

# Liquid Price of Anarchy

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## Abstract

Incorporating budget constraints into the analysis of auctions has become increasingly important, as they model practical settings more accurately. The social welfare function, which is the standard measure of efficiency in auctions, is inadequate for settings with budgets, since there may be a large disconnect between the value a bidder derives from obtaining an item and what can be liquidated from her. The *Liquid Welfare* objective function has been suggested as a natural alternative for settings with budgets. Simple auctions, like simultaneous item auctions, are evaluated by their performance at equilibrium using the Price of Anarchy (PoA) measure – the ratio of the objective function value of the optimal outcome to the worst equilibrium. Accordingly, we evaluate the performance of simultaneous item auctions in budgeted settings by the *Liquid Price of Anarchy* (LPoA) measure – the ratio of the optimal Liquid Welfare to the Liquid Welfare obtained in the worst equilibrium.

Our main result is that the LPoA for mixed Nash equilibria is bounded by a constant when bidders are additive and items can be divided into sufficiently many discrete parts. Our proofs are robust, and can be extended to achieve similar bounds for simultaneous second price auctions as well as Bayesian Nash equilibria. For pure Nash equilibria, we establish tight bounds on the LPoA for the larger class of fractionally-subadditive valuations. To derive our results, we develop a new technique in which some bidders deviate (surprisingly) toward a non-optimal solution. In particular, this technique does not fit into the smoothness framework.

## 1 Introduction

Budget constraints have become an important practical consideration in most existing auctions, as reflected in recent literature (see, e.g., [3, 5, 17, 30]), because they model reality more accurately. The issue of limited liquidity of buyers arises when transaction amounts are large and may exhaust bidders' liquid assets, as is the case for privatization auctions in Eastern Europe and FCC spectrum auctions in the U.S. (see, e.g., [4]). As another example, advertisers in Google Adword auctions are instructed to specify their budget even before specifying their bids and keywords. Many other massive electronic marketplaces have a large number of participants with limited liquidity, which impose budget constraints. Buyers would not borrow money from a bank to partake in multiple auctions on eBay, and even with available credit, they only have a limited amount of attention, so that in aggregate they cannot spend too much money by participating in every auction online. Finally, budget constraints also arise in small scale systems, such as the reality TV show Storage Wars, where people participate in cash-only auctions to win the content of an expired storage locker with an unknown asset.

Maximizing social welfare is a classic objective function that has been extensively studied within the context of resource allocation problems, and auctions in particular. The *social welfare* of an allocation is the sum of agents' valuations for their allocated bundles. Unfortunately, in settings

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where agents have limited budgets (hereafter, *budgeted settings*), the social welfare objective fails to accurately capture what happens in practice. Consider, for example, an auction in which there are two bidders and one item to be allocated among the bidders. One bidder has a high value but a very small budget, while the second bidder has a medium value along with a medium budget. In this case, a high social welfare is achieved by allocating the item to the bidder who values the item highly. In contrast, most Internet advertising and electronic marketplaces (such as Google and eBay) would allocate the item in the opposite way, namely to the bidder with a medium value and budget due to monetary constraints. Hence, the social welfare objective is a poor model for how auctions are executed in reality. Indeed, it seems reasonable to favor participants with substantial investments and engagement in the economical system to maintain a healthy economy.

Following Dobzinski and Paes Leme [18], we measure the efficiency of outcomes in budgeted settings according to their *Liquid Welfare* objective, motivated as follows. In the absence of budgets, the value a buyer obtains from a given bundle captures their *willingness-to-pay* for the bundle. According to this interpretation, the social welfare objective captures the maximum revenue a seller can extract from buyers in a non-strategic setting. In budgeted settings, however, the value a buyer receives from a bundle no longer captures how much revenue can be extracted from them, since the revenue that can be extracted from a buyer is bounded by both the buyer’s value and the buyer’s budget. To reconcile this discrepancy, Dobzinski and Paes Leme [18] proposed to evaluate the welfare of buyers according to their *admissibility-to-pay*; that is, the minimum between the buyer’s value for the allocated bundle and the buyer’s budget. The aggregate welfare according to this definition is termed the *Liquid Welfare* (LW). Indeed, the Liquid Welfare is exactly the revenue a seller can extract from buyers in budgeted, non-strategic settings.

In this work we study the efficiency of simple (non incentive compatible) auctions in budgeted settings. The standard measure for quantifying the efficiency of simple auctions is the *Price of Anarchy* (PoA) [23,28,31], defined as the ratio of the optimal social welfare to the social welfare of the worst equilibrium. In budgeted settings, it is thus natural to quantify the efficiency of simple auctions by the *Liquid Price of Anarchy* (LPoA), defined as the ratio of the optimal Liquid Welfare to the Liquid Welfare of the worst equilibrium.

A prominent auction format, which has been extensively studied recently, is the simultaneous item auction setting. In such auctions, buyers submit bids simultaneously on all items, and the allocation and prices are determined separately for each individual item, based only on the bids submitted for that item. This format is similar to auctions used in practice (e.g., eBay auctions). In the long line of works of Christodoulou et al. [14], Bhawalkar and Roughgarden [6], Hassidim et al. [22], Syrgkanis and Tardos [31], and Feldman et al. [21], it was shown that these simple auctions have nearly optimal social efficiency guarantees for a broad range of equilibrium concepts when buyers have complement-free valuations but are not limited by budgets.

The most common framework for analyzing the Price of Anarchy of games and auctions is the *smoothness* framework (see, e.g., [28,31]). Such techniques usually involve a thought experiment in which each player deviates toward some strategy related to the optimal solution, and hence the total utility of all players can be bounded appropriately. One important and necessary condition for applying the smoothness framework is that the objective function dominates the sum of utilities (which holds for social welfare). However, this technique falls short in the case of Liquid Welfare, since a bidder’s utility can be arbitrarily higher than their value, and in aggregate, bidders may achieve a total utility that is much larger than the Liquid Welfare at equilibrium. To overcome this issue, we develop new techniques to bound the LPoA in budgeted settings. Our techniques include a novel type of hypothetical deviation that is used to *upper bound* the aggregate utility of bidders (in addition to the traditional deviation that is used to lower bound it), and the consideration of a special set of carefully chosen bidders to engage in these hypothetical deviations (see more details

in the “Our Techniques” section below). To the best of our knowledge, most prior techniques, including those that depart from the smoothness framework (e.g., [21]), examine the utility derived when *every* player deviates toward the optimal solution.

With our new techniques at hand, we address the following question: *What is the Liquid Price of Anarchy of simultaneous item auctions in settings with budgets?*

## Our Contributions

We show that simultaneous item auctions achieve nearly optimal performance (i.e., a constant Liquid Price of Anarchy) in many cases of interest.

Our main result concerns the case in which agent valuations are additive (i.e., agent  $i$ ’s value for item  $j$  is  $v_{ij}$  and the value for a set of items is the sum of the individual valuations).

**Main theorem:** For simultaneous item auctions (first and second price) with additive bidders, the LPoA with respect to mixed Nash equilibria and Bayesian Nash equilibria is constant. This assumes a divisible model where items can be divided into sufficiently many discrete parts.

We also show that for pure Nash equilibria, our results hold for more general settings.

**Theorem:** For simultaneous item auctions (first and second price) with fractionally-subadditive bidders, the LPoA of pure Nash equilibria is 2, even in the indivisible model. This is tight.

The following remarks are in order:

1. In settings without budgets, simultaneous item auctions reduce to  $m$  independent auctions (where  $m$  is the number of items). In contrast, when agents have budget constraints, the separate auctions exhibit non-trivial dependencies even under additive valuations.
2. Our main result requires that every item can be divided into at least  $\Omega(n)$  parts (where  $n$  is the number of agents). If items can only be divided into a sublinear number of parts, then the LPoA is super constant.
3. Our LPoA result for pure Nash equilibria (in the indivisible model) holds for deterministic tie-breaking rules. Surprisingly, if the tie-breaking rule is randomized, then the LPoA becomes linear in  $n$  (even if agents play pure strategies and have additive valuations).

## Our Techniques

The most common framework for analyzing the Price of Anarchy of games and auctions is the *smoothness* framework (see, e.g., [28,31]). One important and necessary condition for applying the smoothness framework is that the objective function dominates the sum of utilities, which clearly holds in the typical case of the social welfare objective, but not the Liquid Welfare objective.

To overcome this obstacle, we introduce two new ideas. (1) In addition to deriving a lower bound on the sum of bidders’ utilities (following the traditional *deviations-towards-the-optimum* technique), we also derive an *upper bound* on their utility as a function of the Liquid Welfare, using a novel *boosting deviation*, in which bidders bid more aggressively on items they receive in equilibrium. (2) Instead of summing the utility across all bidders, we consider the utility derived from a carefully selected set of bidders.

