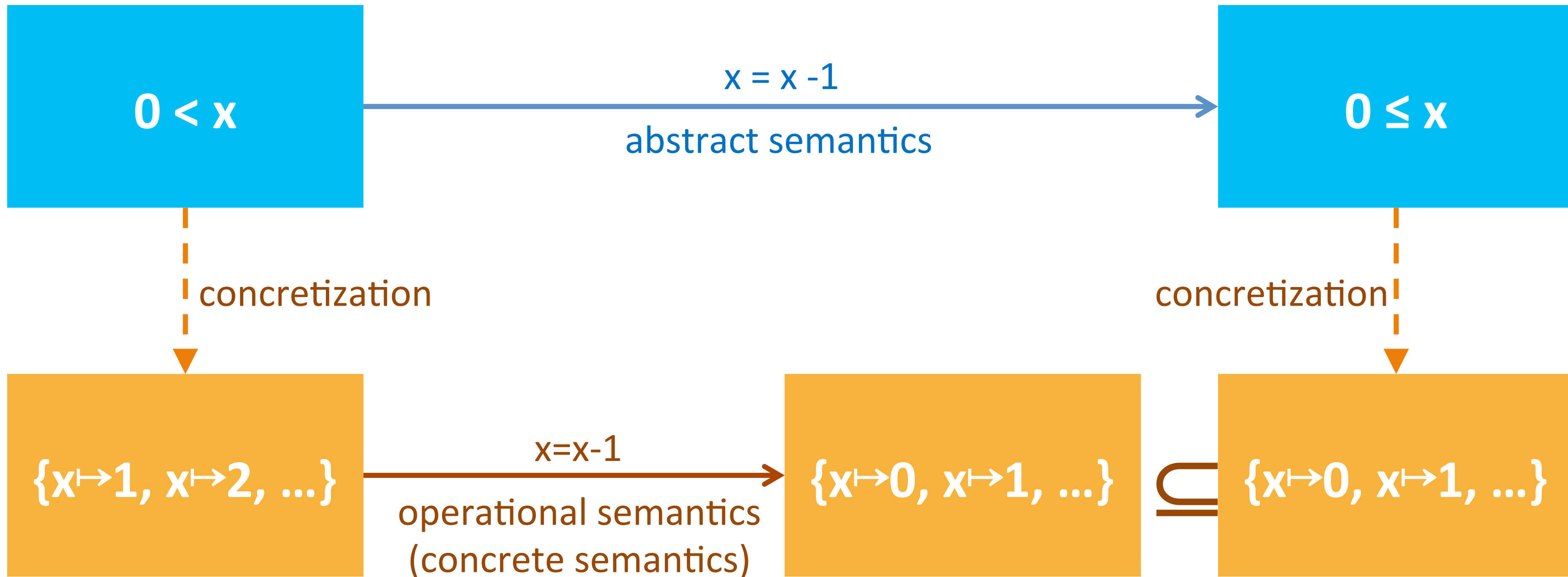
  $(2^{\mathbf{State}}, \subseteq, \cup, \cap, \emptyset, \mathbf{State})$

- The collecting semantics accumulates the (possibly infinite) sets of states generated during the execution
  - Not computable in general

