Program Analysis and Verification

0368-4479

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Lecture 12: Interprocedural Analysis + Numerical Analysis

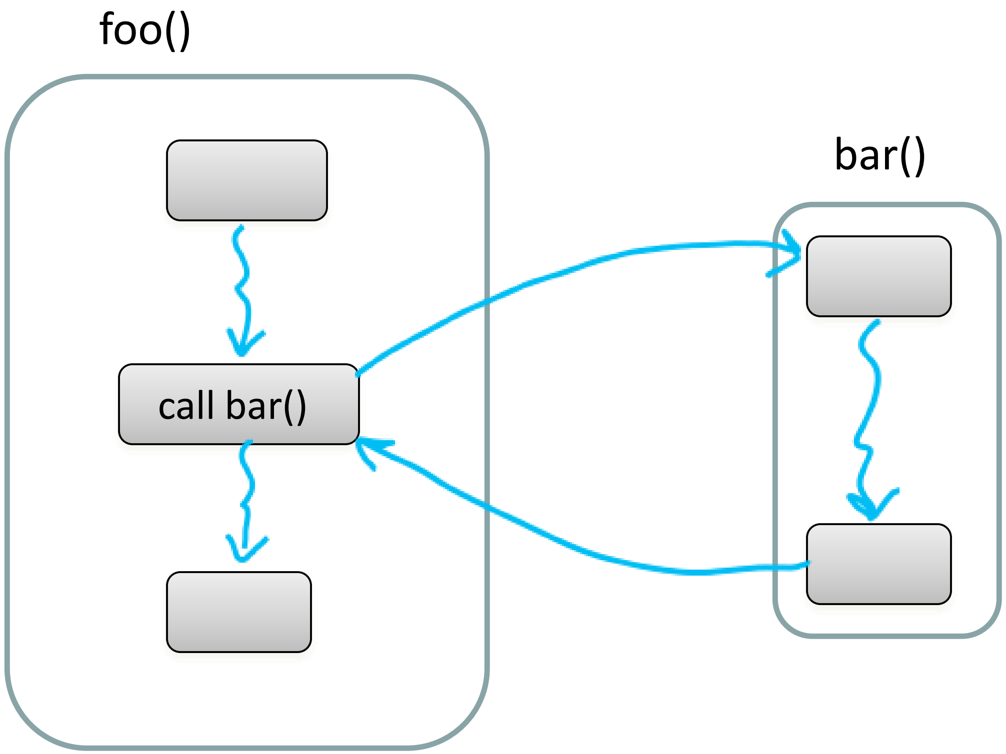
Slides credit: Roman Manevich, Mooly Sagiv, Eran Yahav

Procedural program

```
void main() {
    int x;
    x = p(7);
    x = p(9);
}
```

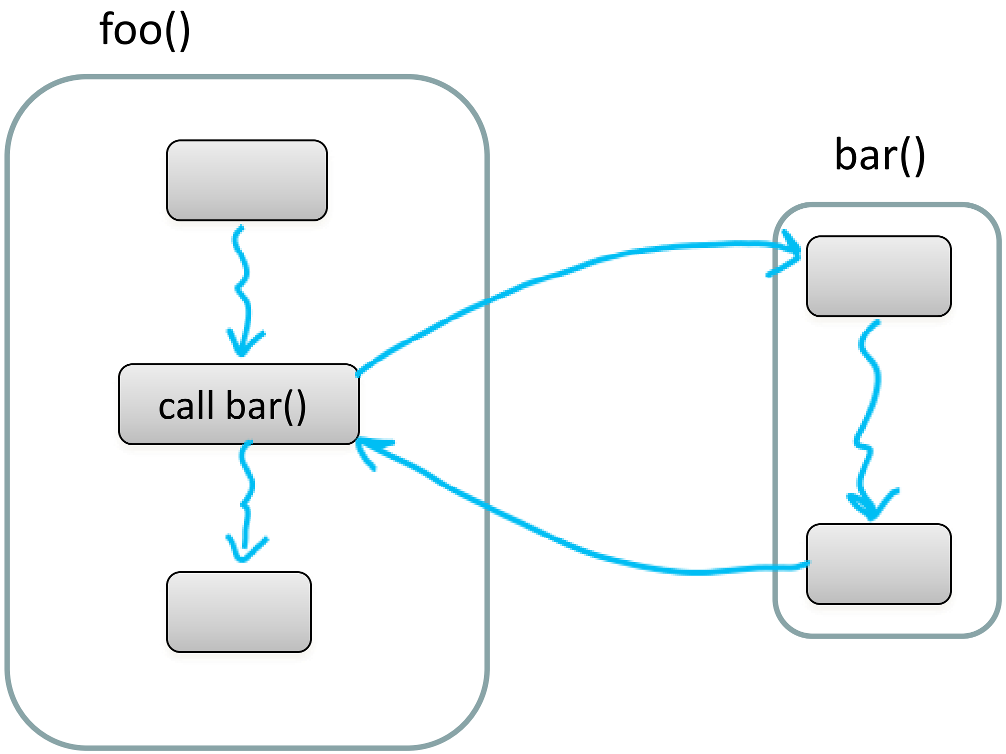
```
int p(int a) {
  return a + 1;
}
```

Effect of procedures



The effect of calling a procedure is the effect of executing its body

Interprocedural Analysis



goal: compute the abstract effect of calling a procedure

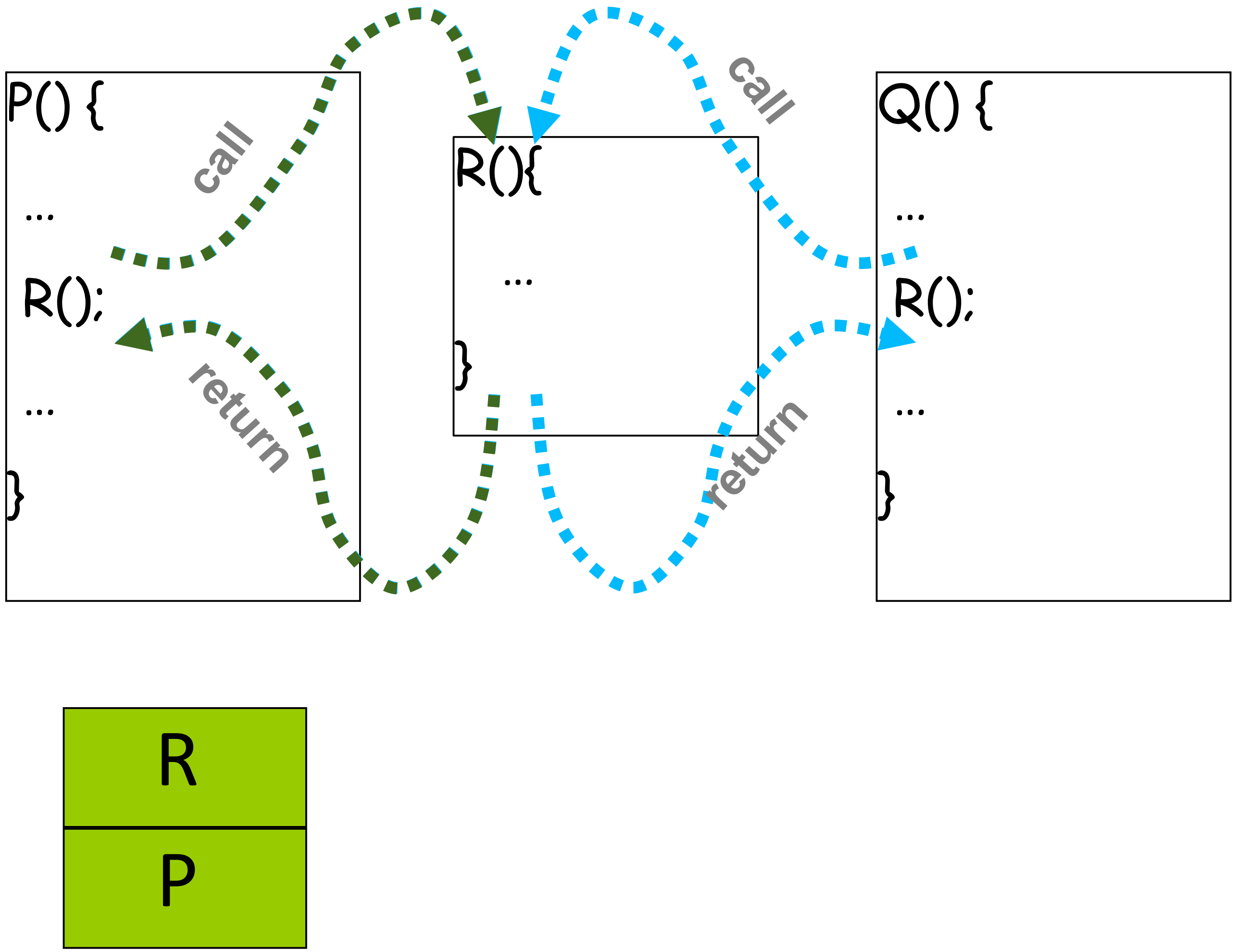
Naïve solutions

- Inilining
- Call/Return as goto

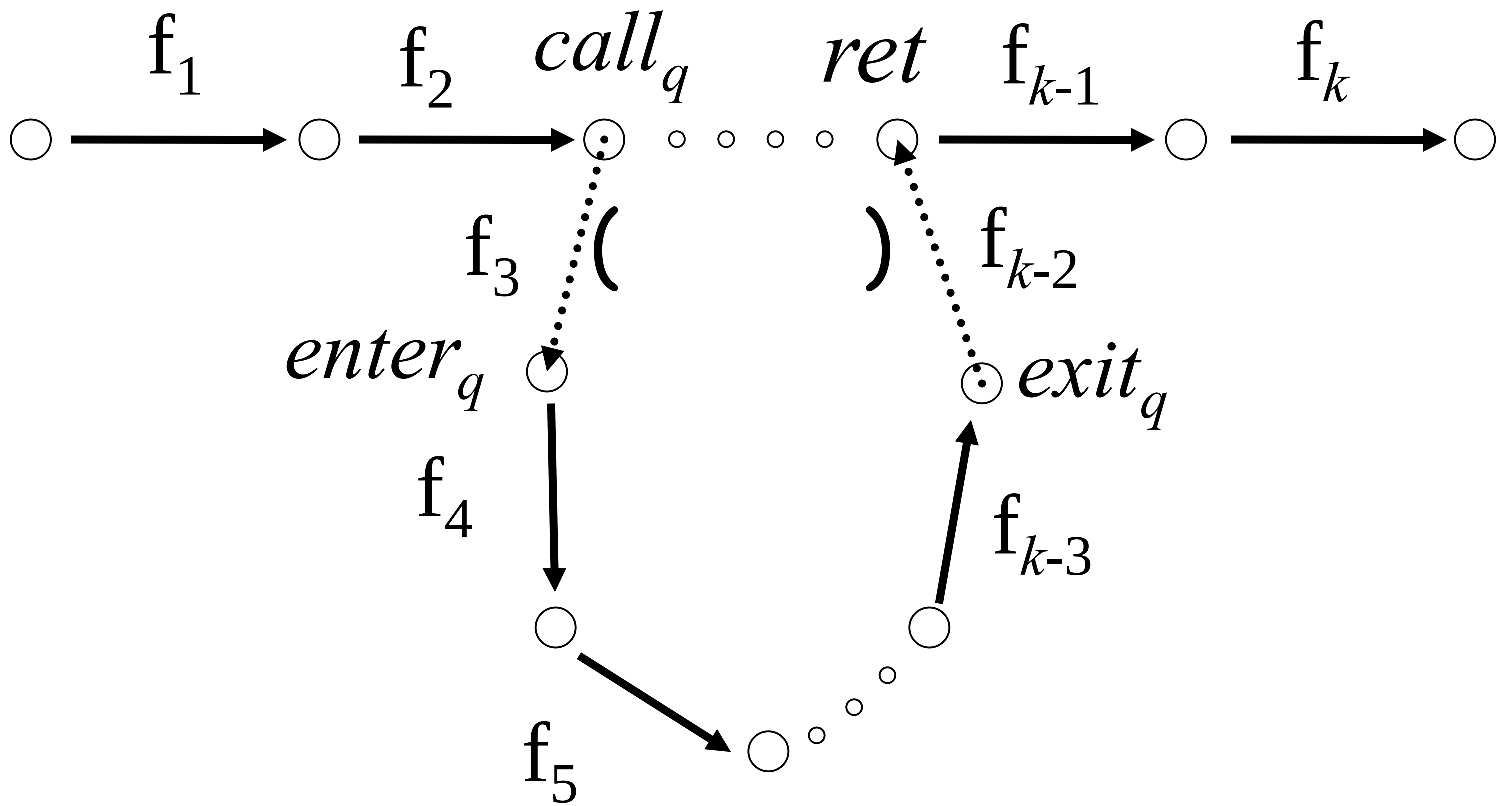
Guiding light

- Exploit stack regime
  - ➔Precision
  - ➔Efficiency

Stack regime

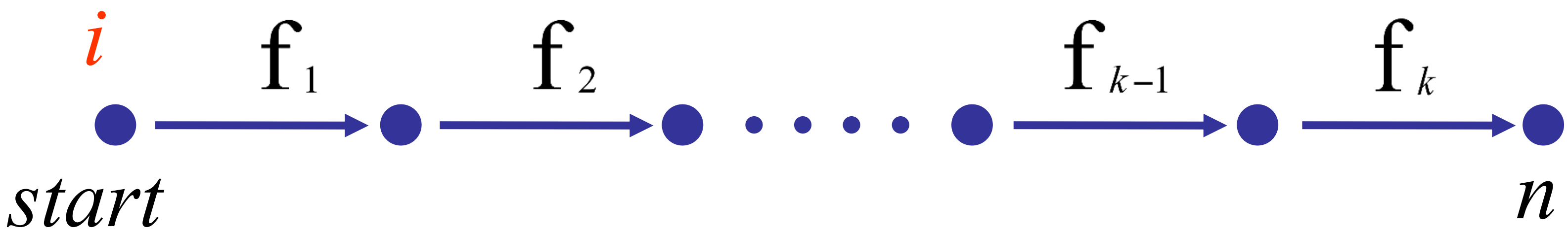


Interprocedural Valid Paths



- IVP: all paths with matching calls and returns
  - And prefixes

Join Over All Paths (JOP)