Our analysis can be summarized by the following inequality: let  $LW(\mathbf{b})$  denote the expected Liquid Welfare of a bid profile  $\mathbf{b}$ ,  $OPT$  denote the optimal Liquid Welfare, and  $u_i(\mathbf{b})$  denote the expected utility of bidder  $i$  under a bid profile  $\mathbf{b}$ . For any equilibrium  $\mathbf{b}$ , there exists a set of bidders

$S$  and constants  $c_1 < 1$  and  $c_2 > 1$ , such that

$$c_1 \cdot \text{OPT} \leq \sum_{i \in S} u_i(\mathbf{b}) \leq c_2 \cdot \text{LW}(\mathbf{b}),$$

where the left inequality follows from the traditional *deviations-towards-the-optimum* technique, and the right inequality follows from the new *boosting deviation* technique.

Syrngkanis and Tardos [31] also addressed the PoA of simple auctions in settings with budgets. They showed that the *social welfare* (SW) at equilibrium is at least a constant fraction of the optimal *Liquid Welfare*. One might be tempted to leverage their results for bounding the LPoA. This approach, however, is inadequate, since the LW at equilibrium can be arbitrarily smaller than the SW at equilibrium, even if items can be divided into arbitrarily many parts (e.g., if all budgets are small and values are large)<sup>1</sup>. For this reason, the LPoA bounds we establish carry over to their setting, but not vice versa<sup>2</sup>. Since the focus of [31] was to bound the SW at equilibrium (as opposed to LW), they established their bounds using the smoothness framework. In particular, they developed a powerful composition framework, in which they first obtained results for single-item auctions, and then showed that such auctions compose well to obtain more general results with any number of players and items. Note that it is not clear whether the composition framework is applicable in our setting.

## Related Work

There is a vast literature in algorithmic game theory that incorporates budgets into the design of incentive compatible mechanisms. The paper of [5] showed that, in the case of one infinitely divisible good, the adaptive clinching auction is incentive compatible under some assumptions. Moreover, the work of [30] initiated the design of incentive compatible mechanisms in the context of reverse auctions, where the payments of the auctioneer cannot exceed a hard budget constraint (follow-up works include [1,3,13,19]). A great deal of work focused on designing incentive compatible mechanisms that approximately maximize the auctioneer’s revenue in various settings with budget-constrained bidders [8, 11, 24, 26, 27]. Some works analyzed how budgets affect markets and non-truthful mechanisms [4, 12].

The earlier work [17] on multi-unit auctions with budget constraints concerns the design of incentive compatible mechanisms that always produce Pareto-optimal allocation. The results in this line of work are mostly negative with a notable exception of mechanisms based on Ausubel’s adaptive clinching auction framework [2].

Some recent results concern the design of incentive compatible mechanisms with respect to the Liquid Welfare objective, introduced by Dobzinski and Paes Leme [18]. They gave a constant approximation for the auction that sells a single divisible good to additive buyers with budgets. In a follow-up work, Lu and Xiao [25] gave an  $O(1)$ -approximation for bidders with general valuations in the single-item setting.

A large body of literature is concerned with simultaneous item bidding auctions. These simple auctions have been studied from a computational perspective, including the papers of Cai and

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<sup>1</sup>Note that the SW at equilibrium can be arbitrarily larger than the optimal LW, so the ratio studied by Syrgkanis and Tardos [31] can be either smaller or greater than 1. Our results imply that whenever the LW at equilibrium is more than a constant factor smaller than the SW at equilibrium, it must be the case that the optimal LW is also more than a constant factor smaller than the SW at equilibrium.

<sup>2</sup>Note, however, that in some sense our results and those of [31] are incomparable. Although the bounds we establish on the LPoA imply bounds according to their PoA measure, they achieved results for more general equilibrium concepts and valuation functions in some settings.

Papadimitriou [9] and Dobzinski et al. [16]. There is also extensive work addressing the Price of Anarchy of such simple auctions (see [29] for more general Price of Anarchy results). The work of Christodoulou et al. [14] initiated the study of simultaneous item auctions within the Price of Anarchy framework. The authors showed that, for second price auctions, the social welfare of every Bayesian Nash equilibrium is a 2-approximation to the optimal social welfare, even for players with fractionally-subadditive valuation functions. A large amount of follow-up work [6, 7, 15, 20–22, 31] made significant progress in our understanding of simultaneous item auctions, but none of these works measures inefficiency with respect to the Liquid Welfare objective.

The two most closely related works to ours are those of Syrgkanis and Tardos [31], along with Caragiannis and Voudouris [10], which both take the Liquid Welfare objective into account when measuring the inefficiency of equilibria. The work of [31] gave a variety of Price of Anarchy results, focusing on the development of a smoothness framework for broad solution concepts such as correlated equilibria and Bayesian Nash equilibria, and exploring composition properties of various mechanisms. They extended their results to the setting where players are budget-constrained, achieving similar approximation guarantees when comparing the *social welfare achieved at equilibrium* to the *optimal Liquid Welfare*. In particular, their results imply an  $\frac{\epsilon}{\epsilon-1}$ -approximation for simultaneous first price auctions, and a 2-approximation for all-pay auctions and simultaneous second price auctions under the no-overbidding assumption. While [31] show that the social welfare at equilibrium cannot be much worse than the optimal Liquid Welfare, one should note that the social welfare at equilibrium can be arbitrarily better than the optimal Liquid Welfare (e.g., if all budgets are small, the optimal Liquid Welfare is small). It is useful to note that, in general, the ratio between the Liquid Welfare at equilibrium and the social welfare at equilibrium can be arbitrarily bad (if all budgets are small, then the Liquid Welfare of any allocation is small, while players' values for received goods can be arbitrarily large).

The paper of Caragiannis and Voudouris [10] also considered the scenario where players have budgets and studied the same ratio we consider in this paper, namely the *Liquid Welfare at equilibrium* to the *optimal Liquid Welfare*. They studied the proportional allocation mechanism, which concerns auctioning off one divisible item proportionally according to the bids that players submit. They showed that, assuming players have concave non-decreasing valuation functions, the Liquid Welfare at coarse-correlated equilibria and Bayesian Nash equilibria achieve at least a constant fraction of the optimal Liquid Welfare. It should be noted that, for random allocations, they measure the benchmark at equilibrium ex-ante over the randomness of the allocation, i.e.,  $\sum_{i=1}^n \min\{\mathbb{E}_{\mathbf{v}_i, B_i}[v_i(x_i)], B_i\}$ , where  $v_i$  is player  $i$ 's valuation,  $B_i$  is player  $i$ 's budget, and  $x_i$  denotes the allocation player  $i$  receives. In contrast, for random allocations, we use the stronger ex-post measure of the expected Liquid Welfare at equilibrium given by  $\sum_{i=1}^n \mathbb{E}[\min\{v_i(x_i), B_i\}]$ .

## 2 Model and Preliminaries

We consider *simultaneous item auctions*, in which  $m$  heterogeneous items are sold to  $n$  bidders (or players) in  $m$  independent auctions. A bidder's *strategy* is a bid vector  $b_i \in \mathbb{R}_{\geq 0}^m$ , where  $b_{ij}$  represents player  $i$ 's bid for item  $j$ . We use  $\mathbf{b}$  to denote the bid profile  $\mathbf{b} = (b_1, \dots, b_n)$ , and we will often use the notation  $\mathbf{b} = (b_i, \mathbf{b}_{-i})$  to denote the strategy profile where player  $i$  bids  $b_i$  and the remaining players bid according to  $\mathbf{b}_{-i} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$ .

The outcome of an auction consists of an allocation rule  $\mathbf{x}$  and payment rule  $\mathbf{p}$ . The allocation rule  $\mathbf{x}$  maps bid profiles to an allocation vector for each individual bidder  $i$ , where  $x_i(\mathbf{b}) = (x_{i1}, \dots, x_{im})$  denotes the set of items won by player  $i$ . In a *simultaneous first price auction*, each item  $j$  is allocated to the highest bidder (breaking ties according to some rule) and

the winner pays their bid. The total payment of bidder  $i$  is  $p_i(\mathbf{b}) = \sum_{j \in x_i(\mathbf{b})} b_{ij}$ .