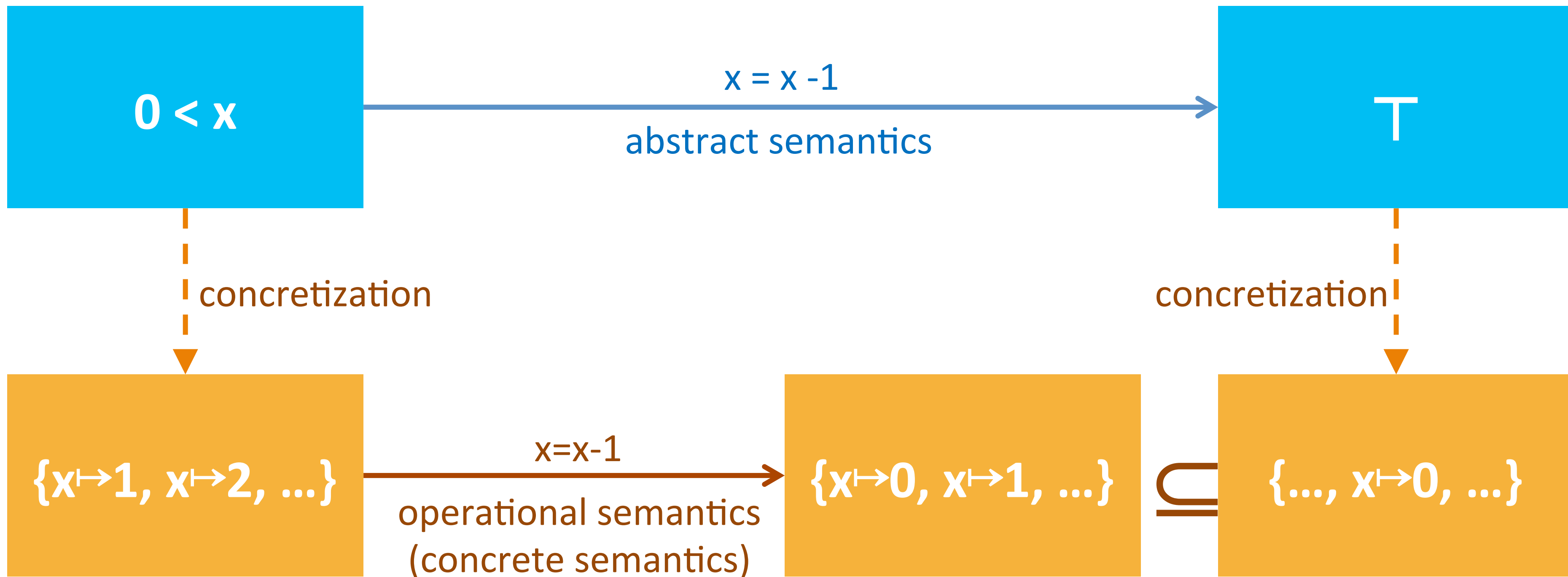
Abstract (conservative) interpretation



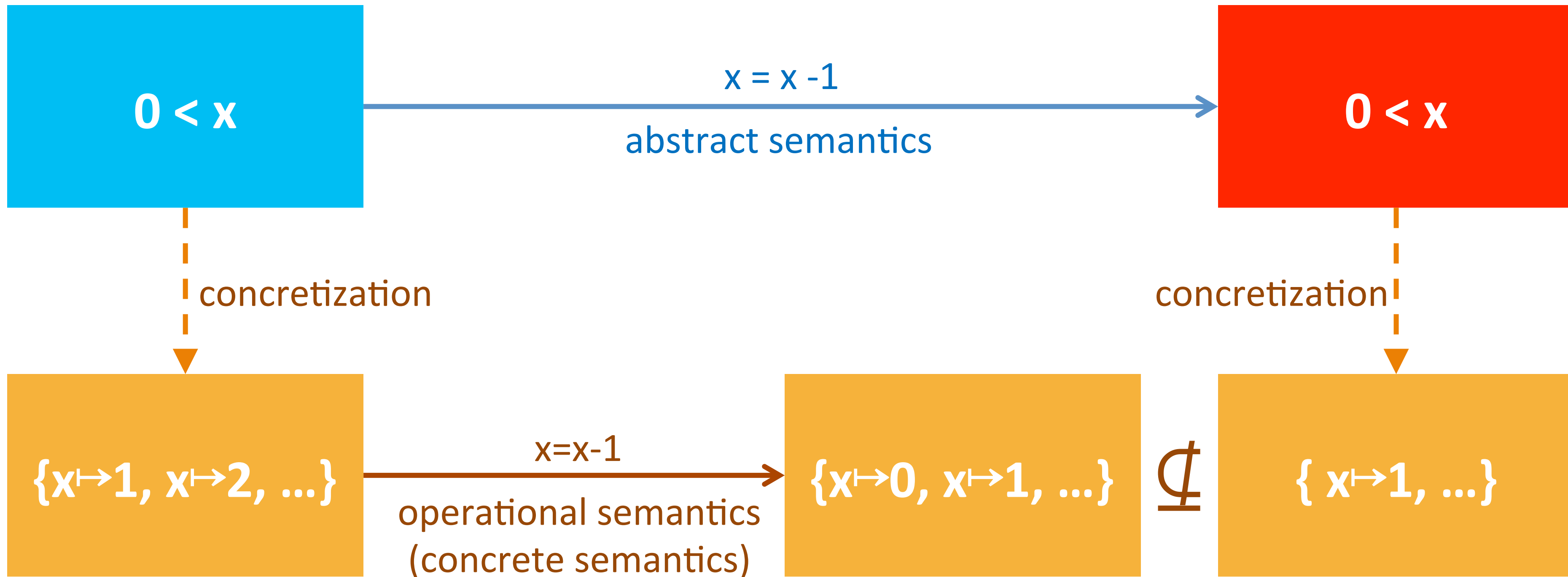
Abstract (conservative) interpretation



Abstract (conservative) interpretation



Abstract (non-conservative) interpretation



Abstract Interpretation by Example

Motivating Application: Optimization

- A compiler optimization is defined by a program transformation:

  $$T : Prog \longrightarrow Prog$$

- The transformation is semantics-preserving:

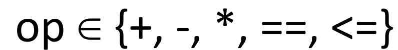
  $$\forall s \in State.\ S_{sos} [\![C]\!] s = S_{sos} [\![T(C)]\!] s$$

- The transformation is applied to the program only if an *enabling condition* is met
- We use static analysis for inferring enabling conditions

Common Subexpression Elimination

- If we have two variable assignments

  ```
  x := a op b
  ...
  y := a op b
  ```

  and the values of x, a, and b have not changed between the assignments, rewrite the code as

  ```
  x = a op b
  ...
  y := x
  ```

- Eliminates useless recalculation
- Paves the way for more optimizations
  - e.g., dead code elimination

What do we need to prove?

```
{ true }
C1
x := a op b
C2
{ x = a op b }
y := a op b
C3
```

```
{ true }
C1
x := a op b
C2
{ x = a op b }
y := x
C3
```

Available Expressions Analysis

- A static analysis that infers for every program point a set of facts of the form
  AV = $\{ x = y \mid x, y \in \text{Var} \} \cup$
  $\{ x = - y \mid x, y \in \text{Var} \} \cup$
  $\{ x = y \; op \; z \mid y, z \in \text{Var}, op \in \{+, -, *, <=\} \}$

- For every program with n=|Var| variables number of possible facts is finite: $|AV|=O(n^3)$
  - Yields a trivial algorithm … but, is it efficient?

Which proof is more desirable?

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ x=a+b }
y := a + b
...
```

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ z=a+c }
y := a + b
...
```

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ x=a+b ∧ z=a+c }
y := a + b
...
```

Which proof is more desirable?

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ x=a+b }
y := a + b
...
```

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ z=a+c }
y := a + b
...
```

More detailed predicate = more optimization opportunities