$$[\![f_k \circ \ldots \circ f_1]\!] \in L \rightarrow L$$

- $JOP[v] = \sqcup\{[[e_1, e_2, \ldots, e_n]](\iota) \mid (e_1, \ldots, e_n) \in paths(v)\}$
- $JOP \sqsubseteq LFP$
  - Sometimes JOP = LFP
    - precise up to "**symbolic execution**"
    - Distributive problem

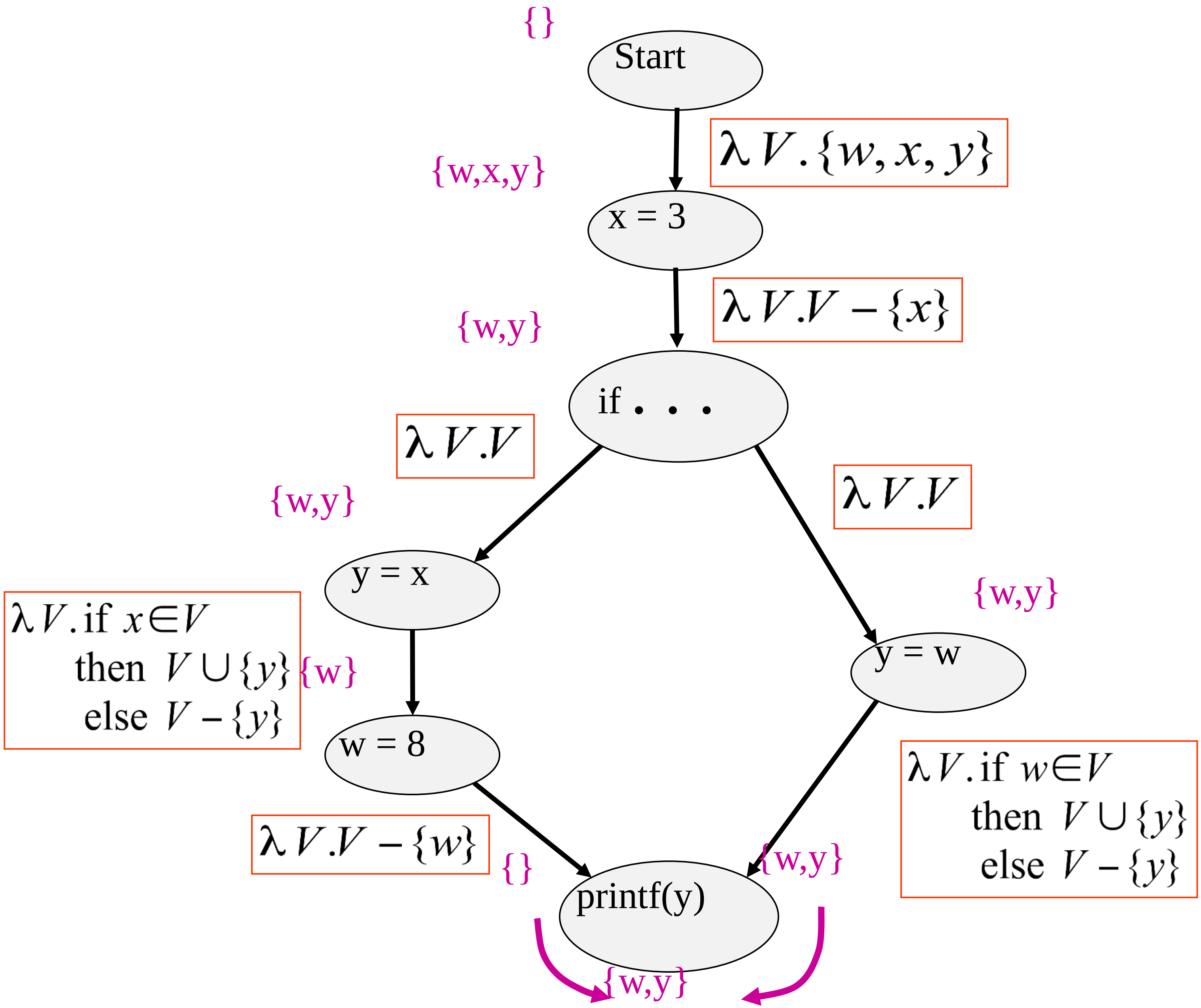
CFL-Graph reachability

- Special cases of functional analysis
- Finite distributive lattices
- Provides more efficient analysis algorithms
- Reduce the interprocedural analysis problem to finding context free reachability

IDFS / IDE

- **IDFS** Interprocedural Distributive Finite Subset
  Precise interprocedural dataflow analysis via graph reachability. *Reps, Horowitz, and Sagiv, POPL' 95*
- **IDE** Interprocedural Distributive Environment
  Precise interprocedural dataflow analysis with applications to constant propagation. *Reps, Horowitz, and Sagiv, FASE' 95, TCS' 96*
  - ***More general solutions exist***

Possibly Uninitialized Variables



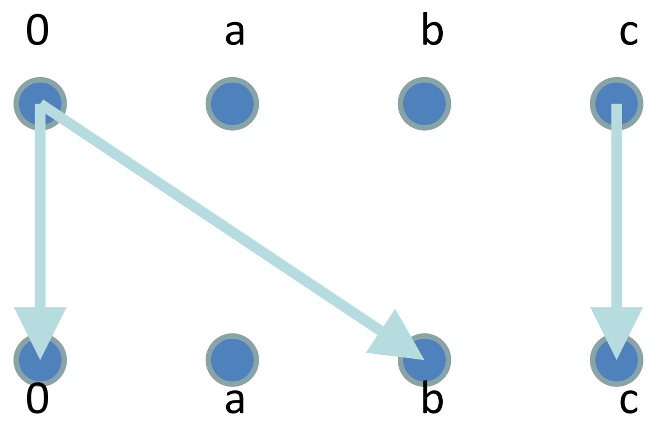
IFDS Problems

- Finite subset distributive
  - Lattice L = $\wp$(D)
  - $\sqsubseteq$ is $\subseteq$
  - $\sqcup$ is $\cup$
  - Transfer functions are distributive

- Efficient solution through formulation as CFL reachability

Encoding Transfer Functions

- Enumerate all input space and output space
- Represent functions as graphs with 2(D+1) nodes
- Special symbol "0" denotes empty sets (sometimes denoted Λ)
- Example: D = { a, b, c }
  f(S) = (S – {a}) U {b}



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Efficiently Representing Functions

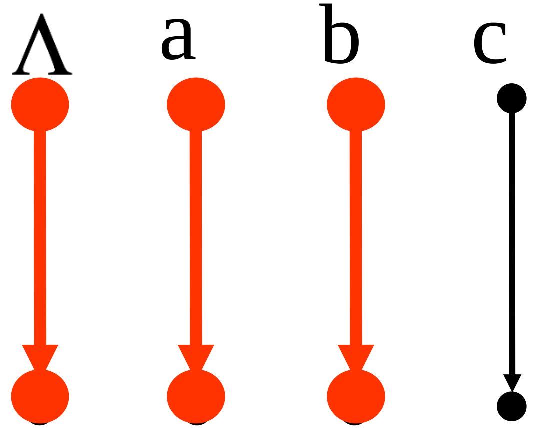
- Let $f: 2^D \rightarrow 2^D$ be a distributive function
- Then:
  - $f(X) = f(\emptyset) \cup (\bigcup \{ f(\{z\}) \mid z \in X \})$
  - $f(X) = f(\emptyset) \cup (\bigcup \{ f(\{z\}) \setminus f(\emptyset) \mid z \in X \})$