Each player  $i$  has a *valuation function*  $v_i$ , which maps sets of items to  $\mathbb{R}_{\geq 0}$  ( $v_i$  captures how much player  $i$  values item bundles), and a budget  $B_i$ . We assume that all valuations are normalized and monotone, i.e.,  $v_i(\emptyset) = 0$  and  $v_i(S) \leq v_i(T)$  for any  $i \in [n]$  and  $S \subseteq T \subseteq [m]$ . We mostly consider bidders with additive valuations, i.e.,  $v_i(S) = \sum_{j \in S} v_{ij}$  (where  $v_{ij}$  denotes agent  $i$ 's value for item  $j$ ). The *utility*  $u_i(x_i(\mathbf{b}))$  of each player  $i$  is  $v_i(x_i(\mathbf{b})) - p_i(\mathbf{b}) = \sum_j v_{ij} \cdot x_{ij} - p_i(\mathbf{b})$  if  $p_i(\mathbf{b}) \leq B_i$ ; and  $u_i(x_i(\mathbf{b})) = -\infty$  if  $p_i(\mathbf{b}) > B_i$ . Buyers select their bids strategically in order to maximize utility.

**Share model:** For our results beyond pure Nash equilibria with deterministic tie-breaking rules, we focus on bidders with additive valuations and consider a *share model*, in which item  $j$  is divided into  $h$  identical shares and player  $i$  values each share at  $\frac{v_{ij}}{h}$ .

**Definition 1** (Pure Nash Equilibrium). *A bid profile  $\mathbf{b}$  is a Pure Nash Equilibrium (PNE) if, for any player  $i$  and any deviating bid  $b'_i$ :  $u_i(b_i, \mathbf{b}_{-i}) \geq u_i(b'_i, \mathbf{b}_{-i})$ .*

A mixed Nash equilibrium is defined similarly, except that bidding strategies can be randomized  $b_i \sim s_i$  and utility is measured in expectation over the joint bid distribution  $\mathbf{s} = s_1 \times \dots \times s_n$ .

**Definition 2** (Mixed Nash Equilibrium). *A bid profile  $\mathbf{s}$  is a mixed Nash equilibrium if, for any player  $i$  and any deviating bid  $b'_i$ :  $\mathbb{E}_{\mathbf{b} \sim \mathbf{s}}[u_i(b_i, \mathbf{b}_{-i})] \geq \mathbb{E}_{\mathbf{b}_{-i} \sim \mathbf{s}_{-i}}[u_i(b'_i, \mathbf{b}_{-i})]$ .*

Note that, in general, we assume that the bidding space is discretized (i.e., each player can only bid in multiples of a sufficiently small value  $\varepsilon$ ). This is done to ensure that there always exists a mixed Nash equilibrium, as otherwise we do not have a finite game.

We now give a definition about the welfare function we seek to optimize.

**Definition 3** (Liquid Welfare). *The Liquid Welfare, denoted by LW, of an allocation  $\mathbf{x}$  is given by  $\text{LW}(\mathbf{x}) = \sum_{i \in [n]} \min\{v_i(x_i), B_i\}$ . For random allocations, we use the measure given by  $\text{LW}(\mathbf{x}) = \sum_{i \in [n]} \mathbb{E}[\min\{v_i(x_i), B_i\}]$ .*

For a given vector of valuations  $\mathbf{v} = (v_1, \dots, v_n)$ , we use  $\text{OPT}(\mathbf{v})$  to denote the value of the optimal outcome given by  $\text{OPT}(\mathbf{v}) = \max_{S_1, \dots, S_n} \sum_i \min\{v_i(S_i), B_i\}$ , where the sets  $S_i$  form a partition of  $[m]$  (i.e.,  $\cup_i S_i = [m]$  and  $\forall i \neq j : S_i \cap S_j = \emptyset$ ). We often use  $\text{OPT}$  instead of  $\text{OPT}(\mathbf{v})$  when the context is clear.

**Definition 4** (Liquid Price of Anarchy). *Given a fixed valuation profile  $\mathbf{v}$ , the Liquid Price of Anarchy (LPoA) is the worst-case ratio between the optimal Liquid Welfare and the expected Liquid Welfare at a Nash equilibrium (pure or mixed) and is given by*

$$\text{LPoA}(\mathbf{v}) = \sup_{\mathbf{s}} \left\{ \frac{\text{OPT}(\mathbf{v})}{\text{LW}(\mathbf{s}(\mathbf{v}))} \mid \mathbf{s} \in \text{Nash Equilibria} \right\}.$$

### 3 Main Result (Liquid Price of Anarchy)

We consider a share model in which each item  $j$  is divided into  $h$  identical shares, where player  $i$  values each share at  $\frac{v_{ij}}{h}$  (here  $v_{ij}$  denotes player  $i$ 's value for item  $j$ ). For the sake of analysis, we treat each share as a separate item, so that buyers can submit different bids on every single share. A more realistic market clearing mechanism for one item would be one where

1. Each buyer specifies how many shares they want to buy and which price they are willing to pay per share.

2. In decreasing order of bids and until the stock of shares lasts, each buyer receives their demanded number of shares while paying their bid per purchase.

We note that our analysis carries over to this “clearing house” item auction with small adjustments, which we mention in Section 5 and discuss at the end of this section.

**Theorem 1.** *The Liquid Price of Anarchy of simultaneous first price auctions is constant (at most 51.5), when every item has at least  $n$  equal shares (copies). If less shares  $h$  are available then the LPoA is  $O\left(\frac{n}{h}\right)$ .*

In what follows we build up notation and intuition toward the proof. Recall that agents have additive valuations and submit bids on shares, and if they receive an  $x_{ij}$  fraction of shares of item  $j$ , then their value is given by  $v_i(x_{ij}) = v_{ij} \cdot x_{ij}$ . We further assume that the buyers bid according to a mixed Nash equilibrium  $\mathbf{b} \sim \mathbf{s}$ . When buyers bid in simultaneous auctions, this essentially induces a distribution of prices over all shares of items  $\mathbf{p} \sim \mathcal{D}$  from a distribution  $\mathcal{D}$  (e.g., winning bids in first price auctions, namely  $p_j^\ell = \max_i b_{ij}^\ell$ , where  $b_{ij}^\ell$  is player  $i$ 's bid for share  $\ell$  of item  $j$ ). In particular, for all items we can define an “expected price per item” at equilibrium or just a “price per item” as  $\bar{\mathbf{p}} = (\bar{p}_1, \dots, \bar{p}_m)$ , where  $\bar{p}_j = \alpha \sum_{\ell=1}^h \mathbb{E}[p_j^\ell]$ , for some  $\alpha > 1$  ( $\alpha = 2$  will be sufficient for us). This induces a natural “expected price per share,” namely  $\frac{\bar{p}_j}{h}$ . One simple observation about  $\bar{\mathbf{p}}$  is the following:

**Observation 3.1.** *Revenue is related to prices:  $\text{REV}(\mathbf{s}) = \frac{1}{\alpha} \sum_{j=1}^m \bar{p}_j$ , where  $\text{REV}(\mathbf{s})$  denotes the expected revenue at the equilibrium profile  $\mathbf{s}$ .*

We next show that if players bid on some fraction of shares of item  $j$  uniformly at random according to  $\bar{p}_j$ , then they win a large number of shares in expectation.

**Claim 3.1.** *For any item  $j$ , if a player bids on a  $\delta$ -fraction of shares chosen uniformly at random of item  $j$  at a given price  $\frac{\bar{p}_j}{h}$  per share, then the player receives in expectation at least  $h \cdot \delta \cdot \left(1 - \frac{1}{\alpha}\right)$  shares of the item (i.e., at least a  $\delta \cdot \left(1 - \frac{1}{\alpha}\right)$ -fraction of item  $j$ ).*

*Proof.* Suppose towards a contradiction that the expected number of shares won by bidder  $i$  is less than  $\delta \cdot h \cdot \left(1 - \frac{1}{\alpha}\right)$ . In particular, it means that

$$\sum_{\ell=1}^h \Pr [i \text{ bids on share } \ell] \cdot \Pr \left[ p_j^\ell < \frac{\bar{p}_j}{h} \right] = \sum_{\ell=1}^h \delta \cdot \Pr \left[ p_j^\ell < \frac{\bar{p}_j}{h} \right] < \delta \cdot h \cdot \left(1 - \frac{1}{\alpha}\right).$$

We further use the definition of  $\bar{p}_j$  and Markov's inequality to obtain a contradiction as follows:

$$\frac{\bar{p}_j}{\alpha} = \sum_{\ell=1}^h \mathbb{E} \left[ p_j^\ell \right] \geq \sum_{\ell=1}^h \frac{\bar{p}_j}{h} \cdot \Pr \left[ p_j^\ell \geq \frac{\bar{p}_j}{h} \right] = \sum_{\ell=1}^h \frac{\bar{p}_j}{h} \left(1 - \Pr \left[ p_j^\ell < \frac{\bar{p}_j}{h} \right]\right) > \bar{p}_j - \frac{\bar{p}_j}{h} \cdot h \cdot \left(1 - \frac{1}{\alpha}\right).$$