```
{ true }
x := a + b
{ x=a+b }
z := a + c
{ x=a+b ∧ z=a+c }
y := a + b
...
```

$x=a+b \land z=a+c \Rightarrow x=a+b$

$x=a+b \land z=a+c \Rightarrow z=a+c$

Implication formalizes “more detailed” relation between predicates

Developing a theory of approximation

- Formulae are suitable for many analysis-based proofs but we may want to represent predicates in other ways:
  - Sets of “facts”
  - Automata
  - Linear (in)equalities
  - … ad-hoc representation

- Wanted: a uniform theory to represent semantic values and approximations

Preorder

- We say that a binary order relation $\sqsubseteq$ over a set D is a preorder if the following conditions hold for every d, d’, d’’ $\in$ D
  - Reflexive: d $\sqsubseteq$ d
  - Transitive: d $\sqsubseteq$ d’ and d’ $\sqsubseteq$ d’’ implies d $\sqsubseteq$ d’’

- There may exist d, d’ such that

  d $\sqsubseteq$ d’ and d’ $\sqsubseteq$ d yet d $\neq$ d’

Preorder example

- **S**imple **A**vailable **E**xpressions
- Define SAV = $\{ x = y \mid x, y \in \text{Var} \} \cup$
  $\{ x = y + z \mid y, z \in \text{Var} \}$
- For D=2^{SAV} (sets of available expressions) define (for two subsets $A_1, A_2 \in$ D)
  $A_1 \sqsubseteq^{\text{imp}} A_2$ if and only if $\bigwedge A_1 \Rightarrow \bigwedge A_2$
- A_1 is "more detailed" if it implies all facts of A_2

Can we decide $A_1 \sqsubseteq^{\text{imp}} A_2$?

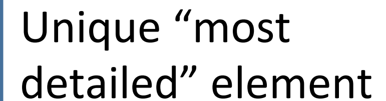
- Compare {x=y $\wedge$ x=a+b} with {x=y $\wedge$ y=a+b}
  - Which one should we choose?

The meaning of implication

- A predicate P represents the set of states
  $models(P) = \{ s \mid s \models P \}$
- $P \Rightarrow Q$ means
  $models(P) \subseteq models(Q)$

Partially ordered sets

- A partially ordered set (poset) is a pair $(D, \sqsubseteq)$
  - D is a set of elements – a (semantic) domain
  - $\sqsubseteq$ is a partial order between pairs of elements from D. That is $\sqsubseteq : D \times D$ with the following properties, for all d, d', d'' in D
    - Reflexive: $d \sqsubseteq d$
    - Transitive: $d \sqsubseteq d'$ and $d' \sqsubseteq d''$ implies $d \sqsubseteq d''$
    - **Anti-symmetric: $d \sqsubseteq d'$ and $d' \sqsubseteq d$ implies $d = d'$**



- Notation: if $d \sqsubseteq d'$ and $d \neq d'$ we write $d \sqsubset d'$

From preorders to partial orders

- We can transform a preorder into a poset by
  1. Coarsening the ordering

  2. Switching to a canonical form by choosing a representative for the set of equivalent elements
     d^* for { d' | d $\sqsubseteq$ d' and d' $\sqsubseteq$ d }

Coarsening for SAV

- For $D=2^{SAV}$ (sets of available expressions) define (for two subsets $A_1, A_2 \in D$)
  $A_1 \sqsubseteq^{coarse} A_2$ if and only if $A_1 \supseteq A_2$
- Notice that if $A_1 \supseteq A_2$ then $\bigwedge A_1 \Rightarrow \bigwedge A_2$
- Compare {x=y $\wedge$ x=a+b} with {x=y $\wedge$ y=a+b}
- How about {x=y $\wedge$ x=a+b $\wedge$ y=a+b} ?

Canonical form for SAV

- For an available expressions element A define $\mathit{Explicate}(A)$ = minimal set B such that:
  1. $A \subseteq B$
  2. x=y $\in B$ implies y=x $\in B$
  3. x=y $\in B$ and y=z $\in B$ implies x=z $\in B$
  4. x=y+z $\in B$ implies x=z+y $\in B$
  5. x=y $\in B$ and x=z+w $\in B$ implies y=z+w $\in B$
  6. x=y $\in B$ and z=x+w $\in B$ implies z=y+w $\in B$
  7. x=z+w $\in B$ and y=z+w $\in B$ implies x=y $\in B$
- Makes all implicit facts explicit

- Define $A^* = \mathit{Explicate}(A)$
- Define (for two subsets $A_1, A_2 \in$ D)
  $A_1 \sqsubseteq^{exp} A_2$ if and only if $A_1^* \supseteq A_2^*$
- **Lemma:** $A_1 \sqsubseteq^{exp} A_2$ if and only $A_1 \sqsubseteq^{imp} A_2$

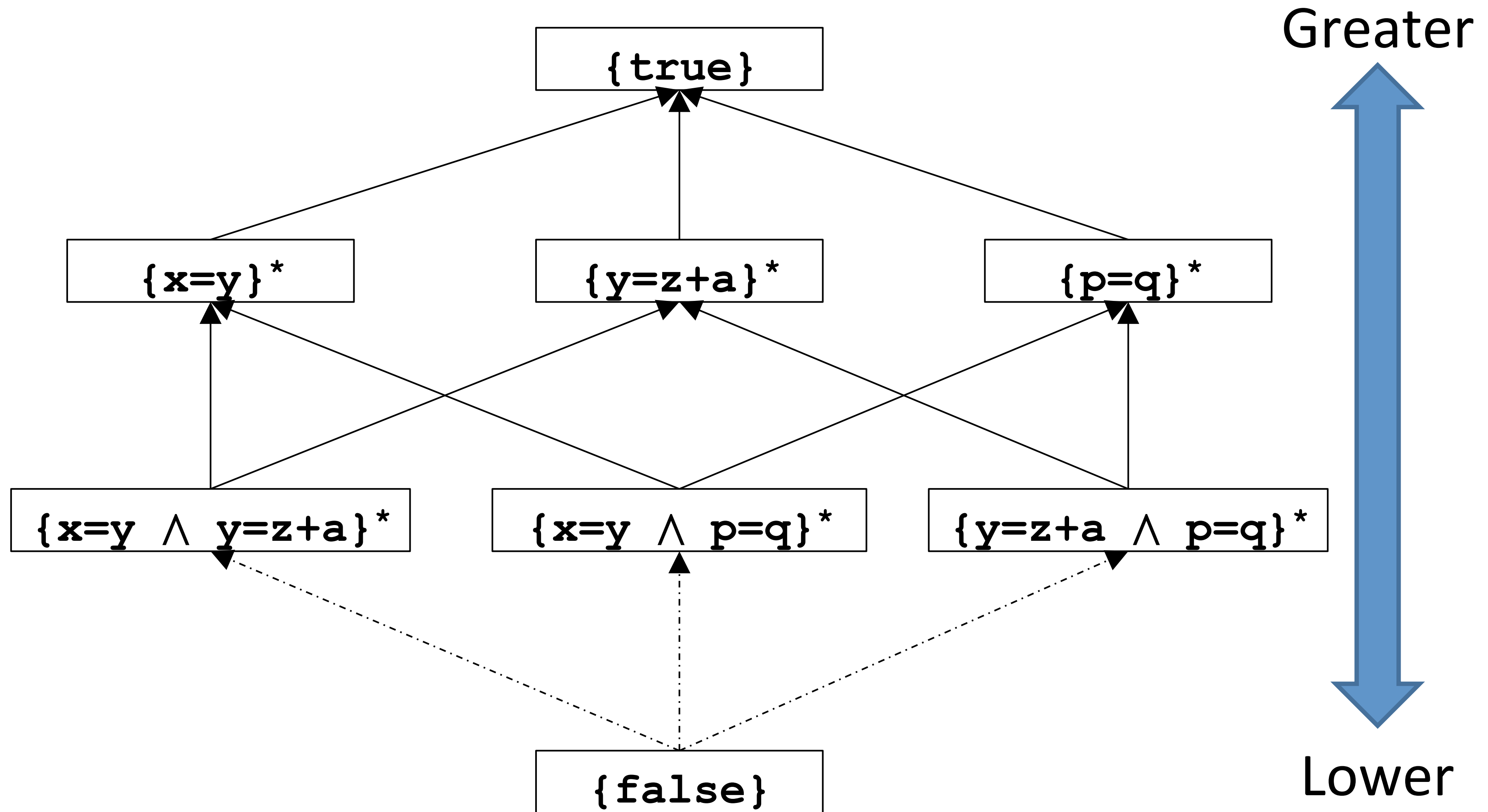
Therefore
$A_1 \sqsubseteq^{imp} A_2$ is decidable

Some posets-related terminology

- If $x \sqsubseteq y$ we can say
  - x is *lower* than y
  - x is *more precise* than y
  - x is *more concrete* than y
  - x *under-approximates* y

  - y is *greater* than x
  - y is *less precise* than x
  - y is *more abstract* than x
  - y *over-approximates* x

Visualizing ordering for SAV



D={x=y, y=x, p=q, q=p, y=z+a, y=a+z, z=y+z, x=z+a}

Pointed poset

- A poset $(D, \sqsubseteq)$ with a least element $\perp$ is called a pointed poset