Representing Dataflow Functions

Identity Function

$$f = \lambda V.V$$

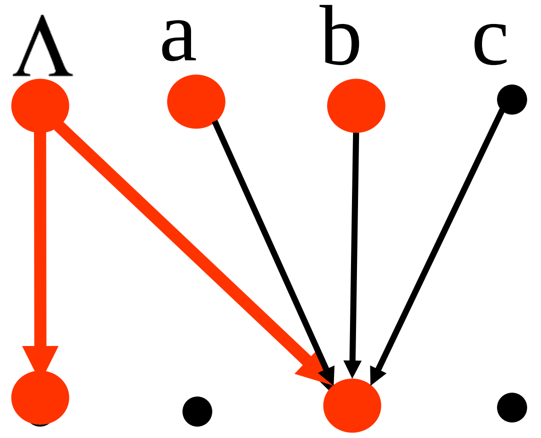
$$f(\{a,b\}) = \{a,b\}$$



Constant Function

$$f = \lambda V.\{b\}$$

$$f(\{a,b\}) = \{b\}$$

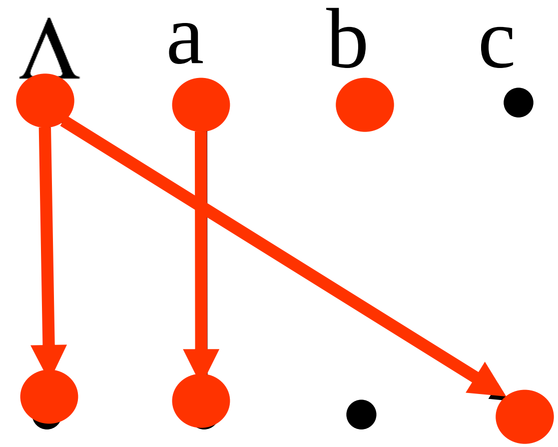


Representing Dataflow Functions

"Gen/Kill" Function

$$f = \lambda V.(V - \{b\}) \cup \{c\}$$

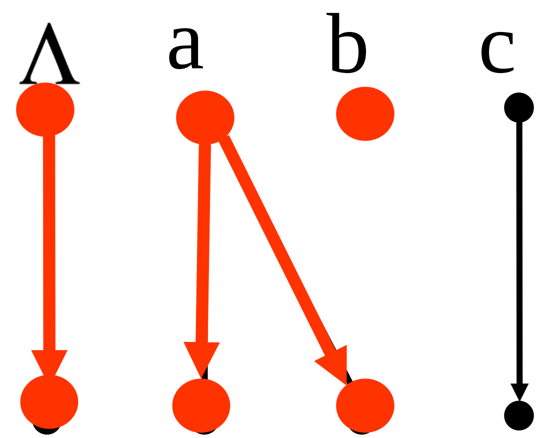
$$f(\{a,b\}) = \{a,c\}$$

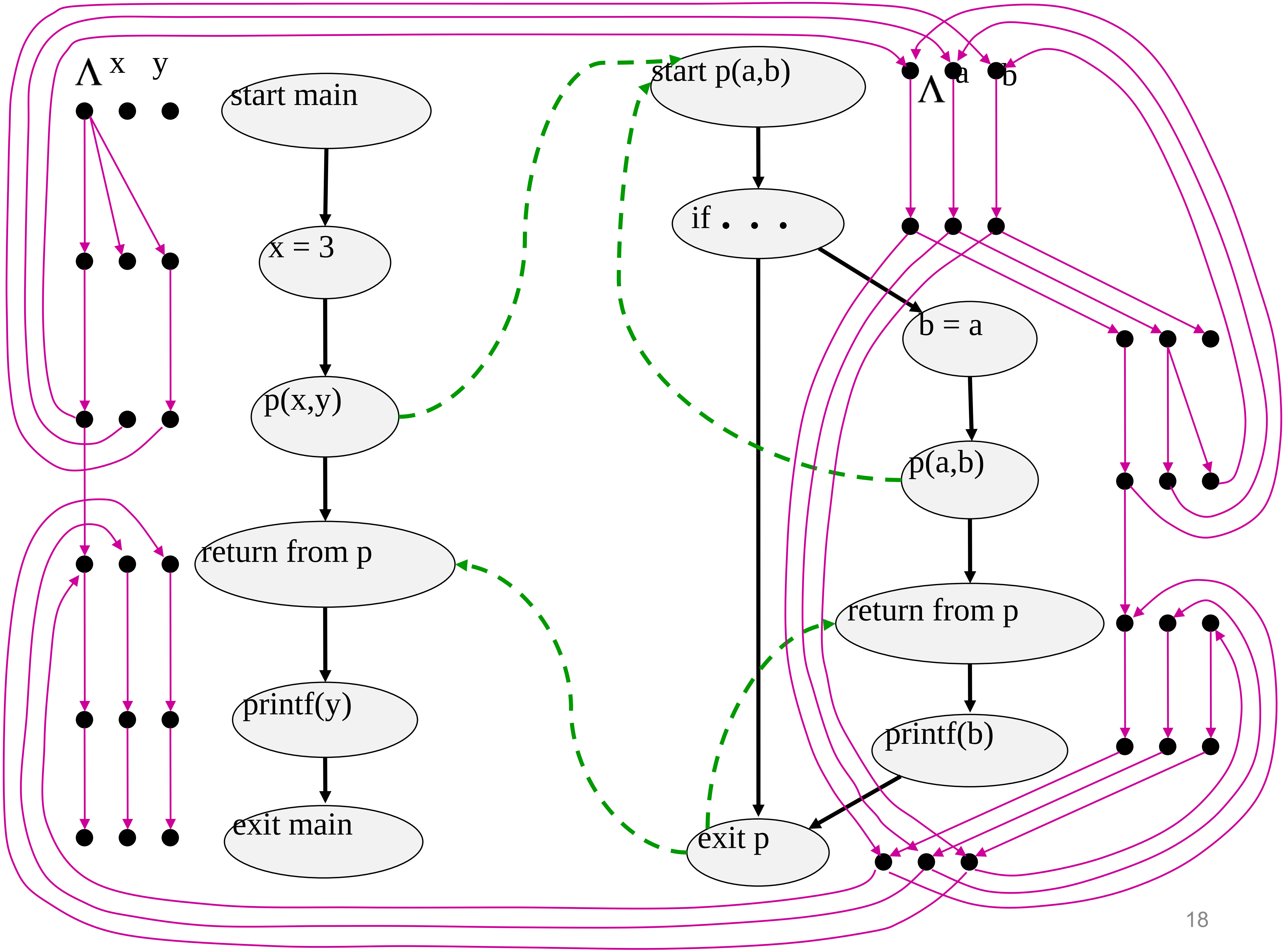


Non-"Gen/Kill" Function

$$f = \lambda V.\text{if } a \in V \text{ then } V \cup \{b\} \text{ else } V - \{b\}$$

$$f(\{a,b\}) = \{a,b\}$$

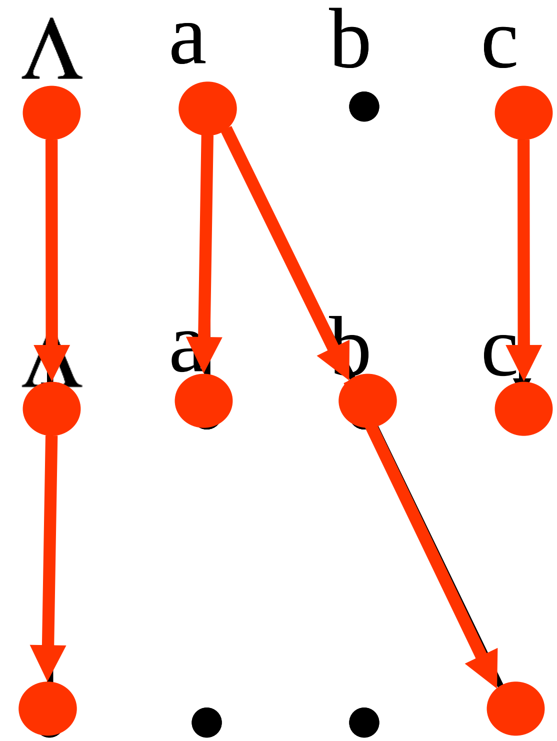



Λ x y
start main
x = 3
p(x,y)
return from p
printf(y)
exit main
start p(a,b)
Λ a b
if . . .
b = a
p(a,b)
return from p
printf(b)
exit p

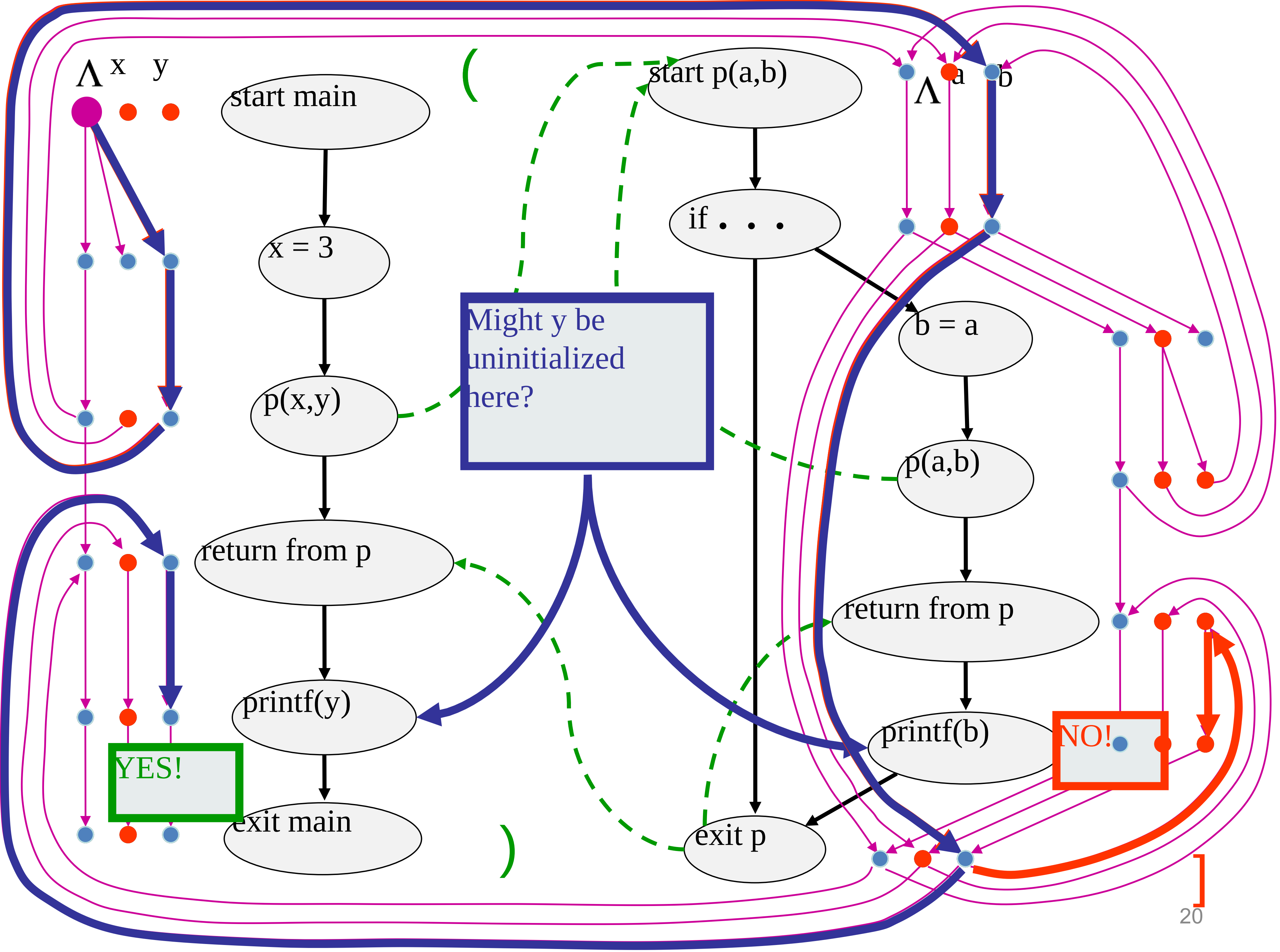
Composing Dataflow Functions

$$f_1 = \lambda V.\text{if } a \in V \text{ then } V \cup \{b\} \text{ else } V - \{b\}$$

$$f_2 = \lambda V.\text{if } b \in V \text{ then } \{c\} \text{ else } \phi$$



$$f_2 \circ f_1(\{a, c\}) = \boxed{\{c\}}$$



Λ x y
start main
x = 3
p(x,y)
return from p
printf(y)
exit main
YES!
Might y be uninitialized here?
start p(a,b)
if . . .
b = a
p(a,b)
return from p
printf(b)
exit p
Λ a b
NO!

The Tabulation Algorithm

- Worklist algorithm, start from entry of "main"
- Keep track of
  - Path edges: matched paren paths from procedure entry
  - Summary edges: matched paren call-return paths
- At each instruction
  - Propagate facts using transfer functions; extend path edges
- At each call
  - Propagate to procedure entry, start with an empty path
  - If a summary for that entry exits, use it
- At each exit
  - Store paths from corresponding call points as summary paths
  - When a new summary is added, propagate to the return node

Interprocedural Dataflow Analysis via CFL-Reachability

- Graph: Exploded control-flow graph
- *L*: *L*(*unbalLeft*)
  - unbalLeft = valid
- Fact *d* holds at *n* iff there is an *L*(*unbalLeft*)-path from $\langle start_{main}, \Lambda \rangle$ to $\langle n, d \rangle$

Asymptotic Running Time

- CFL-reachability
  - Exploded control-flow graph: ND nodes
  - Running time: $O(N^3D^3)$
- Exploded control-flow graph → Special structure

Running time: $O(ED^3)$

Typically: $E \approx N$, hence $O(ED^3) \approx O(ND^3)$

"Gen/kill" problems: $O(ED)$

IDE

- Goes beyond IFDS problems
  - Can handle unbounded domains
- Requires special form of the domain
- Can be **much** more efficient than IFDS

Example Linear Constant Propagation

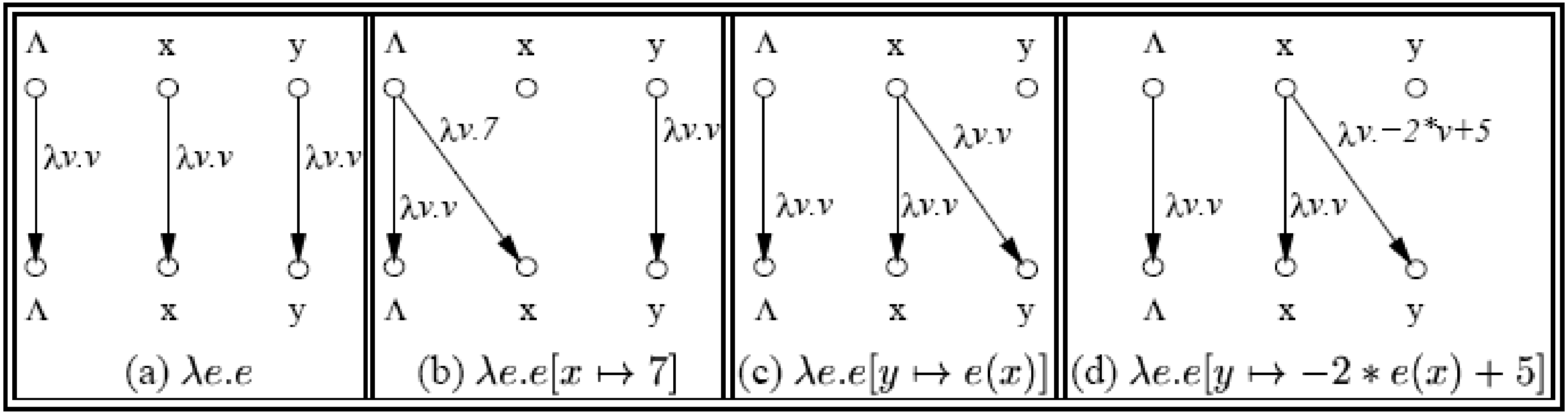
- Consider the constant propagation lattice
- The value of every variable y at the program exit can be represented by:

  $y = \sqcup \{(a_x x + b_x) \mid x \in Var_* \} \sqcup c$

  $a_x, c \in Z \cup \{\bot, \top\} \qquad b_x \in Z$

- Supports efficient composition and "functional" join
  - [z := a * y + b]
  - What about [z:=x+y]?