□

When relating prices to Liquid Welfare we notice that

**Observation 3.2.** *Revenue is bounded by the Liquid Welfare:  $\text{REV}(\mathbf{s}) \leq \text{LW}(\mathbf{s})$ , where  $\text{LW}(\mathbf{s})$  denotes the expected Liquid Welfare at the equilibrium profile  $\mathbf{s}$ .*

We consider the following fractional relaxation of the allocation problem with the goal of optimizing Liquid Welfare.

$$\begin{aligned} & \text{Maximize} && \sum_{i=1}^n \sum_{j=1}^m v_{ij} \cdot z_{ij} && \text{Liquid linear program (LLP)} \\ & \text{Subject to} && \sum_j v_{ij} \cdot z_{ij} \leq B_i \quad \forall i; && \sum_i z_{ij} \leq 1 \quad \forall j; && z_{ij} \geq 0 \quad \forall i, j. \end{aligned}$$

We denote by  $\mathbf{y} = (y_{ij})$  the optimal solution to LLP. Notice that the optimal fractional solution for the Liquid Welfare would never benefit from allocating a set of items to a player such that their value for the set exceeds their budget. The solution to LLP gives an upper bound on the optimal Liquid Welfare OPT.

**Observation 3.3.** *The optimal fractional solution to LLP is better than the optimal allocation:*  

$$\sum_{i=1}^n \sum_{j=1}^m v_{ij} \cdot y_{ij} \geq \text{OPT}.$$

We now define some notation that will be useful in order to obtain our result. We let  $q_{ij}$  be the expected fraction of shares that player  $i$  receives from item  $j$  at an equilibrium strategy  $\mathbf{s}$ . In addition, for each agent  $i$ , we consider a set of high value items  $J_i \stackrel{\text{def}}{=} \{j \mid v_{ij} \geq \bar{p}_j\}$ . We further define  $Q_i$  to be the probability that  $v_i(x_i) \geq B_i$  at equilibrium (recall that  $x_i$  denotes the random set that player  $i$  receives in the mixed Nash equilibrium). We also define three sets of bidders, the first two of which are for budget feasibility reasons and the last of which is for bidders that often fall under their budget in equilibrium (these sets need not be disjoint). In particular, for a fixed parameter  $\gamma > 1$  to be determined later, we define sets  $\mathcal{I}_1$ ,  $\mathcal{I}_2$ , and  $\mathcal{I}_3$ :

$$\mathcal{I}_1 \stackrel{\text{def}}{=} \left\{ i \mid \gamma \sum_{j \in J_i} \bar{p}_j \cdot q_{ij} \leq B_i \right\}, \quad \mathcal{I}_2 \stackrel{\text{def}}{=} \left\{ i \mid \sum_{j \in [m]} \frac{\bar{p}_j}{h} \leq B_i \right\}, \quad \text{and} \quad \mathcal{I}_3 \stackrel{\text{def}}{=} \left\{ i \mid Q_i \leq \frac{1}{2\gamma} \right\}.$$

Throughout our proof, we focus on bidders in the set  $\mathcal{I} \stackrel{\text{def}}{=} \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$ . We define sets  $\bar{\mathcal{I}}_1 \stackrel{\text{def}}{=} [n] \setminus \mathcal{I}_1$ ,  $\bar{\mathcal{I}}_2 \stackrel{\text{def}}{=} [n] \setminus \mathcal{I}_2$ ,  $\bar{\mathcal{I}}_3 \stackrel{\text{def}}{=} [n] \setminus \mathcal{I}_3$ , and  $\bar{\mathcal{I}} \stackrel{\text{def}}{=} [n] \setminus \mathcal{I}$ . To this end, we need to argue that bidders outside of the set  $\mathcal{I}$  do not contribute a lot to the Liquid Welfare at equilibrium  $\mathbf{s}$ .

**Claim 3.2.** *The total budget of players in  $\bar{\mathcal{I}}$  is small:  $\sum_{i \in \bar{\mathcal{I}}} B_i < \alpha \cdot \left(\gamma + \frac{n}{h}\right) \cdot \text{REV}(\mathbf{s}) + \sum_{i \in \bar{\mathcal{I}}_3} B_i$ .*

*Proof.* We first consider agents in  $\bar{\mathcal{I}}_1$  and obtain

$$\sum_{i \in \bar{\mathcal{I}}_1} B_i < \gamma \sum_{i \in \bar{\mathcal{I}}_1} \sum_{j \in J_i} \bar{p}_j \cdot q_{ij} \leq \gamma \sum_i \sum_{j \in J_i} \bar{p}_j \cdot q_{ij} \leq \gamma \sum_j \bar{p}_j \sum_{i: j \in J_i} q_{ij} \leq \gamma \sum_j \bar{p}_j \leq \gamma \cdot \alpha \cdot \text{REV}(\mathbf{s}),$$

where the second to last inequality follows from the fact that  $\sum_i q_{ij} \leq 1$  since we have a valid fractional allocation. Next, we consider agents in  $\bar{\mathcal{I}}_2$  and obtain

$$\sum_{i \in \bar{\mathcal{I}}_2} B_i < \sum_{i \in \bar{\mathcal{I}}_2} \sum_j \frac{\bar{p}_j}{h} \leq \frac{n}{h} \sum_j \bar{p}_j \leq \frac{n}{h} \cdot \alpha \cdot \text{REV}(\mathbf{s}).$$

Combining these bounds for agents in  $\bar{\mathcal{I}}_1$  and  $\bar{\mathcal{I}}_2$  we have

$$\sum_{i \in \bar{\mathcal{I}}} B_i \leq \sum_{i \in \bar{\mathcal{I}}_1} B_i + \sum_{i \in \bar{\mathcal{I}}_2} B_i + \sum_{i \in \bar{\mathcal{I}}_3} B_i \leq \alpha \cdot \left(\gamma + \frac{n}{h}\right) \cdot \text{REV}(\mathbf{s}) + \sum_{i \in \bar{\mathcal{I}}_3} B_i. \quad \square$$



To achieve our result, we essentially consider two main ideas for player deviations in set  $\mathcal{I}$  (each idea actually consists of two parts). The first idea is to use the solution to LLP as guidance to claim that players can extract a large amount of value relative to the optimal solution. However, for players to deviate, we must first round the fractional solution  $\mathbf{y}$  into an integral solution (here, by integral, we mean a multiple of  $\frac{1}{h}$  since this represents the fraction of shares a player receives out of  $h$  copies). Define the first LLP deviation (integral part) to be  $\mathbf{b}_1^{[i]} = (b'_i, \mathbf{b}_{-i})$ , where in  $b'_i$  buyer  $i$  bids on a random  $\lfloor y_{ij} \rfloor_h$ -fraction of each item  $j \in J_i$  with price  $\bar{p}_j$  (here,  $\lfloor y \rfloor_h = \frac{\lfloor y \cdot h \rfloor}{h}$  and the bid per share is  $\frac{\bar{p}_j}{h}$ ). Define the second LLP deviation (fractional part) to be  $\mathbf{b}_1^{\{i\}} = (b'_i, \mathbf{b}_{-i})$ , where in  $b'_i$  buyer  $i$  bids on a random  $\{y_{ij}\}_h$ -fraction of each item  $j \in J_i$  with price  $\bar{p}_j$  (here,  $\{y\}_h = \frac{1}{h}$  if  $y > 0$ , and  $\{y\}_h = 0$  otherwise). We note that both LLP deviations  $\mathbf{b}_1^{[i]}$  and  $\mathbf{b}_1^{\{i\}}$  are feasible, since  $v_{ij} \geq \bar{p}_j$  for every  $j \in J_i$ , and  $\sum_j v_{ij} \cdot y_{ij} \leq B_i$  as  $\mathbf{y}$  is a solution to LLP along with  $\mathcal{I} \subseteq \mathcal{I}_2$ . Moreover, for any  $y_{ij}$ , we have  $\lfloor y_{ij} \rfloor_h + \{y_{ij}\}_h \geq y_{ij}$ .