  - For all $d \in D$ we have that $\perp \sqsubseteq d$
- The pointed poset is denoted by $(D, \sqsubseteq, \perp)$
- We can always transform a poset $(D, \sqsubseteq)$ into a pointed poset by adding a special bottom element

  $$(D \cup \{\perp\}, \sqsubseteq \cup \{\perp \sqsubseteq d \mid d \in D\}, \perp)$$

- Greatest element for SAV = {**true** = ?}
- Least element for SAV = {**false**= ?}

Annotating conditions

$$[\text{if}_p] \quad \frac{\{\, b \wedge P \,\}\, S_1 \,\{\, Q \,\}, \quad \{\, \neg b \wedge P \,\}\, S_2 \,\{\, Q \,\}}{\{\, P \,\}\ \texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2\ \{\, Q \,\}}$$

$\{\, P \,\}$
`if` b `then`
    $\{\, b \wedge P \,\}$
    S_1
    $\{\, Q_1 \,\}$
`else`
    $\{\, b \wedge P \,\}$
    S_2
    $\{\, Q_2 \,\}$
$\{\, Q \,\}$

Q approximates Q_1 and Q_2

We need a general way to approximate a set of semantic elements by a single semantic element

Join operator

- Assume a poset $(D, \sqsubseteq)$
- Let $X \subseteq D$ be a subset of D (finite/infinite)
- The join of X is defined as
  - $\bigsqcup X$ = the least upper bound (LUB) of all elements in X *if it exists*
  - $\bigsqcup X = \min_{\sqsubseteq}\{ b \mid \text{forall } x \in X \text{ we have that } x \sqsubseteq b \}$
  - The supremum of the elements in X
  - A kind of abstract union (disjunction) operator
- Properties of a join operator
  - Commutative: $x \sqcup y = y \sqcup x$
  - Associative: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
  - Idempotent: $x \sqcup x = x$

Meet operator

- Assume a poset $(D, \sqsubseteq)$
- Let $X \subseteq D$ be a subset of D (finite/infinite)
- The meet of X is defined as
  - $\sqcap X$ = the greatest lower bound (GLB) of all elements in X *if it exists*
  - $\sqcap X = \max_{\sqsubseteq}\{ b \mid \text{forall } x \in X \text{ we have that } b \sqsubseteq x\}$
  - The infimum of the elements in X
  - A kind of abstract intersection (conjunction) operator
- Properties of a join operator
  - Commutative: $x \sqcap y = y \sqcap x$
  - Associative: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
  - Idempotent: $x \sqcap x = x$

Complete lattices

- A complete lattice $(D, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ is
- A set of elements D
- A partial order $x \sqsubseteq y$
- A join operator $\sqcup$
- A meet operator $\sqcap$
- A bottom element
  $\bot = ?$
- A top element
  $\top = ?$

Complete lattices

- A complete lattice $(D, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ is
- A set of elements D
- A partial order $x \sqsubseteq y$
- A join operator $\sqcup$
- A meet operator $\sqcap$
- A bottom element
  $\bot = \sqcup \emptyset$
- A top element
  $\top = \sqcup D$

Transfer Functions

- Mathematical foundations

Towards an automatic proof

- **Goal:** automatically compute an annotated program proving as many facts of the form
  $x = y + z$ as possible
- **Decision 1:** develop a forward-going proof
- **Decision 2:** draw predicates from a finite set D
  - "looking under the light of the lamp"
  - A compromise that simplifies problem by focusing attention – possibly miss some facts that hold
- **Challenge 1:** handle straight-line code
- **Challenge 2:** handle conditions
- **Challenge 3:** handle loops

Domain for SAV

- Define *atomic facts* (for SAV) as
  $\theta = \{ x = y \mid x, y \in \text{Var} \} \cup \{ x = y + z \mid x, y, z \in \text{Var} \}$
  - For n=|Var| number of atomic facts is $O(n^3)$
- Define *sav-predicates* as $\Pi = 2^\theta$
- For $D \subseteq \theta$, Conj(D) = $\wedge$D
  - Conj(*{a=b, c=b+d, b=c}*) = (a=b) $\wedge$ (c=b+d) $\wedge$ (b=c)
- Note:
  - $\text{Conj}(D_1 \cup D_2) = \text{Conj}(D_1) \wedge \text{Conj}(D_1)$
  - Conj({}) $\Leftrightarrow$ true

Challenge 2: handling straight-line code

handling straight-line code: Goal

- Given a program of the form

$$x_1 := a_1; \ldots x_n := a_n$$

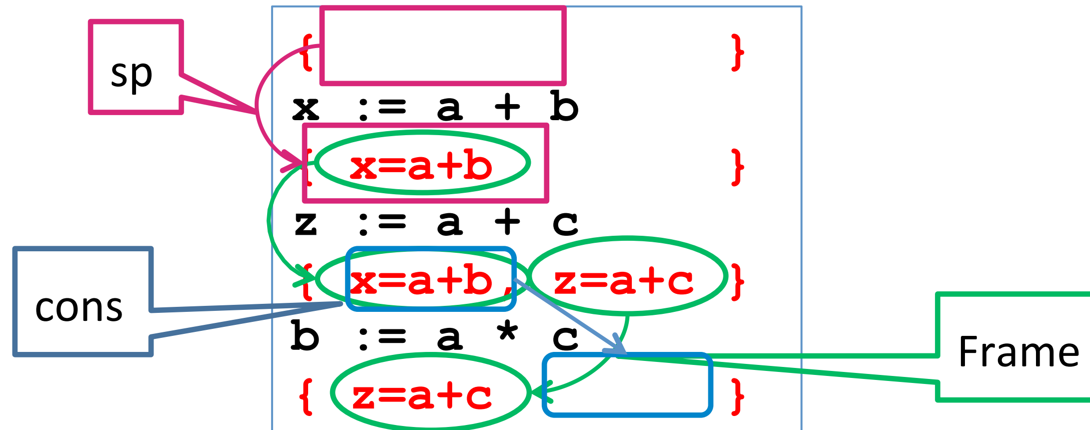
- Find predicates $P_0, \ldots, P_n$ such that
  1. $\{P_0\}\ x_1 := a_1\ \{P_1\} \ldots \{P_{n-1}\}\ x_n := a_n\ \{P_n\}$ is a proof
     - **sp**$(x_i := a_i, P_{i-1}) \Longrightarrow P_i$
  2. $P_i = \mathrm{Conj}(D_i)$
     - D_i is a set of simple (SAV) facts

Example