Linear constant propagation



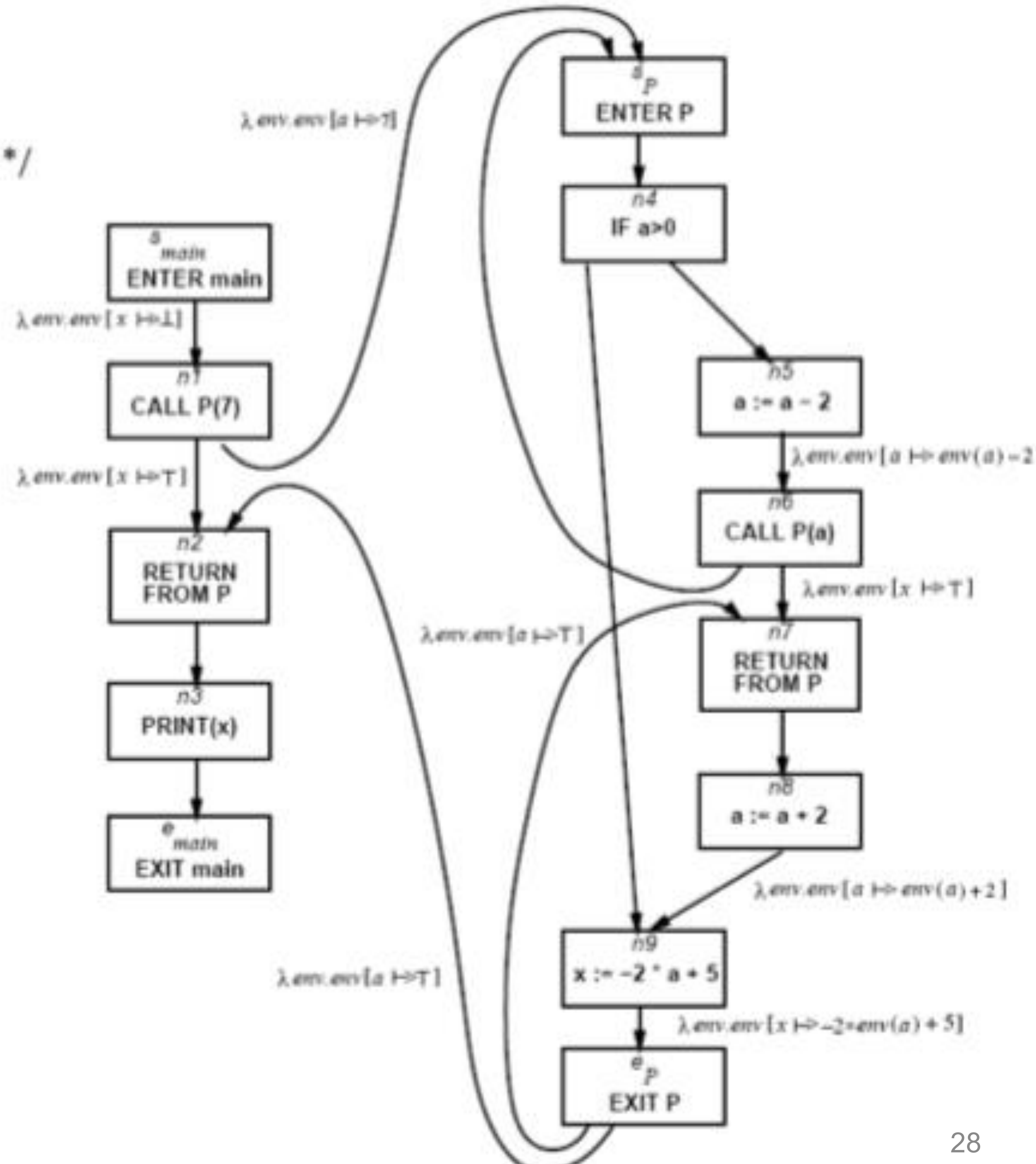
Point-wise representation of environment transformers

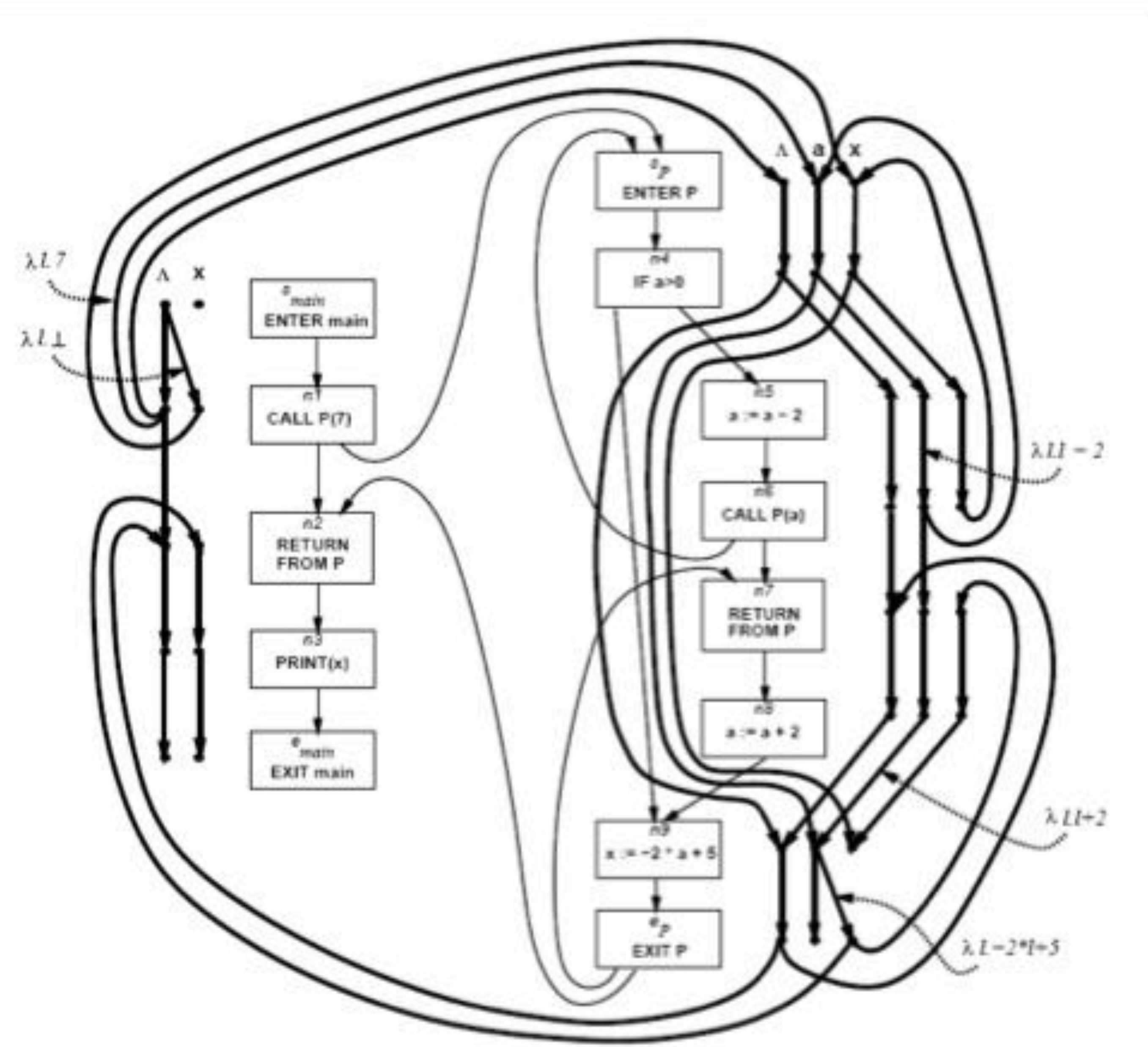
IDE Analysis

- Point-wise representation closed under composition
- CFL-Reachability on the exploded graph
- Compose functions

```
declare x: integer
program main
begin
        call P(7)
        print (x) /* x is a constant here */
end

procedure P (value a : integer)
begin /* a is not a constant here */
        if a > 0 then
           a := a − 2
           call P (a)
           a := a + 2
        fi
        x := −2 ∗ a + 5
        /* x is not a constant here */
end
```




λl.7
λl.⊥
Λ
x
s_{main}
ENTER main
n1
CALL P(7)
n2
RETURN FROM P
n3
PRINT(x)
e_{main}
EXIT main
s_P
ENTER P
n4
IF a>0
n5
a := a - 2
n6
CALL P(a)
n7
RETURN FROM P
n8
a := a + 2
n9
x := -2 * a + 5
e_P
EXIT P
Λ
a
x
λl.l - 2
λl.l+2
λl.-2*l+5

Costs

- $O(ED^3)$
- Class of value transformers $F \subseteq L \rightarrow L$
  - $id \in F$
  - Finite height
- Representation scheme with (efficient)
    - Application
    - Composition
    - Join
    - Equality
    - Storage

Conclusion

- Handling functions is crucial for abstract interpretation
- Virtual functions and exceptions complicate things
- But scalability is an issue
  - Small call strings
  - Small functional domains
  - Demand analysis

Challenges in Interprocedural Analysis

- Respect call-return mechanism
- Handling recursion
- Local variables
- Parameter passing mechanisms
- The called procedure is not always known
- The source code of the called procedure is not always available

A trivial treatment of procedure

- Analyze a single procedure
- After every call continue with conservative information
  - Global variables and local variables which “may be modified by the call” have unknown values
- Can be easily implemented
- Procedures can be written in different languages
- Procedure inline can help

Disadvantages of the trivial solution

- Modular (object oriented and functional) programming encourages small frequently called procedures
- Almost all information is lost

Bibliography

- **Textbook 2.5**
- Patrick Cousot & Radhia Cousot. Static determination of dynamic properties of recursive procedures In *IFIP Conference on Formal Description of Programming Concepts*, E.J. Neuhold, (Ed.), pages 237-277, St-Andrews, N.B., Canada, 1977. North-Holland Publishing Company (1978).
- **Two Approaches to interprocedural analysis by Micha Sharir and Amir Pnueli**
- IDFS Interprocedural Distributive Finite Subset Precise interprocedural dataflow analysis via graph reachability. *Reps, Horowitz, and Sagiv, POPL' 95*
- **IDE** Interprocedural Distributive Environment Precise interprocedural dataflow analysis with applications to constant propagation. *Sagiv, Reps, Horowitz, and TCS' 96*

A Semantics for Procedure Local Heaps and its Abstractions

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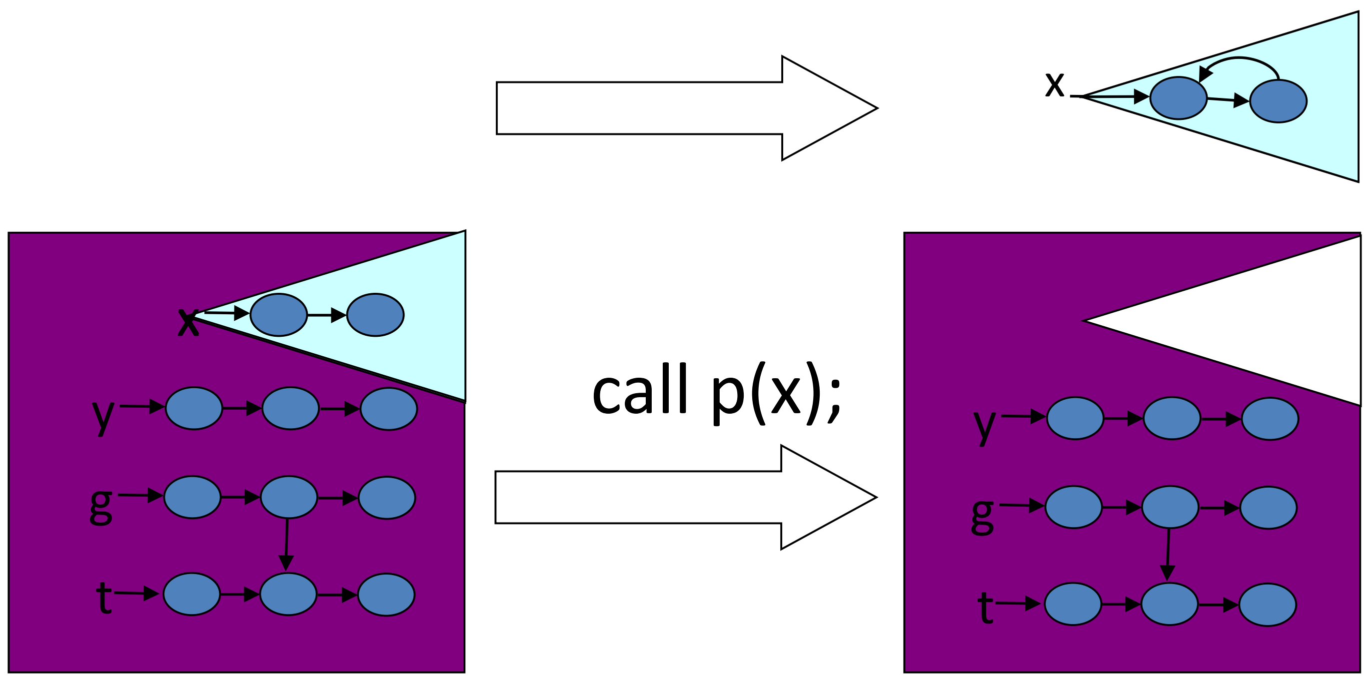
Motivation