**Lemma 3.1** (LLP deviations). *Buyers in  $\mathcal{I}$  at equilibrium  $\mathbf{s}$  derive large value:*

$$\sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} \geq \left( \frac{1}{2} - \frac{1}{2\alpha} \right) \left( \text{OPT} - \alpha \left( 1 + \gamma + \frac{n}{h} \right) \text{REV}(\mathbf{s}) - \sum_{i \in \bar{\mathcal{I}}_3} B_i \right).$$

*Proof.* For the integral part of the LLP deviation, since  $\mathbb{E}_{\mathbf{b} \sim \mathbf{s}}[v_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b} \sim \mathbf{s}}[u_i(\mathbf{b})]$  and  $\mathbf{s}$  is a mixed Nash equilibrium, we have:

$$\begin{aligned} \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} &\geq \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbf{b} \sim \mathbf{s}}[u_i(\mathbf{b})] \geq \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbf{b}_{-i} \sim \mathbf{s}_{-i}} \left[ u_i \left( \mathbf{b}_1^{[i]} \right) \right] \\ &\geq \sum_{i \in \mathcal{I}} \sum_{j \in J_i} \left( 1 - \frac{1}{\alpha} \right) \cdot \lfloor y_{ij} \rfloor_h (v_{ij} - \bar{p}_j), \end{aligned} \quad (1)$$

where to derive the last inequality we use Claim 3.1. Similarly, for the fractional part of the LLP deviation we have:

$$\begin{aligned} \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} &\geq \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbf{b} \sim \mathbf{s}}[u_i(\mathbf{b})] \geq \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbf{b}_{-i} \sim \mathbf{s}_{-i}} \left[ u_i \left( \mathbf{b}_1^{\{i\}} \right) \right] \\ &\geq \sum_{i \in \mathcal{I}} \sum_{j \in J_i} \left( 1 - \frac{1}{\alpha} \right) \cdot \{y_{ij}\}_h (v_{ij} - \bar{p}_j). \end{aligned} \quad (2)$$

Combining Equation (1) and Equation (2) we get

$$\begin{aligned} 2 \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} &\geq \sum_{i \in \mathcal{I}} \sum_{j \in J_i} \left( 1 - \frac{1}{\alpha} \right) \cdot (\lfloor y_{ij} \rfloor_h + \{y_{ij}\}_h) (v_{ij} - \bar{p}_j) \\ &\geq \sum_{i \in \mathcal{I}} \sum_{j \in J_i} \left( 1 - \frac{1}{\alpha} \right) \cdot y_{ij} (v_{ij} - \bar{p}_j) = \left( 1 - \frac{1}{\alpha} \right) \left( \sum_{i \in \mathcal{I}} \sum_{j \in J_i} v_{ij} \cdot y_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \in J_i} \bar{p}_j \cdot y_{ij} \right). \end{aligned} \quad (3)$$

We further estimate

$$\begin{aligned} \sum_{i \in \mathcal{I}} \sum_{j \in J_i} v_{ij} \cdot y_{ij} &= \sum_{i,j} v_{ij} \cdot y_{ij} - \sum_{i \in \bar{\mathcal{I}}} \sum_j v_{ij} \cdot y_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \notin J_i} v_{ij} \cdot y_{ij} \\ &\geq \sum_{i,j} v_{ij} \cdot y_{ij} - \sum_{i \in \bar{\mathcal{I}}} B_i - \sum_{i \in \mathcal{I}} \sum_{j \notin J_i} \bar{p}_j \cdot y_{ij}, \end{aligned}$$

where in the last inequality we used the condition from the LLP that  $\sum_j v_{ij} \cdot y_{ij} \leq B_i$  and that  $v_{ij} \leq \bar{p}_j$  for each  $j \notin J_i$ . We substitute the last estimate into Equation (3) and get

$$\begin{aligned} 2 \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} &\geq \left(1 - \frac{1}{\alpha}\right) \left( \sum_{i,j} v_{ij} \cdot y_{ij} - \sum_{i \in \bar{\mathcal{I}}} B_i - \sum_{i \in \mathcal{I}} \sum_{j \notin J_i} \bar{p}_j \cdot y_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \in J_i} \bar{p}_j \cdot y_{ij} \right) \\ &= \left(1 - \frac{1}{\alpha}\right) \left( \sum_{i,j} v_{ij} \cdot y_{ij} - \sum_{i \in \mathcal{I}} \sum_j \bar{p}_j \cdot y_{ij} - \sum_{i \in \bar{\mathcal{I}}} B_i \right) \\ &\geq \left(1 - \frac{1}{\alpha}\right) \left( \text{OPT} - \alpha \left(1 + \gamma + \frac{n}{h}\right) \text{REV}(\mathbf{s}) - \sum_{i \in \bar{\mathcal{I}}_3} B_i \right), \end{aligned}$$

where the last inequality follows from the LLP constraint that  $\sum_i y_{ij} \leq 1$  for each  $j$ , the observation  $\sum_j \bar{p}_j = \alpha \cdot \text{REV}(\mathbf{s})$ , and Claim 3.2.  $\square$

We now turn to our second type of deviation, but we need to further restrict the set of items that players bid on. In particular, we let  $\Gamma_i = \left\{j \mid q_{ij} \leq \frac{1}{\gamma}\right\}$ , and define  $G_i = J_i \cap \Gamma_i$ . We now define the  $\gamma$ -boosting deviation (integral part) as  $\mathbf{b}_2^{\lfloor i \rfloor} = (b'_i, \mathbf{b}_{-i})$ , where in  $b'_i$  buyer  $i$  bids on a random  $\lfloor \gamma \cdot q_{ij} \rfloor_h$ -fraction of each item  $j \in G_i$  with price  $\bar{p}_j$ , where  $\gamma > 1$  is a constant to be determined later. Note that each  $\mathbf{b}_2^{\lfloor i \rfloor}$  deviation for every  $i \in \mathcal{I}$  is feasible since  $\mathcal{I} \subseteq \mathcal{I}_1$ . Similarly, we define the fractional part of the  $\gamma$ -boosting deviation as  $\mathbf{b}_2^{\{i\}}$ , which is also a feasible deviation since  $\mathcal{I} \subseteq \mathcal{I}_2$ . Also, since players bid on items in  $G_i \subseteq \Gamma_i$ , we have  $\lfloor \gamma \cdot q_{ij} \rfloor_h \leq 1$  (we also have  $\{\gamma \cdot q_{ij}\}_h \leq 1$ , which holds for all items by definition).

**Lemma 3.2** ( $\gamma$ -boosting deviation). *The value derived by buyers in  $\mathcal{I}$  is comparable to the Liquid Welfare obtained at equilibrium:*

$$\left(1 - \frac{2\alpha}{\gamma(\alpha - 1)}\right) \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} \leq \alpha \cdot \text{REV}(\mathbf{s}) + 2 \cdot \text{LW}(\mathbf{s}) - \frac{1}{\gamma} \sum_{i \in \bar{\mathcal{I}}_3} B_i.$$

*Proof.* For the integral part of the  $\gamma$ -boosting deviation, we can obtain bounds via the Nash equilibrium condition and Claim 3.1:

$$\begin{aligned} \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} &\geq \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbf{b} \sim \mathbf{s}} [u_i(\mathbf{b})] \geq \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbf{b}_{-i} \sim \mathbf{s}_{-i}} \left[ u_i \left( \mathbf{b}_2^{\lfloor i \rfloor} \right) \right] \\ &\geq \sum_{i \in \mathcal{I}} \sum_{j \in G_i} \left(1 - \frac{1}{\alpha}\right) \lfloor \gamma \cdot q_{ij} \rfloor_h (v_{ij} - \bar{p}_j). \end{aligned}$$

Similarly, for the fractional part of the  $\gamma$ -boosting deviation we get:

$$\sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} \geq \sum_{i \in \mathcal{I}} \mathbb{E}_{\mathbf{b}_{-i} \sim \mathbf{s}_{-i}} \left[ u_i \left( \mathbf{b}_2^{\{i\}} \right) \right] \geq \sum_{i \in \mathcal{I}} \sum_{j \in G_i} \left(1 - \frac{1}{\alpha}\right) \{\gamma \cdot q_{ij}\}_h (v_{ij} - \bar{p}_j).$$

Together these two deviations give us

$$2 \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} \geq \sum_{i \in \mathcal{I}} \sum_{j \in G_i} \left(1 - \frac{1}{\alpha}\right) \gamma \cdot q_{ij} (v_{ij} - \bar{p}_j). \quad (4)$$