```
{                   }
x := a + b
{                   }
z := a + c
{                   }
b := a * c
{                   }
```

- Find a proof that satisfies both conditions

Example



- Can we make this into an algorithm?

Algorithm for straight-line code

- **Goal:** find predicates $P_0, \ldots, P_n$ such that
  1. $\{P_0\}\ x_1 := a_1\ \{P_1\} \ldots \{P_{n\text{-}1}\}\ x_n := a_n\ \{P_n\}$ is a proof
     That is: $\mathbf{sp}(x_i := a_\mathrm{i}, P_{i\text{-}1}) \Rightarrow P_i$
  2. Each P_i has the form Conj(D_i) where D_i is a set of simple (SAV) facts
- **Idea:** define a function $\mathrm{F}^{\mathrm{SAV}}[x{:=}a] : \Pi \rightarrow \Pi$ s.t.
  if $\quad \mathrm{F}^{\mathrm{SAV}}[x{:=}a](D) = D'$
  then $\mathbf{sp}(x := a, \mathrm{Conj}(D)) \Rightarrow \mathrm{Conj}(D')$
  – We call F the abstract transformer for $x{:=}a$
- Initialize D_0={}
- For each i: compute $D_{\mathrm{i+1}} = \mathrm{Conj}(\mathrm{F}^{\mathrm{SAV}}[x_i := a_\mathrm{i}]\ D_\mathrm{i})$
- Finally $P_\mathrm{i} = \mathrm{Conj}(D_\mathrm{i})$

Defining an SAV abstract transformer

- **Goal:** define a function $F^{SAV}[x{:=}a] : \Pi \rightarrow \Pi$ s.t.
  if $F^{SAV}[x{:=}a](D) = D'$
  then **sp**($x := a$, Conj(D)) $\Rightarrow$ Conj(D')

Defining an SAV abstract transformer

- **Goal:** define a function $F^{SAV}[x{:=}a] : \Pi \rightarrow \Pi$ s.t.
  if $F^{SAV}[x{:=}a](D) = D'$
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- **Idea:** define rules for individual facts and generalize to sets of facts by the conjunction rule

Defining an SAV abstract transformer

- **Goal:** define a function $F^{SAV}[x{:=}a] : \Pi \rightarrow \Pi$ s.t.
  if $F^{SAV}[x{:=}a](D) = D'$
  then **sp**($x := a$, Conj(D)) $\Rightarrow$ Conj(D')
- **Idea:** define rules for individual facts and generalize to sets of facts by the conjunction rule

[kill-lhs]	$\{ x{=}\omega \}\ x{:=}a\ \{ \}$
[kill-rhs-1]	$\{ y{=}x{+}w \}\ x{:=}a\ \{ \}$
[kill-rhs-2]	$\{ y{=}w{+}x \}\ x{:=}a\ \{ \}$
[gen]	$\{ \}\ x{:=}\ \omega\ \{ x{=}\omega \}$
[preserve]	$\{ y{=}z{+}w \}\ x{:=}a\ \{ y{=}z{+}w \}$

ω Is either a variable v or an addition expression $v{+}w$

SAV abstract transformer example