- Interprocedural shape analysis
  - Conservative static pointer analysis
  - Heap intensive programs
    - Imperative programs with procedures
    - Recursive data structures
- Challenge
  - Destructive update
  - Localized effect of procedures

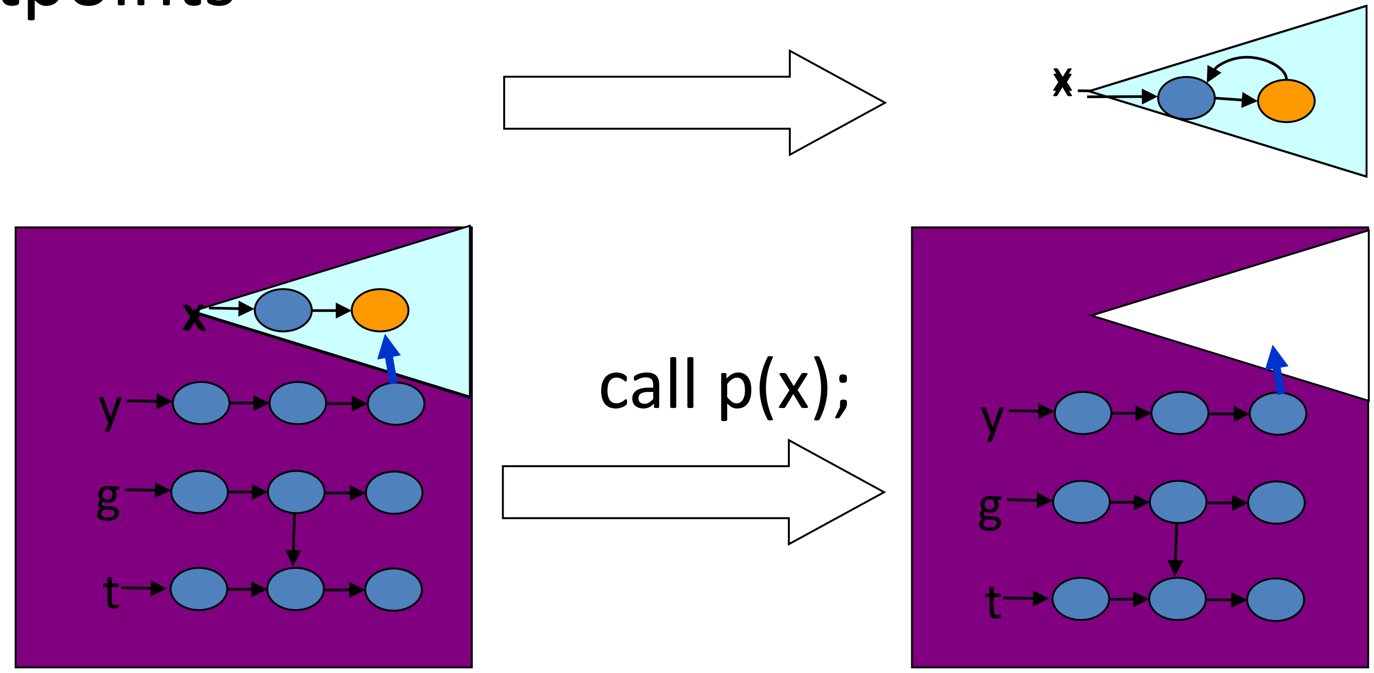
Main idea

- Local heaps



Main idea

- Local heaps
- Cutpoints

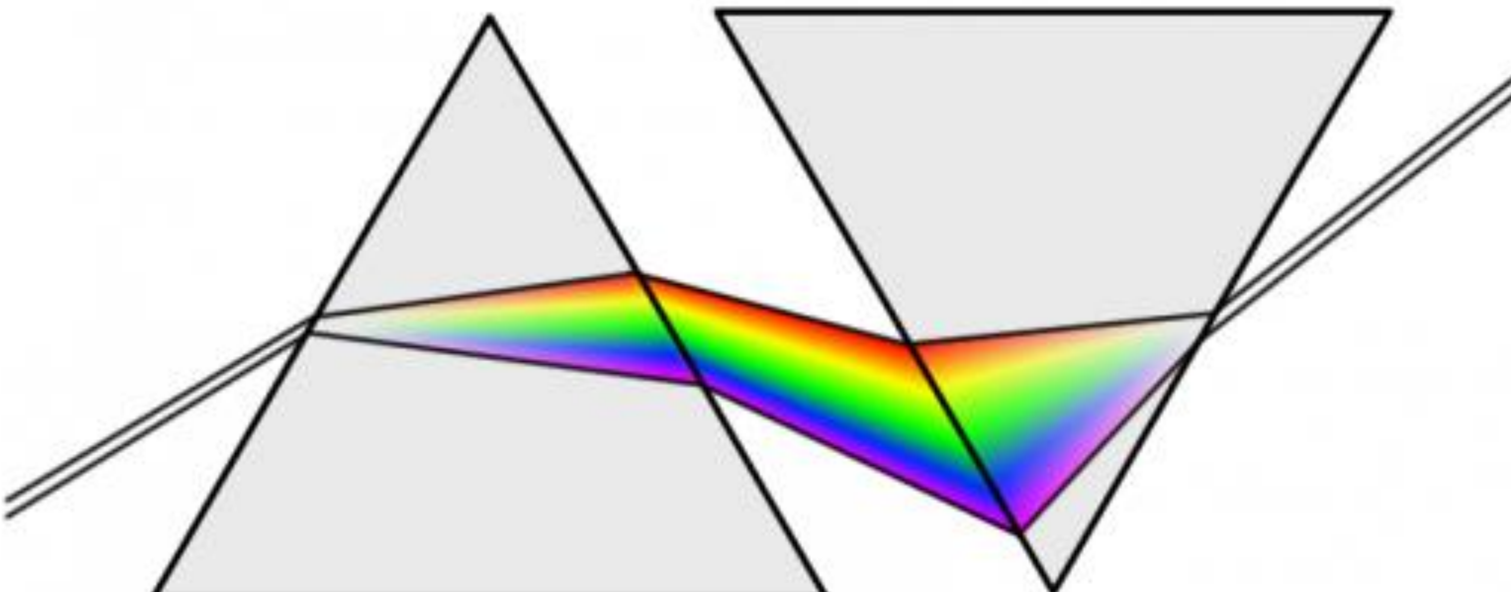


Numerical Analysis

Abstract Interpretation [Cousot'77]

- Mathematical foundation of static analysis

Widening/Narrowing



How can we prove this automatically?

```
public void loopExample() {
    int x = 7;
    while (x < 1000) {
        ++x;
    }
    if (!(x == 1000))
        error("Unable to prove x == 1000!");
}
```

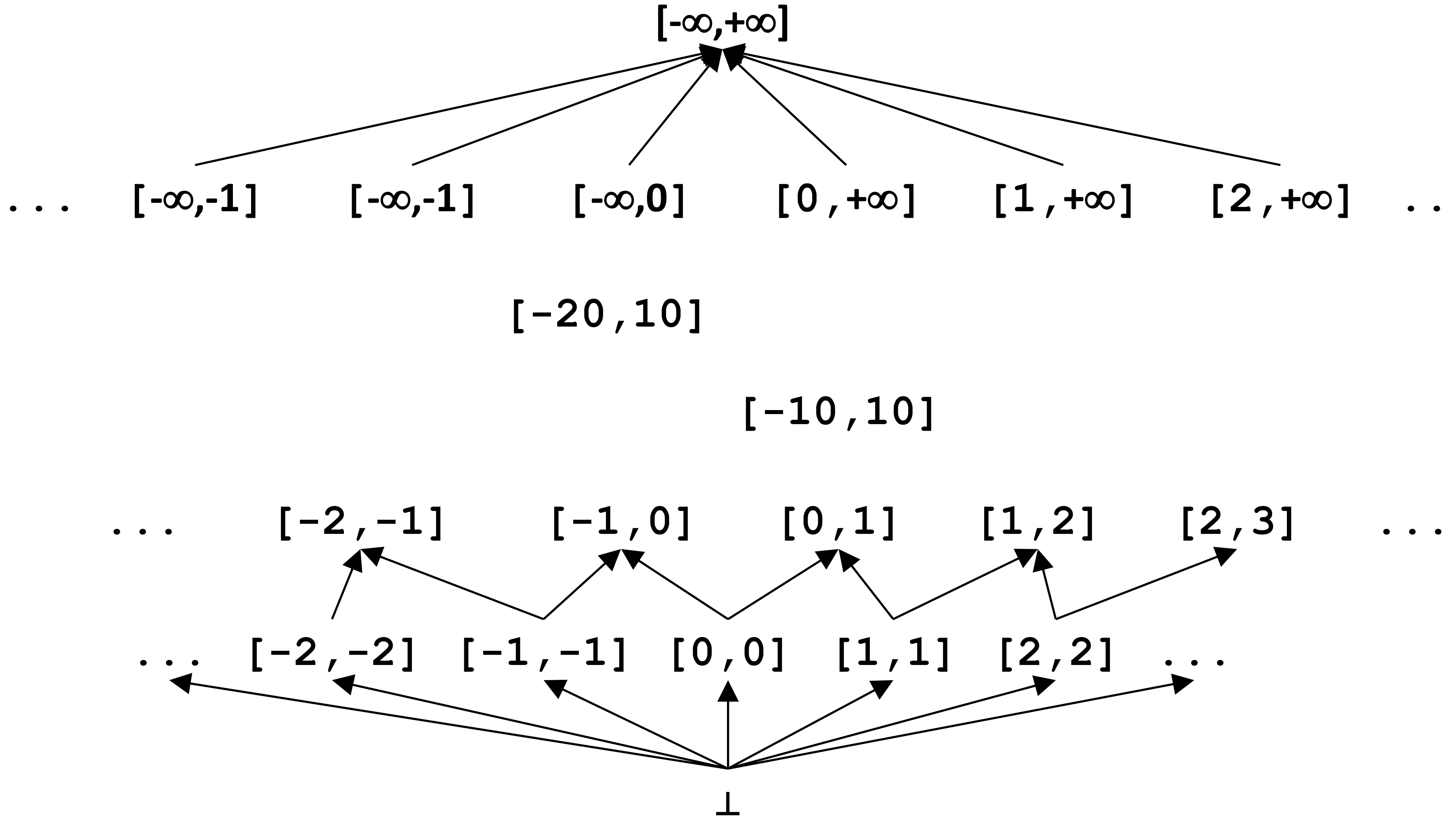
RelProd(CP, VE)

```
Reached fixed-point after 19 iterations.
Solution = {
  V[0] : (true, true)
  V[1] : (true, true)
  V[2] : (x=7, true)
  V[3] : (x=7, true)
  V[4] : (true, true)
  V[7] : (true, true)
  V[5] : (true, true)
  V[6] : (true, true)
  V[8] : (true, true)
  V[9] : (true, true)
  V[10] : (true, true)
  V[12] : (true, true)
  V[11] : (true, true)
}
1 possible errors found.
```

Intervals domain

- One of the simplest numerical domains
- Maintain for each variable *x* an interval [*L*,*H*]
  - *L* is either an integer of -∞
  - *H* is either an integer of +∞
- A (non-relational) numeric domain

Intervals lattice for variable *x*



Intervals lattice for variable *x*

- $D^{int}[x]$ = { (L,H) | L∈-∞,**Z** and H∈**Z**,+∞ and $L \leq H$}
- ⊥
- ⊤=[-∞,+∞]
- ⊑ = ?
  - [1,2] ⊑ [3,4] ?
  - [1,4] ⊑ [1,3] ?
  - [1,3] ⊑ [1,4] ?
  - [1,3] ⊑ [-∞,+∞] ?
- What is the lattice height?