We further estimate the term  $\sum_{i \in \mathcal{I}} \sum_{j \in G_i} (v_{ij} - \bar{p}_j) \cdot q_{ij}$  on the RHS of Equation (4):

$$\begin{aligned}
\sum_{i \in \mathcal{I}} \sum_{j \in G_i} (v_{ij} - \bar{p}_j) \cdot q_{ij} &= \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \notin G_i} v_{ij} \cdot q_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \in G_i} \bar{p}_j \cdot q_{ij} \\
&\geq \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \notin J_i} \bar{p}_j \cdot q_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \notin \Gamma_i} v_{ij} \cdot q_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \in G_i} \bar{p}_j \cdot q_{ij} \\
&\geq \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \notin \Gamma_i} v_{ij} \cdot q_{ij} - \sum_j \bar{p}_j \sum_{i \in \mathcal{I}} q_{ij} \\
&\geq \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} - \sum_{i \in \mathcal{I}} \sum_{j \notin \Gamma_i} v_{ij} \cdot q_{ij} - \alpha \cdot \text{REV}(\mathbf{s}), \tag{5}
\end{aligned}$$

where the first inequality holds as  $\sum_{j \notin G_i} v_{ij} q_{ij} \leq \sum_{j \notin J_i} v_{ij} q_{ij} + \sum_{j \notin \Gamma_i} v_{ij} q_{ij}$  and  $v_{ij} < \bar{p}_j$  for every  $j \notin J_i$ . Our next goal will be to bound the term  $\sum_{i \in \mathcal{I}} \sum_{j \notin \Gamma_i} v_{ij} \cdot q_{ij}$  on the RHS of Equation (5). Before that we need to do some preparations. To ease the notations we denote by  $j^{\{\ell\}}$  the  $\ell^{\text{th}}$  share of item  $j$ . We observe that the expected Liquid Welfare at equilibrium is at least

$$\begin{aligned}
\text{LW}(\mathbf{s}) &= \sum_i \Pr_{\mathbf{b} \sim \mathbf{s}} [v_i(x_i) > B_i] \cdot B_i + \sum_i \sum_j \sum_{\ell=1}^h \Pr_{\mathbf{b} \sim \mathbf{s}} \left[ \{v_i(x_i) \leq B_i\} \wedge \{i \text{ wins } j^{\{\ell\}}\} \right] \cdot \frac{v_{ij}}{h} \\
&= \sum_i Q_i \cdot B_i + \sum_{i,j} v_{ij} \left( \frac{1}{h} \sum_{\ell=1}^h \Pr \left[ i \text{ wins } j^{\{\ell\}} \right] - \frac{1}{h} \sum_{\ell=1}^h \Pr \left[ \{v_i(x_i) > B_i\} \wedge \{i \text{ wins } j^{\{\ell\}}\} \right] \right) \\
&= \sum_i Q_i \cdot B_i + \sum_{i,j} v_{ij} \cdot \max \left\{ 0, q_{ij} - \frac{1}{h} \sum_{\ell=1}^h \Pr \left[ \{v_i(x_i) > B_i\} \wedge \{i \text{ wins } j^{\{\ell\}}\} \right] \right\} \\
&\geq \sum_i Q_i \cdot B_i + \sum_{i,j} v_{ij} \cdot \max \{0, q_{ij} - \Pr [v_i(x_i) > B_i]\} \\
&= \sum_i Q_i \cdot B_i + \sum_{i,j} \max \{0, q_{ij} - Q_i\} \cdot v_{ij} \geq \sum_{i \in \bar{\mathcal{I}}_3} \frac{1}{2\gamma} \cdot B_i + \sum_{i \in \mathcal{I}} \sum_{j \notin \Gamma_i} v_{ij} \cdot \frac{q_{ij}}{2}, \tag{6}
\end{aligned}$$

where the third equality holds true as the expression inside the max cannot be negative and  $q_{ij} = \frac{1}{h} \sum_{\ell=1}^h \Pr [i \text{ wins } j^{\{\ell\}}]$  by definition of  $q_{ij}$ , the first inequality holds since  $\Pr [v_i(x_i) > B_i] \geq \Pr [\{v_i(x_i) > B_i\} \wedge \{i \text{ wins } j^{\{\ell\}}\}]$ , the last equality holds true by definition of  $Q_i$ , and the last inequality holds since players  $i \in \bar{\mathcal{I}}_3$  have  $Q_i > \frac{1}{2\gamma}$ , while for players  $i \in \mathcal{I} \subseteq \bar{\mathcal{I}}_3$  and items  $j \notin \Gamma_i$  we have  $Q_i \leq \frac{1}{2} \cdot \frac{1}{\gamma} \leq \frac{q_{ij}}{2}$ . Now we rearrange terms from Equation (6) to get  $\sum_{i \in \mathcal{I}} \sum_{j \notin \Gamma_i} v_{ij} q_{ij} \leq 2 \cdot \text{LW}(\mathbf{s}) - \frac{1}{\gamma} \sum_{i \in \bar{\mathcal{I}}_3} B_i$ . Combining Equation (4) and Equation (5), we can substitute this upper bound to get:

$$2 \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} \geq \left(1 - \frac{1}{\alpha}\right) \gamma \left( \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} - \alpha \cdot \text{REV}(\mathbf{s}) - 2 \cdot \text{LW}(\mathbf{s}) + \frac{1}{\gamma} \sum_{i \in \bar{\mathcal{I}}_3} B_i \right).$$

Dividing both sides by  $(1 - \frac{1}{\alpha}) \gamma$  and rearranging terms gives the lemma:

$$\left(1 - \frac{2\alpha}{\gamma(\alpha - 1)}\right) \sum_{i \in \mathcal{I}} \sum_j v_{ij} \cdot q_{ij} \leq \alpha \cdot \text{REV}(\mathbf{s}) + 2 \cdot \text{LW}(\mathbf{s}) - \frac{1}{\gamma} \sum_{i \in \bar{\mathcal{I}}_3} B_i. \quad \square$$

Now we have all necessary components to conclude the proof of Theorem 1 and show that the Liquid Price of Anarchy of any mixed Nash equilibrium is bounded.

*Proof of Theorem 1.* We combine the bounds from Lemma 3.1 and Lemma 3.2 and obtain

$$\alpha \cdot \text{REV}(\mathbf{s}) + 2 \cdot \text{LW}(\mathbf{s}) - \frac{1}{\gamma} \sum_{i \in \bar{\mathcal{I}}_3} B_i \geq \left(1 - \frac{2\alpha}{\gamma(\alpha - 1)}\right) \left(\frac{1}{2} - \frac{1}{2\alpha}\right) \left(\text{OPT} - \alpha \left(1 + \gamma + \frac{n}{h}\right) \text{REV}(\mathbf{s}) - \sum_{i \in \bar{\mathcal{I}}_3} B_i\right).$$

Since  $\text{LW}(\mathbf{s}) \geq \text{REV}(\mathbf{s})$  we further derive that

$$\left(\alpha + 2 + \frac{1}{2} \left(1 - \frac{1}{\alpha} - \frac{2}{\gamma}\right) \alpha \left(1 + \gamma + \frac{n}{h}\right)\right) \text{LW}(\mathbf{s}) \geq \frac{1}{2} \left(1 - \frac{1}{\alpha} - \frac{2}{\gamma}\right) \text{OPT} + \left(\frac{1}{\gamma} - \frac{1}{2} \left(1 - \frac{1}{\alpha} - \frac{2}{\gamma}\right)\right) \sum_{i \in \bar{\mathcal{I}}_3} B_i.$$

As long as the factor in front of  $\sum_{i \in \bar{\mathcal{I}}_3} B_i$  is nonnegative, we have  $\text{OPT} \leq O\left(\frac{n}{h}\right) \cdot \text{LW}(\mathbf{s})$  for any  $1 \leq h \leq n$  for a particular choice of parameters (e.g.,  $\alpha = 2.26, \gamma = 7.16$ ). In particular, when  $h \geq n$ , we have that the LPoA is at most 51.5.  $\square$

*Remark 1.* The bound on the Liquid Price of Anarchy derived in Theorem 1 holds for simultaneous first price auctions with the house clearing item bidding mechanism.

*Proof.* We observe that the bidding strategy defined in Claim 3.1 extends naturally to the new house clearing mechanism. Moreover, for a new equilibrium  $\mathbf{s}$  with appropriately redefined item prices  $\bar{\mathbf{p}}(\mathbf{s})$ , the argument from the proof of Claim 3.1 gives us exactly the same guarantee on the expected number of shares won by a bidder. Indeed, in the new mechanism when a bidder faces competitors' bids or equivalently the set of share prices  $\{p_j^\ell\}_{\ell=1}^h$ , they avoid the uncertainty of bidding on all shares  $\ell$  with high prices  $\frac{\bar{p}_j}{h} < p_j^\ell$  and thus they receive at least as many copies as they would get in the independent share bidding auction.

We further note that all the remaining parts of the proof of Theorem 1 do not depend on the market clearing format of first price item bidding auctions.  $\square$

## 4 Pure Nash Equilibria

We also study *pure* Nash equilibria of simultaneous first price auctions with deterministic tie-breaking rules. The proofs of the next two theorems are given in Appendices B and C, respectively.