```
{                 }
x := a + b
{ x=a+b           }
z := a + c
{ x=a+b, z=a+c }
b := a * c
{ z=a+c           }
```

[kill-lhs] $\{ x=\omega \}\ x:=a\ \{ \}$

[kill-rhs-1] $\{ y=x+w \}\ x:=a\ \{ \}$

[kill-rhs-2] $\{ y=w+x \}\ x:=a\ \{ \}$

[gen] $\{ \}\ x:=\omega\ \{ x=\omega \}$

[preserve] $\{ y=z+w \}\ x:=a\ \{ y=z+w \}$

ω Is either a variable v or an addition expression $v+w$

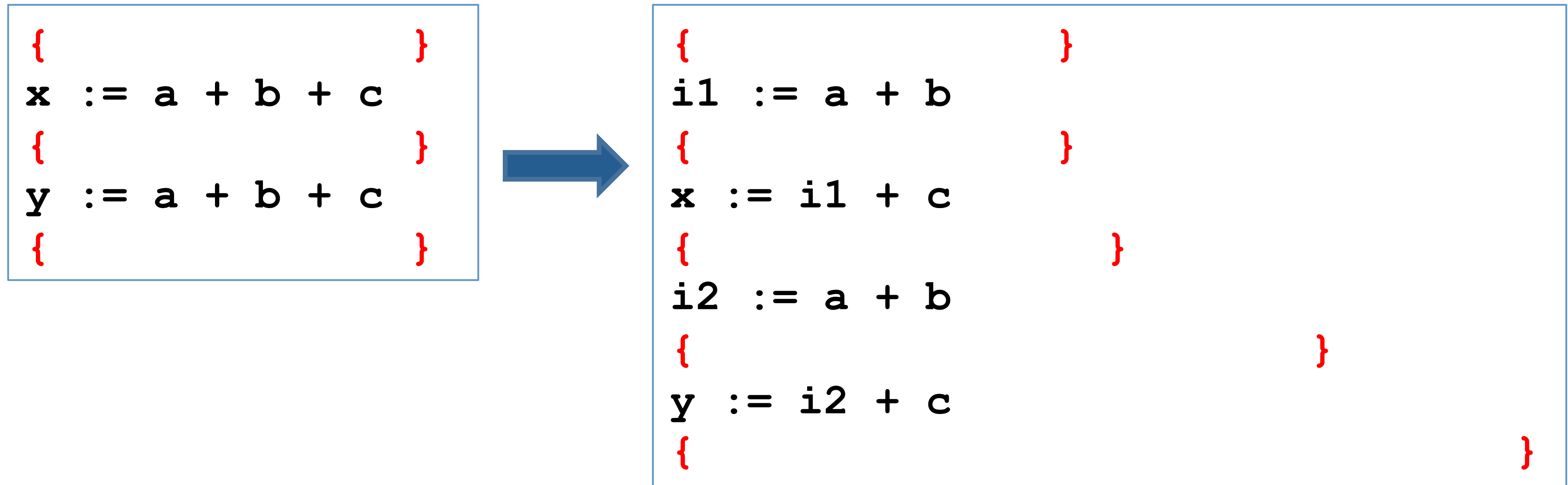
Problem 1: large expressions