Intervals lattice for variable *x*

- $D^{int}[x] = \{ (L,H) \mid L \in -\infty, \mathbf{Z}$ and $H \in \mathbf{Z}, +\infty$ and $L \leq H \}$
- $\bot$
- $\top = [-\infty, +\infty]$
- $\sqsubseteq$ = ?
  - $[1,2] \sqsubseteq [3,4]$ no
  - $[1,4] \sqsubseteq [1,3]$ no
  - $[1,3] \sqsubseteq [1,4]$ yes
  - $[1,3] \sqsubseteq [-\infty, +\infty]$ yes
- What is the lattice height? Infinite

Joining/meeting intervals

- $[a,b] \sqcup [c,d]$ = ?
  - $[1,1] \sqcup [2,2]$ = ?
  - $[1,1] \sqcup [2, +\infty]$ = ?
- $[a,b] \sqcap [c,d]$ = ?
  - $[1,2] \sqcap [3,4]$ = ?
  - $[1,4] \sqcap [3,4]$ = ?
  - $[1,1] \sqcap [1,+\infty]$ = ?
- Check that indeed $x \sqsubseteq y$ if and only if $x \sqcup y = y$

Joining/meeting intervals

- $[a,b] \sqcup [c,d] = [\min(a,c), \max(b,d)]$
  - $[1,1] \sqcup [2,2] = [1,2]$
  - $[1,1] \sqcup [2,+\infty] = [1,+\infty]$
- $[a,b] \sqcap [c,d] = [\max(a,c), \min(b,d)]$ if a proper interval and otherwise $\perp$
  - $[1,2] \sqcap [3,4] = \perp$
  - $[1,4] \sqcap [3,4] = [3,4]$
  - $[1,1] \sqcap [1,+\infty] = [1,1]$
- Check that indeed $x \sqsubseteq y$ if and only if $x \sqcup y = y$

Interval domain for programs

- $D^{int}[x] = \{ (L,H) \mid L \in -\infty,\mathbf{Z}$ and $H \in \mathbf{Z},+\infty$ and $L \leq H\}$
- For a program with variables $Var=\{x_1,\ldots,x_k\}$
- $D^{int}[Var] = ?$

Interval domain for programs

- $D^{int}[x] = \{ (L,H) \mid L \in -\infty,\mathbf{Z}$ and $H \in \mathbf{Z},+\infty$ and $L \leq H\}$
- For a program with variables $Var=\{x_1,\ldots,x_k\}$
- $D^{int}[Var] = D^{int}[x_1] \times \ldots \times D^{int}[x_k]$
- How can we represent it in terms of formulas?

Interval domain for programs

- $D^{int}[x]$ = { (L,H) | L∈-∞,**Z** and H∈**Z**,+∞ and $L \leq H$}
- For a program with variables *Var*={$x_1,\ldots,x_k$}
- $D^{int}[Var] = D^{int}[x_1] \times \ldots \times D^{int}[x_k]$
- How can we represent it in terms of formulas?
  - Two types of factoids $x \geq c$ and $x \leq c$
  - Example: $S = \wedge\{x \geq 9, y \geq 5, y \leq 10\}$
  - Helper operations
    - c + +∞ = +∞
    - remove(S, x) = S without any x-constraints
    - lb(S, x) =

Assignment transformers

- $[\![x := c]\!]^\# S = ?$
- $[\![x := y]\!]^\# S = ?$
- $[\![x := y+c]\!]^\# S = ?$
- $[\![x := y+z]\!]^\# S = ?$
- $[\![x := y*c]\!]^\# S = ?$
- $[\![x := y*z]\!]^\# S = ?$

Assignment transformers

- $[\![x := c]\!]^\# S$ = remove(S,x) $\cup$ {$x \geq c$, $x \leq c$}
- $[\![x := y]\!]^\# S$ = remove(S,x) $\cup$ {$x \geq$lb(S,y), $x \leq$ub(S,y)}
- $[\![x := y+c]\!]^\# S$ = remove(S,x) $\cup$ {$x \geq$lb(S,y)+c, $x \leq$ub(S,y)+c}
- $[\![x := y+z]\!]^\# S$ = remove(S,x) $\cup$ {$x \geq$lb(S,y)+lb(S,z),
  $x \leq$ub(S,y)+ub(S,z)}
- $[\![x := y*c]\!]^\# S$ = remove(S,x) $\cup$ if c>0 {$x \geq$lb(S,y)*c, $x \leq$ub(S,y)*c}
  else {$x \geq$ub(S,y)*-c, $x \leq$lb(S,y)*-c}
- $[\![x := y*z]\!]^\# S$ = remove(S,x) $\cup$?

`assume` transformers

- $[\![$**`assume`** $x=c]\!]^\# S$ = ?
- $[\![$**`assume`** $x<c]\!]^\# S$ = ?
- $[\![$**`assume`** $x=y]\!]^\# S$ = ?
- $[\![$**`assume`** $x\neq c]\!]^\# S$ = ?

assume transformers

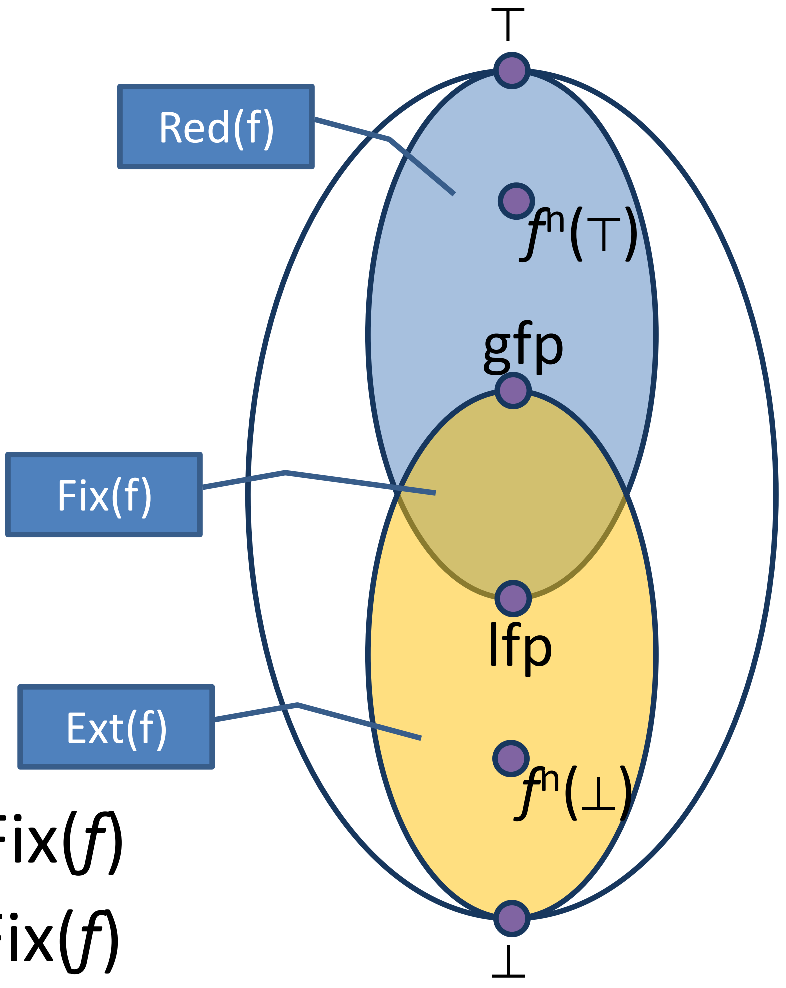
- $[\![$**assume** x=$c]\!]^\#\ S = S \sqcap \{x \geq c,\ x \leq c\}$
- $[\![$**assume** x<$c]\!]^\#\ S = S \sqcap \{x \leq c\text{-}1\}$
- $[\![$**assume** x=$y]\!]^\#\ S = S \sqcap \{x \geq \mathrm{lb}(S,y),\ x \leq \mathrm{ub}(S,y)\}$
- $[\![$**assume** x≠$c]\!]^\#\ S$ = ?

assume transformers

- $[\![$**assume** $x=c]\!]^\# S = S \sqcap \{x \geq c, x \leq c\}$
- $[\![$**assume** $x<c]\!]^\# S = S \sqcap \{x \leq c\text{-}1\}$
- $[\![$**assume** $x=y]\!]^\# S = S \sqcap \{x \geq \mathrm{lb}(S,y), x \leq \mathrm{ub}(S,y)\}$
- $[\![$**assume** $x \neq c]\!]^\# S = (S \sqcap \{x \leq c\text{-}1\}) \sqcup (S \sqcap \{x \geq c+1\})$

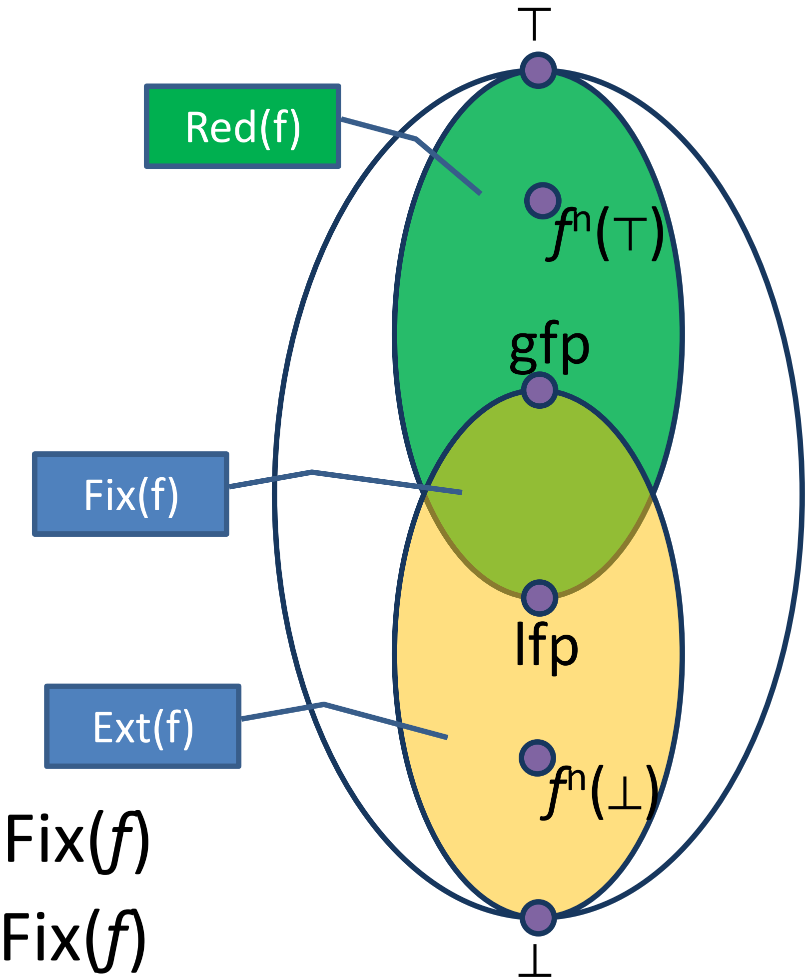
Effect of function *f* on lattice elements

- $L = (D, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$
- $f : D \rightarrow D$ **monotone**
- $\text{Fix}(f) = \{ d \mid f(d) = d \}$
- $\text{Red}(f) = \{ d \mid f(d) \sqsubseteq d \}$
- $\text{Ext}(f) = \{ d \mid d \sqsubseteq f(d) \}$
- **Theorem** [Tarski 1955]
  - $\text{lfp}(f) = \sqcap\text{Fix}(f) = \sqcap\text{Red}(f) \in \text{Fix}(f)$
  - $\text{gfp}(f) = \sqcup\text{Fix}(f) = \sqcup\text{Ext}(f) \in \text{Fix}(f)$



Effect of function *f* on lattice elements

- $L = (D, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$
- $f : D \to D$ **monotone**
- $\text{Fix}(f) = \{ d \mid f(d) = d \}$
- $\text{Red}(f) = \{ d \mid f(d) \sqsubseteq d \}$
- $\text{Ext}(f) = \{ d \mid d \sqsubseteq f(d) \}$
- **Theorem** [Tarski 1955]
  - $\text{lfp}(f) = \sqcap\text{Fix}(f) = \sqcap\text{Red}(f) \in \text{Fix}(f)$
  - $\text{gfp}(f) = \sqcup\text{Fix}(f) = \sqcup\text{Ext}(f) \in \text{Fix}(f)$



Continuity and ACC condition

- Let $L = (D, \sqsubseteq, \sqcup, \bot)$ be a complete partial order
  - Every ascending chain has an upper bound
- A function f is continuous if for every increasing chain $Y \subseteq D^*$,

$$f(\sqcup Y) = \sqcup\{ f(y) \mid y \in Y \}$$

- L satisfies the ascending chain condition (ACC) if every ascending chain eventually stabilizes:

$$d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n = d_{n+1} = \ldots$$

Fixed-point theorem [Kleene]