**Theorem 2.** *Consider a simultaneous first price auction where budgeted bidders have fractionally-subadditive valuations<sup>3</sup>. If  $\mathbf{b}$  is a pure Nash equilibrium, then  $\text{LW}(\mathbf{b}) \geq \frac{\text{OPT}}{2}$ .*

A complementary tightness result for Theorem 2 is given in Appendix E.

Unfortunately, this result is not quite satisfying compared to mixed Nash equilibria. The first reason is that pure Nash equilibria might not even exist. Consider the following auction with  $n = 2$

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<sup>3</sup>Valuation  $v$  is fractionally-subadditive or equivalently XOS if there is a set of additive valuations  $A = \{a_1, \dots, a_\ell\}$  such that  $v_i(S) = \max_{a \in A} a(S)$  for every  $S \subseteq [m]$ . XOS is a super class of submodular and additive valuations.

players and  $m = 10$  identical items, where player 1 values each item at 1 and has a budget of 1, while the second player values each item at 1.1, and has a budget of 1.1. It is easy to see that there can be no pure Nash equilibrium, for the following reason. If player 2 knows what player 1 is bidding, player 2 has the budget to simply outbid player 1 and win all items. On the other hand, if player 2 is winning all items, one of their bids must be at most 0.11 (due to their budget constraint), and therefore player 1 can outbid player 2 on this item. Hence, there is no pure Nash equilibrium. Moreover, it holds true for both simultaneous first price and second price auctions. Second, the LPoA guarantees strongly depend on the tie-breaking rules used in the auction. In particular, if we allow randomized tie-breaking rules, the LPoA is no longer a constant.

**Theorem 3.** *With a randomized tie-breaking rule, there are simultaneous first and second price auction games which have an  $\Omega(n)$  Liquid Price of Anarchy, even when agents play pure strategies.*

**Lower Bound: Divisible Items.** In order to reconcile this big gap between the Liquid Welfare of the optimal allocation and pure equilibria with randomized tie-breaking rules (and mixed Nash equilibria in general), we considered in Section 3 a share model in which each item  $j$  is divided into  $h$  identical shares and obtained an upper bound of  $O\left(1 + \frac{n}{h}\right)$  on the LPoA. We now show a complementary lower bound that the Liquid Price of Anarchy is super constant for such equilibria when the number of shares is  $h = o(n)$ . The proof of Theorem 4 is deferred to Appendix C.

**Theorem 4.** *There are some simultaneous first price and second price auction games for which the Liquid Price of Anarchy is  $\Omega\left(\frac{n}{h}\right)$  when the number of shares is  $h$ .*

## 5 Extensions

*Remark 1.* The bound on the Liquid Price of Anarchy derived in Theorem 1 holds for simultaneous first price auctions with the house clearing item bidding mechanism.

The result of Theorem 1 also extends to Bayesian first price auctions and very similar results hold true for the second price auction format, under standard necessary restrictions on the bidding strategy of the buyers (we describe and discuss these restrictions in Appendix A). The formal proof of Theorem 5 is given in Appendix D and closely follows the proof of Theorem 1.

**Theorem 5.** *In simultaneous first price auctions with  $n$  additive bidders and budgets, where every item has  $h$  equal shares (copies), the Liquid Price of Anarchy of Bayesian Nash equilibria is  $O\left(1 + \frac{n}{h}\right)$  (at most 51.5, when  $h \geq n$ ).*

**Theorem 6.** *In simultaneous second price auctions with  $n$  additive bidders and budgets, where every item has  $h$  equal shares (copies), the Liquid Price of Anarchy of Bayesian Nash equilibria is  $O\left(1 + \frac{n}{h}\right)$  (at most 51.5, when  $h \geq n$ ), under the no over-bidding and no over-budgeting assumptions.*

*Proof.* The proof that Bayesian Nash equilibria achieve a good Liquid Price of Anarchy for simultaneous second price auctions follows similarly to the proof of Theorem 5 in Appendix D. In the following we only discuss the differences that are related to the second price format. In first price simultaneous auctions we have  $\text{REV}(\mathbf{s}) = \frac{1}{\alpha} \sum_{j=1}^m \bar{p}_j$ , while in the second price auction each bidder pays less than  $\frac{1}{\alpha} \bar{p}_j$  per item  $j$ . However, the proof for simultaneous second price auctions goes through if we substitute  $\text{REV}(\mathbf{s})$  with the sum of prices  $\frac{1}{\alpha} \sum_{j=1}^m \bar{p}_j$ , for the reason that the property  $\frac{1}{\alpha} \sum_{j=1}^m \bar{p}_j \leq \text{LW}(\mathbf{s})$  still holds. To see why, fix a pure bidding profile  $\mathbf{b} = (b_1, \dots, b_n)$  of

the players. Recall that  $j^{\{\ell\}}$  denotes the  $\ell^{\text{th}}$  share of item  $j$ , and that  $x_i(\mathbf{b})$  denotes the set of shares of items that player  $i$  wins when players bid according to  $\mathbf{b}$ . In particular, the no over-bidding and no over-budgeting assumptions (see Appendix A for the definitions of these assumptions) imply  $\sum_{j^{\{\ell\}} \in x_i(\mathbf{b})} b_{ij}^\ell \leq v_i(x_i(\mathbf{b}))$  and  $\sum_{j^{\{\ell\}} \in x_i(\mathbf{b})} b_{ij}^\ell \leq B_i$ . Hence, similarly to the simultaneous first price auction setting, we get:

$$\begin{aligned} \text{REV}(\mathbf{s}) &= \frac{1}{\alpha} \sum_{j=1}^m \bar{p}_j = \sum_{j=1}^m \sum_{\ell=1}^h \mathbb{E} [p_j^\ell] = \sum_{j=1}^m \sum_{\ell=1}^h \mathbb{E} \left[ \max_i b_{ij}^\ell \right] \\ &= \sum_{i=1}^n \mathbb{E} \left[ \sum_{j^{\{\ell\}} \in x_i(\mathbf{b})} b_{ij}^\ell \right] \leq \sum_{i=1}^n \mathbb{E} [\min\{v_i(x_i(\mathbf{b})), B_i\}] = \text{LW}(\mathbf{s}), \end{aligned}$$

where the expectation is taken over players' valuation profiles and randomized bidding strategies.

The last observation we mention is that the utility a player receives when performing a deviation in simultaneous second price auctions is at least as high as the utility they would receive if they were forced to pay their bid (since players pay the second highest bid). Thus, all inequalities involving the utility each player derives when performing a deviation still hold. It is not hard to verify that the rest of the proof goes through as well.  $\square$

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## A Second Price Auctions.

We also study *simultaneous second price auctions*, where each item  $j$  again is allocated to the highest bidder, but the winner pays the second highest bid on item  $j$ . The total payment of bidder  $\ell$  in this case is  $p_\ell(\mathbf{b}) = \sum_{j \in x_\ell(\mathbf{b})} \max_{i \neq \ell} b_{ij}$ . As before one can study the Liquid Price of Anarchy for the second price format. However, in general, the Price of Anarchy of second price auctions can be arbitrarily large even when bidders do not have budget constraints and there is only one item for sale<sup>4</sup>. To prevent such pathological equilibria, it is standard [6, 14, 21] to assume that each bidder is guaranteed to derive non-negative utility, no matter how the other bidders behave, i.e.,  $\sum_{j \in S} b_{ij} \leq v_i(S) \forall S \subseteq [m]$ . In other words, no bidder would want to “overbid” on any set of items. In the budgeted setting, in addition to this *no over-bidding* assumption, we also require that  $\sum_{j \in [m]} b_{ij} \leq B_i$  (i.e., *no over-budgeting*). The latter is a necessary assumption even in the single-item case to exclude pathological equilibria. The no over-budgeting assumption can also be motivated by risk-averse attitudes of buyers, who try to eliminate any chance of exceeding their budget and deriving infinite disutility.

## B Pure Nash Equilibria: Beyond Additive Valuations

In this section we consider buyers with complex combinatorial valuations and study the efficiency of pure Nash equilibria of simultaneous item bidding auctions for bidders with budgets. We first show that the Liquid Price of Anarchy for second price auctions, in which bidders have fractionally-subadditive valuations, is 2. The proof is inspired by [14]. We recall the definition of fractionally-subadditive valuations.

**Definition 5.** *Valuation  $v_i$  is fractionally-subadditive if there is a set of additive valuations  $A = \{a_1, \dots, a_\ell\}$  (for some  $\ell \geq 0$ ) such that  $v_i(S) = \max_{a \in A} a(S)$  for every  $S \subseteq [m]$ .*

Valuation  $a \in A$  is called a *maximizing additive valuation* for a particular set  $S$  if  $v_i(S) = a(S)$ .