```
{               }
x := a + b + c
{               }
y := a + b + c
{               }
```

Missed CSE opportunity

- Large expressions on the right hand sides of assignments are problematic
  - Can miss optimization opportunities
  - Require complex transformers
- Solution: transform code to normal form where right-hand sides have bounded size

Solution: Simplify Prog. Lang.



- Main idea: simplify expressions by storing intermediate results in new temporary variables
  - Three-address code
- Number of variables in simplified statements ≤ 3

Solution: Simplify Prog. Lang.

```
{                }
x := a + b + c
{                }
y := a + b + c
{                }
```

```
{                }
i1 := a + b
{ i1=a+b         }
x := i1 + c
{ i1=a+b, x=i1+c }
i2 := a + b
{ i1=a+b, x=i1+c, i2=a+b }
y := i2 + c
{ i1=a+b, x=i1+c, i2=a+b, y=i2+c }
```

Need to infer `i1=i2`

- Main idea: simplify expressions by storing intermediate results in new temporary variables
  - Three-address code
- Number of variables in simplified statements ≤ 3

Problem 2: Transformer Precision

```
{                   }
i1 := a + b
{ i1=a+b            }
x := i1 + c
{ i1=a+b, x=i1+c }
i2 := a + b
{ i1=a+b, x=i1+c, i2=a+b }
y := i2 + c
{ i1=a+b, x=i1+c, i2=a+b, y=i2+c }
```

Need to infer
`i1=i2`

- Our transformer only infers syntactically available expressions – ones that appear in the code explicitly
- We want a transformer that looks deeper into the semantics of the predicates
  - Takes equalities into account

Solution: Use Canonical Form

- **Idea:** make as many implicit facts explicit by
  - Using symmetry and transitivity of equality
  - Commutativity of addition
  - Meaning of equality – can substitute equal variables
- For P=Conj(D) let *Explicate*(D) = minimal set D^* such that:
  1. $D \subseteq D^*$
  2. $x=y \in D^*$ implies $y=x \in D^*$
  3. $x=y \in D^*$ $y=z \in D^*$ implies $x=z \in D^*$
  4. $x=y+z \in D^*$ implies $x=z+y \in D^*$
  5. $x=y \in D^*$ and $x=z+w \in D^*$ implies $y=z+w \in D^*$
  6. $x=y \in D^*$ and $z=x+w \in D^*$ implies $z=y+w \in D^*$
  7. $x=z+w \in D^*$ and $y=z+w \in D^*$ implies $x=y \in D^*$
- Notice that *Explicate*(D) $\Leftrightarrow$ D
  - *Explicate* is a special case of a reduction operator

Sharpening the transformer

- **Define:** $F^*[x:=a] = \text{Explicate} \circ F^{SAV}[x:=a]$

```
{                  }
i1 := a + b
{ i1=a+b, i1=b+a         }
x := i1 + c
{ i1=a+b, i1=b+a, x=i1+c, x=c+i1 }
i2 := a + b
{ i1=a+b, i1=b+a, x=i1+c, x=c+i1, i2=a+b,
  i2=b+a, i1=i2, i2=i1, x=i2+c, x=c+i2, }
y := i2 + c
{ ... }
```

Since sets of facts and their conjunction are isomorphic we will use them interchangeably

An algorithm for annotating SLP

- Annotate(P, x:=a) = {P} x:=a $F^*[x:=a](P)$

- Annotate(P, S_1; S_2) = {P} S_1; {Q_1} S_2 {Q_2
  - Annotate(P, S_1) = {P} S_1 {Q_1}
  - Annotate(Q_1, S_2) = {Q_1} S_2 {Q_2}