- Let $L = (D, \sqsubseteq, \sqcup, \bot)$ be a complete partial order and a **continuous** function f: $D \to D$ then

$$\text{lfp}(f) = \bigsqcup_{n \in N} f^n(\bot)$$

Resulting algorithm

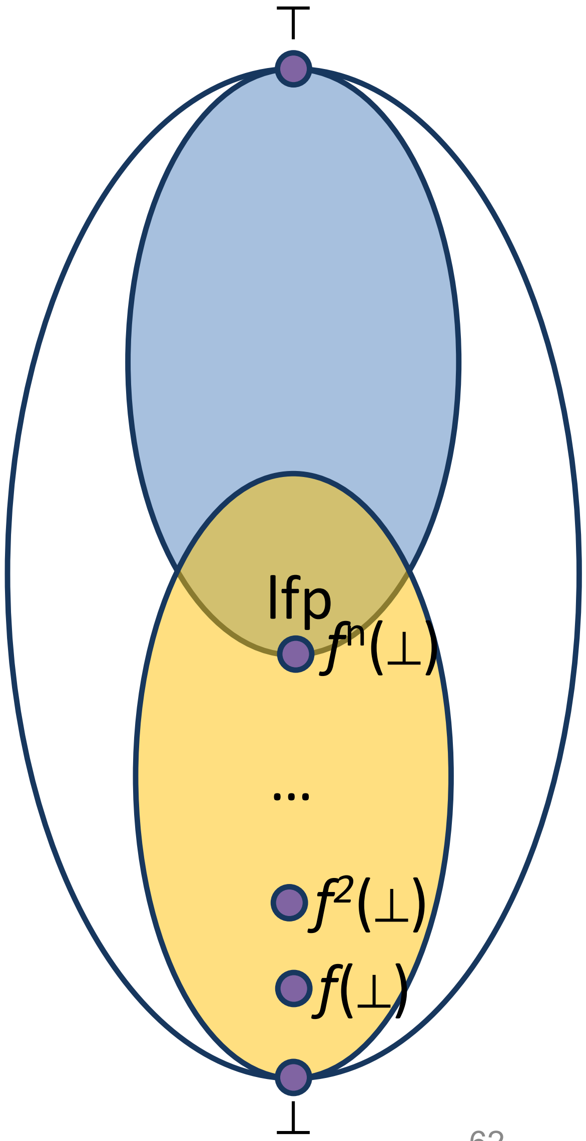
- Kleene's fixed point theorem gives a constructive method for computing the lfp

Mathematical definition

$$\text{lfp}(f) = \bigsqcup_{n \in N} f^n(\bot)$$

Algorithm

$d := \bot$
while $f(d) \neq d$ **do**
    $d := d \sqcup f(d)$
return d



Chaotic iteration

- Input:
  - A cpo $L = (D, \sqsubseteq, \sqcup, \perp)$ satisfying ACC
  - $L^n = L \times L \times \ldots \times L$
  - A monotone function $f : D^n \to D^n$
  - A system of equations $\{ X[\mathrm{i}] \mid f(X) \mid 1 \leq i \leq n \}$
- Output: lfp(f)
- A worklist-based algorithm

```
for i:=1 to n do
   X[i] := ⊥
WL = {1,…,n}
while WL ≠ ∅ do
   j := pop WL // choose index non-deterministically
   N := F[i](X)
   if N ≠ X[i] then
      X[i] := N
      add all the indexes that directly depend on i to WL
      (X[j] depends on X[i] if F[j] contains X[i])
return X
```

Concrete semantics equations

```
public void loopExample() {
R[0]  int x = 7;  R[1]
R[2]  while (x < 1000) {
R[3]      ++x;  R[4]
      }
R[5]  if (!(x == 1000))
R[6]      error("Unable to prove x == 1000!");
}
```

- $R[0] = \{\mathbf{x} \in \mathbf{Z}\}$
  $R[1] = [\![\mathbf{x:=7}]\!]$
  $R[2] = R[1] \cup R[4]$
  $R[3] = R[2] \cap \{s \mid s(x) < 1000\}$
  $R[4] = [\![\mathbf{x:=x+1}]\!]\ R[3]$
  $R[5] = R[2] \cap \{s \mid s(x) \geq 1000\}$
  $R[6] = R[5] \cap \{s \mid s(x) \neq 1001\}$

Abstract semantics equations

```
public void loopExample() {
R[0]  int x = 7;  R[1]
R[2]  while (x < 1000) {
R[3]       ++x;  R[4]
      }
R[5]  if (!(x == 1000))
R[6]       error("Unable to prove x == 1000!");
}
```

- $R[0] = \alpha(\{\mathbf{x} \in \mathbf{Z}\})$
  $R[1] = [\![\mathbf{x:=7}]\!]^{\#}$
  $R[2] = R[1] \sqcup R[4]$
  $R[3] = R[2] \sqcap \alpha(\{s \mid s(x) < 1000\})$
  $R[4] = [\![\mathbf{x:=x+1}]\!]^{\#}\ R[3]$
  $R[5] = R[2] \sqcap \alpha(\{s \mid s(x) \geq 1000\})$
  $R[6] = R[5] \sqcap \alpha(\{s \mid s(x) \geq 1001\}) \sqcup R[5] \sqcap \alpha(\{s \mid s(x) \leq 999\})$

Abstract semantics equations

```
public void loopExample() {
R[0]  int x = 7;  R[1]
R[2]  while (x < 1000) {
R[3]      ++x;  R[4]
      }
R[5]  if (!(x == 1000))
R[6]      error("Unable to prove x == 1000!");
}
```

- $R[0] = \top$
  $R[1] = [7,7]$
  $R[2] = R[1] \sqcup R[4]$
  $R[3] = R[2] \sqcap [-\infty,999]$
  $R[4] = R[3] + [1,1]$
  $R[5] = R[2] \sqcap [1000,+\infty]$
  $R[6] = R[5] \sqcap [999,+\infty] \sqcup R[5] \sqcap [1001,+\infty]$

Too many iterations to converge

```
Iteration 3981: processing V[8] = Interval[x==1000](V[6]) // if x == 1000 goto return
              V[8] : false
              V[6] : and(x=1000)
              V[8]' : and(x=1000)
              Adding [V[12] = Join_IntervalDomain(V[8], V[10]) // return]
              workSet = {V[12]}
Iteration 3982: processing V[12] = Join_IntervalDomain(V[8], V[10]) // return
              V[12] : false
              V[8] : and(x=1000)
              V[10] : false
              V[12]' : and(x=1000)
              Adding [V[11] = V[12] // return]
              workSet = {V[11]}
Iteration 3983: processing V[11] = V[12] // return
              V[11] : false
              V[12] : and(x=1000)
              V[11]' : and(x=1000)
              Adding []
Reached fixed-point after 3983 iterations.
Solution = {
  V[0] : true
  V[1] : true
  V[2] : and(x=7)
  V[3] : and(x=7)
  V[4] : and(8<=x<=1000)
  V[7] : and(7<=x<=1000)
  V[5] : and(7<=x<=999)
  V[6] : and(x=1000)
  V[8] : and(x=1000)
  V[9] : false
  V[10] : false
  V[12] : and(x=1000)
  V[11] : and(x=1000)
}
0 possible errors found.
Writing to sootOutput\IntervalExample.jimple
Soot finished on Wed Jun 12 06:24:14 IDT 2013
Soot has run for 0 min. 1 sec.
```

How many iterations for this one?

```
public void loopExample2(int y) {
    int x = 7;
    if (x < y) {
        while (x < y) {
            ++x;
        }

        if (x != y)
            error("Unable to prove x = y!");
    }
}
```

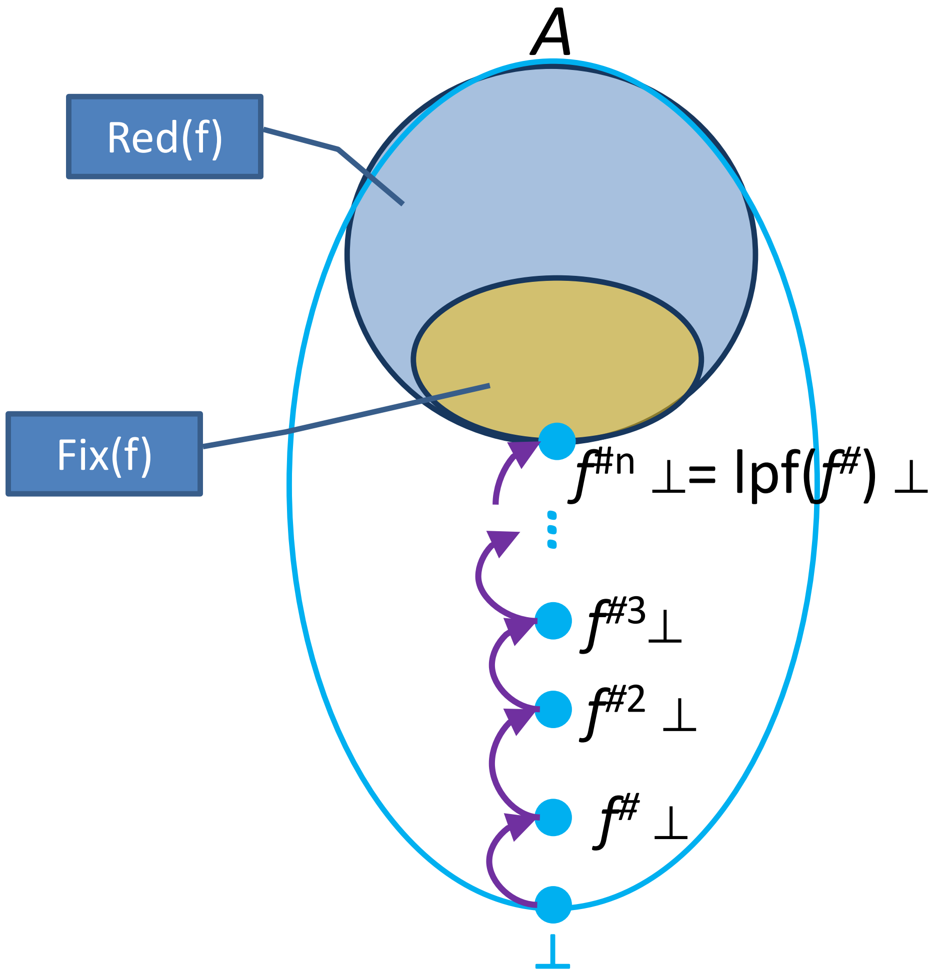
Widening

- Introduce a new binary operator to ensure termination
  - A kind of extrapolation
- Enables static analysis to use infinite height lattices
  - Dynamically adapts to given program
- Tricky to design
- Precision less predictable then with finite-height domains (widening non-monotone)

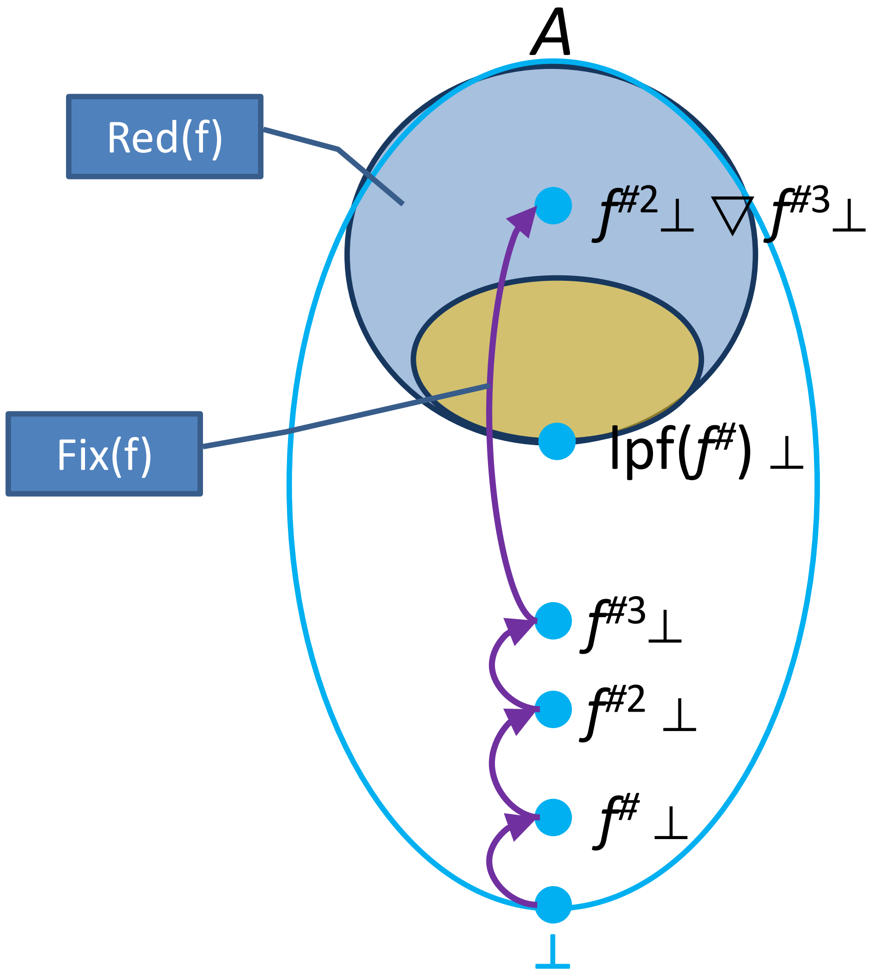
Formal definition

- For all elements $d_1 \sqcup d_2 \sqsubseteq d_1 \triangledown d_2$
- For all ascending chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ the following sequence eventually stabilizes
  - $y_0 = d_0$
  - $y_{i+1} = y_i \triangledown d_{i+1}$
- For a monotone function $f : D \rightarrow D$ define
  - $x_0 = \bot$
  - $x_{i+1} = x_i \triangledown f(x_i)$
- Theorem:
  - There exits k such that $x_{k+1} = x_k$
  - $x_k \in Red(f) = \{ d \mid d \in D \text{ and } f(d) \sqsubseteq d \}$

Analysis with finite-height lattice



Analysis with widening



Widening for Intervals Analysis

- $\bot \nabla [c, d] = [c, d]$
- $[a, b] \ \nabla [c, d] = [$

  if $a \leq c$

  then a

  else $-\infty$,

  if $b \geq d$

  then b

  else ∞

Semantic equations with widening

```
public void loopExample() {
R[0]  int x = 7;  R[1]
R[2]  while (x < 1000) {
R[3]      ++x;  R[4]
      }
R[5]  if (!(x == 1000))
R[6]      error("Unable to prove x == 1000!");
}
```

- $R[0] = \top$
  $R[1] = [7,7]$
  $R[2] = R[1] \sqcup R[4]$
  $R[2.1] = R[2.1] \triangledown R[2]$
  $R[3] = R[2.1] \sqcap [-\infty,999]$
  $R[4] = R[3] + [1,1]$
  $R[5] = R[2] \sqcap [1001,+\infty]$
  $R[6] = R[5] \sqcap [999,+\infty] \sqcup R[5] \sqcap [1001,+\infty]$

Non monotonicity of widening

- $[0,1] \bigtriangledown [0,2] = ?$
- $[0,2] \bigtriangledown [0,2] = ?$

Non monotonicity of widening

- $[0,1] \triangledown [0,2] = [0, \infty]$
- $[0,2] \triangledown [0,2] = [0,2]$

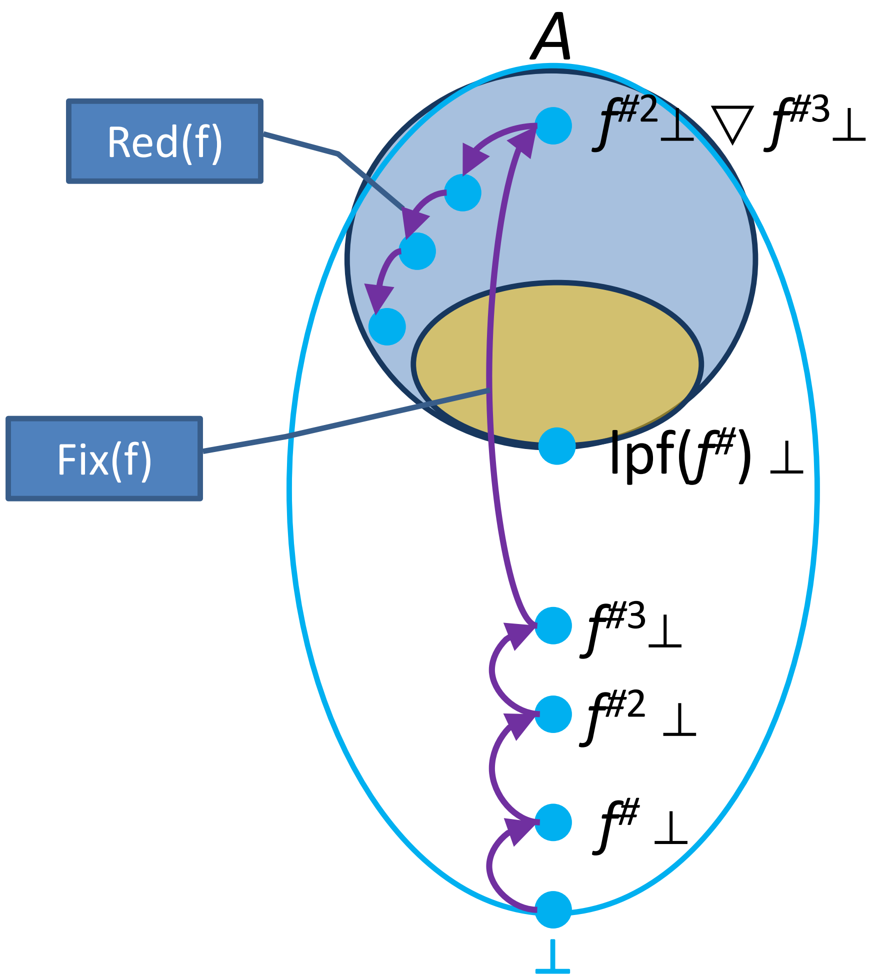
Analysis results with widening

```
Analyzing method loopExample
-------------------------------------------
Solving the following equation system =
V[0] = true // this := @this: IntervalExample
V[1] = AssignTopTransformer(V[0]) // this := @this: IntervalExample
V[2] = AssignConstantToVarTransformer(V[1]) // x = 7
V[3] = V[2] // goto [?= (branch)]
V[4] = AssignAddExprToVarTransformer(V[5]) // x = x + 1
V[7] = JoinLoop_IntervalDomain(V[3], V[4]) // if x < 1000 goto x = x + 1
V[8] = IntervalDomain[Widening|Narrowing](V[8], V[7]) // if x < 1000 goto x = x + 1
V[5] = Interval[x<1000](V[8]) // if x < 1000 goto x = x + 1
V[6] = Interval[x>=1000](V[8]) // if x < 1000 goto x = x + 1
V[9] = Interval[x==1000](V[6]) // if x == 1000 goto return
V[10] = Interval[x!=1000](V[6]) // if x == 1000 goto return
V[11] = V[10] // specialinvoke this.<IntervalExample: void error(java.lang.String)>("Unable to prove x == 1000!")
V[13] = Join_IntervalDomain(V[9], V[11]) // return
V[12] = V[13] // return
```

```
Reached fixed-point after 23 iterations.
Solution = {
  V[0] : true
  V[1] : true
  V[2] : and(x=7)
  V[3] : and(x=7)
  V[4] : and(8<=x<=1000)
  V[7] : and(7<=x<=1000)
  V[8] : and(x>=7)
  V[5] : and(7<=x<=999)
  V[6] : and(x>=1000)
  V[9] : and(x=1000)
  V[10] : and(x>=1001)
  V[11] : and(x>=1001)
  V[13] : and(x>=1000)
  V[12] : and(x>=1000)
}
```

Did we prove it?

Analysis with narrowing



Formal definition of narrowing

- Improves the result of widening
- $y \sqsubseteq x \Rightarrow y \sqsubseteq (x \triangle y) \sqsubseteq x$
- For all decreasing chains $x_0 \sqsupseteq x_1 \sqsupseteq \ldots$ the following sequence is finite
  - $y_0 = x_0$
  - $y_{i+1} = y_i \triangle x_{i+1}$
- For a monotone function $f: D \rightarrow D$ and $x_k \in \mathrm{Red}(f) = \{ d \mid d \in D \text{ and } f(d) \sqsubseteq d \}$ define
  - $y_0 = x$
  - $y_{i+1} = y_i \triangle f(y_i)$
- Theorem:
  - There exits k such that $y_{k+1} = y_k$
  - $y_k \in \mathrm{Red}(f) = \{ d \mid d \in D \text{ and } f(d) \sqsubseteq d \}$

Narrowing for Interval Analysis

- $[a, b] \triangle \bot = [a, b]$
- $[a, b] \triangle [c, d] = [$

    if $a = -\infty$

    then c

    else a,

  if $b = \infty$

    then d

    else b

    ]

Semantic equations with narrowing

```
public void loopExample() {
R[0]  int x = 7; R[1]
R[2]  while (x < 1000) {
R[3]      ++x; R[4]
      }
R[5]  if (!(x == 1000))
R[6]      error("Unable to prove x == 1000!");
}
```

- R[0] = $\top$
  R[1] = [7,7]
  R[2] = R[1] $\sqcup$ R[4]
  R[2.1] = R[2.1] $\triangle$ R[2]
  R[3] = R[2.1] $\sqcap$ [-∞,999]
  R[4] = R[3]+[1,1]
  R[5] = R[2]$^{\#}$ $\sqcap$ [1000,+∞]
  R[6] = R[5] $\sqcap$ [999,+∞] $\sqcup$ R[5] $\sqcap$ [1001,+∞]

Analysis with widening/narrowing

- Two phases
  - Phase 1: analyze with widening until converging
  - Phase 2: use values to analyze with narrowing

```
public void loopExample() {
    int x = 7;
    while (x < 1000) {
        ++x;
    }
    if (!(x == 1000))
        error("Unable to prove x == 1000!");
}
```

Phase 1:

$R[0] = \top$

$R[1] = [7,7]$

$R[2] = R[1] \sqcup R[4]$

$R[2.1] = R[2.1] \bigtriangledown R[2]$

$R[3] = R[2.1] \sqcap [-\infty,999]$

$R[4] = R[3] + [1,1]$

$R[5] = R[2] \sqcap [1001,+\infty]$

$R[6] = R[5] \sqcap [999,+\infty] \sqcup R[5] \sqcap [1001,+\infty]$

Phase 2:

$R[0] = \top$

$R[1] = [7,7]$

$R[2] = R[1] \sqcup R[4]$

$R[2.1] = R[2.1] \bigtriangleup R[2]$

$R[3] = R[2.1] \sqcap [-\infty,999]$

$R[4] = R[3]+[1,1]$

$R[5] = R[2]^{\#} \sqcap [1000,+\infty]$

$R[6] = R[5] \sqcap [999,+\infty] \sqcup R[5] \sqcap [1001,+\infty]$

Analysis with widening/narrowing

```
Reached fixed-point after 23 iterations.
Solution = {
  V[0] : true
  V[1] : true
  V[2] : and(x=7)
  V[3] : and(x=7)
  V[4] : and(8<=x<=1000)
  V[7] : and(7<=x<=1000)
  V[8] : and(x>=7)
  V[5] : and(7<=x<=999)
  V[6] : and(x>=1000)
  V[9] : and(x=1000)
  V[10] : and(x>=1001)
  V[11] : and(x>=1001)
  V[13] : and(x>=1000)
  V[12] : and(x>=1000)
}

Starting chaotic iteration: narrowing phase...
              workSet = {V[0], V[1], V[2], V[3], V[4], V[7], V[8], V[5], V[6], V[9], V[10], V[11], V[13], V[12]}
Iteration 24: processing V[0] = true // this := @this: IntervalExample
              V[0] : true
              V[0]' : true
              workSet = {V[12], V[1], V[2], V[3], V[4], V[7], V[8], V[5], V[6], V[9], V[10], V[11], V[13]}
```

Analysis results widening/narrowing

```
Iteration 44: processing V[1] = AssignTopTransformer(V[0]) // this := @this: IntervalExample
              V[1] : true
              V[0] : true
              V[1]' : true
Reached fixed-point after 44 iterations.
Solution = {
  V[0] : true
  V[1] : true
  V[2] : and(x=7)
  V[3] : and(x=7)
  V[4] : and(8<=x<=1000)
  V[7] : and(7<=x<=1000)
  V[8] : and(7<=x<=1000)
  V[5] : and(7<=x<=999)
  V[6] : and(x=1000)
  V[9] : and(x=1000)
  V[10] : false
  V[11] : false
  V[13] : and(x=1000)
  V[12] : and(x=1000)
}
0 possible errors found.
Writing to sootOutput\IntervalExample.jimple
Soot finished on Wed Jun 12 06:47:24 IDT 2013
Soot has run for 0 min. 0 sec.
```

Precise invariant

Project

- 1-2 Students in a group
  - 3-4: Bigger projects
- Theoretical + Practical
- Your choice of topic
  - Contact me in 3 weeks
- Submission – 15/Sep
  - Code + Examples
  - Document
  - 15 minutes presentation

Past projects

- JavaScript Dominator Analysis
- Attribute Analysis for JavaScript
- Simple Pointer Analysis for C
- Adding program counters to Past Abstraction (abstraction of finite state machines.)
- Verification of Asynchronous programs
- Verifying SDNs using TVLA
- Verifying independent accesses to arrays in GO

Past projects

- Detecting index out of bound errors in C programs
- Lattice-Based Semantics for Combinatorial Models Evolution
- Verifying sorting programs
- Cross-array sorting (array of arrays) – use for storage systems version management
- Verifying LTL formulae over TVLA structures
- Worst-case memory consumption

Past projects

- Automatic loop parallelization via dependency tracking
- Handling asynchronous calls

Default Project

- Pick a framework
  - LLVM (C) : http://llvm.org/
  - Soot (Java) : https://sable.github.io/soot/

- Analysis:
  - Refined pointer analysis
  - Invent numerical domain