**Theorem B.1.** *Consider a simultaneous second price auction in which bidders have fractionally-subadditive valuations. If  $\mathbf{b}$  is a pure Nash equilibrium where players’ bids satisfy no over-bidding and no over-budgeting, then we have  $\text{LW}(\mathbf{b}) \geq \frac{\text{OPT}}{2}$ .*

*Proof.* We begin with the following useful lemma.

**Lemma B.1.** *Fix an arbitrary  $S \subseteq [m]$  and a player  $i$  such that  $v_i(S) > 0$ , and let  $a_r$  be a maximizing additive valuation for  $S$ . Consider the alternative bidding strategy  $b_i^*$  for  $i$ , where  $b_{ij}^* = a_r(\{j\}) \frac{\min\{v_i(S), B_i\}}{v_i(S)}$  for  $j \in S$  and  $b_{ij}^* = 0$  for  $j \notin S$ . Then for any pure profile  $\mathbf{b}_{-i}$  we have:*

$$u_i(b_i^*, \mathbf{b}_{-i}) \geq \min\{v_i(S), B_i\} - \sum_{j \in S} \max_{k \neq i} b_{kj}.$$

*Proof.* Let  $T$  be the set of items that player  $i$  wins in the allocation  $x_i(b_i^*, \mathbf{b}_{-i})$ . Note that  $\max_{k \neq i} b_{kj} = 0$  for any  $j \in T \setminus S$  and  $b_{ij}^* - \max_{k \neq i} b_{kj} = a_r(\{j\}) \frac{\min\{v_i(S), B_i\}}{v_i(S)} - \max_{k \neq i} b_{kj} \leq 0$

---

<sup>4</sup>A canonical example is two bidders who value the item at 0 and a large number  $L$ , respectively, but the first bidder bids  $L + 1$  and the second bidder bids 0.

for any  $j \in S \setminus T$ . Then, we have:

$$\begin{aligned}
u_i(b_i^*, \mathbf{b}_{-i}) &= v_i(T) - \sum_{j \in T} \max_{k \neq i} b_{kj} \geq \sum_{j \in T \cap S} a_r(\{j\}) - \sum_{j \in T \cap S} \max_{k \neq i} b_{kj} \\
&\geq \sum_{j \in T \cap S} a_r(\{j\}) \frac{\min\{v_i(S), B_i\}}{v_i(S)} - \sum_{j \in T \cap S} \max_{k \neq i} b_{kj} \\
&\geq \sum_{j \in S} a_r(\{j\}) \frac{\min\{v_i(S), B_i\}}{v_i(S)} - \sum_{j \in S} \max_{k \neq i} b_{kj} \\
&= \min\{v_i(S), B_i\} - \sum_{j \in S} \max_{k \neq i} b_{kj}.
\end{aligned}$$

□

Let  $v_1, \dots, v_n$  denote the valuations of the players, and fix a particular player  $i$ . Let OPT denote the value of the optimal solution, and let  $S_i^*$  denote the set of items that player  $i$  receives in an optimal allocation and  $S_i = x_i(\mathbf{b})$  be  $i$ 's allocation in the pure Nash equilibrium. Suppose that  $v_i(S_i^*) > 0$  (we will handle the case that  $v_i(S_i^*) = 0$  separately). Let  $a_r$  be the maximizing additive valuation for the set  $S_i^*$ , and consider the following deviating strategy for player  $i$ :  $b_{ij}^* = a_r(\{j\}) \frac{\min\{v_i(S_i^*), B_i\}}{v_i(S_i^*)}$  for  $j \in S_i^*$  and  $b_{ij}^* = 0$  for  $j \notin S_i^*$ . Notice that the alternative strategy  $b_i^*$  satisfies no over-bidding and no over-budgeting.

Then, by Lemma B.1, we have  $u_i(b_i^*, \mathbf{b}_{-i}) \geq \min\{v_i(S_i^*), B_i\} - \sum_{j \in S_i^*} \max_{k \neq i} b_{kj}$ . Since  $\mathbf{b}$  is a pure Nash equilibrium, the following must hold:

$$v_i(S_i) \geq u_i(b_i, \mathbf{b}_{-i}) \geq u_i(b_i^*, \mathbf{b}_{-i}) \geq \min\{v_i(S_i^*), B_i\} - \sum_{j \in S_i^*} \max_{k \neq i} b_{kj}.$$

Hence, if  $v_i(S_i^*) > 0$  and  $v_i(S_i) \leq B_i$ , then we have shown that  $\min\{v_i(S_i), B_i\} = v_i(S_i) \geq \min\{v_i(S_i^*), B_i\} - \sum_{j \in S_i^*} \max_{k \neq i} b_{kj}$ . Now, consider a player  $i$  such that  $v_i(S_i^*) = 0$  and  $v_i(S_i) \leq B_i$ . Then again we have  $\min\{v_i(S_i), B_i\} \geq \min\{v_i(S_i^*), B_i\} \geq \min\{v_i(S_i^*), B_i\} - \sum_{j \in S_i^*} \max_{k \neq i} b_{kj}$ . Finally, consider a player  $i$  such that  $v_i(S_i) > B_i$ . In such a case, we have  $\min\{v_i(S_i), B_i\} = B_i \geq \min\{v_i(S_i^*), B_i\} \geq \min\{v_i(S_i^*), B_i\} - \sum_{j \in S_i^*} \max_{k \neq i} b_{kj}$ . Hence, in each case, we have the same lower bound on  $\min\{v_i(S_i), B_i\}$ . Putting these together and summing over all bidders, we get:

$$\begin{aligned}
\text{LW}(\mathbf{b}) &= \sum_{i=1}^n \min\{v_i(S_i), B_i\} \geq \sum_{i=1}^n \min\{v_i(S_i^*), B_i\} - \sum_{i=1}^n \sum_{j \in S_i^*} \max_{k \neq i} b_{kj} \\
&\geq \text{OPT} - \sum_{j=1}^m \max_k b_{kj} \\
&= \text{OPT} - \sum_{i=1}^n \sum_{j \in S_i} b_{ij} \\
&\geq \text{OPT} - \sum_{i=1}^n \min\{v_i(S_i), B_i\} = \text{OPT} - \text{LW}(\mathbf{b}),
\end{aligned}$$

where the last inequality follows since  $\mathbf{b}$  satisfies no over-bidding (i.e.,  $\sum_{j \in S_i} b_{ij} \leq v_i(S_i)$ ) and no over-budgeting (i.e.,  $\sum_{j \in S_i} b_{ij} \leq B_i$ ). □

Similarly, the Liquid Price of Anarchy for pure Nash equilibria of simultaneous first-price auctions, in which bidders have fractionally-subadditive valuations, is arbitrarily close to 2.

**Theorem B.2** (Theorem 2). *Consider a simultaneous first price auction where bidders have fractionally-subadditive valuations. If  $\mathbf{b}$  is a pure Nash equilibrium, then for any  $\epsilon > 0$  we have  $\text{LW}(\mathbf{b}) \geq \frac{\text{OPT}}{2} - \epsilon$ .*

*Proof.* We begin with the following statement, which is similar to Lemma B.1.

**Lemma B.2.** *Fix an arbitrary  $S \subseteq [m]$  and a player  $i$  such that  $v_i(S) > 0$ , and let  $a_r$  be a maximizing additive valuation for  $S$ . For any  $\delta > 0$ , consider the alternative bidding strategy  $b_i^*$  for  $i$ , where  $b_{ij}^* = \min\{a_r(\{j\}) \frac{\min\{v_i(S), B_i\}}{v_i(S)}, \max_{k \neq i} b_{kj} + \delta\}$  for  $j \in S$  and  $b_{ij}^* = 0$  for  $j \notin S$ . Then for any pure profile  $\mathbf{b}_{-i}$  we have:*

$$u_i(b_i^*, \mathbf{b}_{-i}) \geq \min\{v_i(S), B_i\} - \sum_{j \in S} \max_{k \neq i} b_{kj} - \delta|S|.$$

*Proof.* Let  $T$  be the set of items that player  $i$  wins in the allocation  $x_i(b_i^*, \mathbf{b}_{-i})$ . Note that for any  $j \in T \setminus S$ , we have  $b_{ij}^* = 0$  and for any  $j \in T \cap S$ , we have  $b_{ij}^* \leq \max_{k \neq i} b_{kj} + \delta$ . Moreover,  $b_{ij}^* - \max_{k \neq i} b_{kj} \leq 0$  and  $b_{ij}^* = a_r(\{j\}) \frac{\min\{v_i(S), B_i\}}{v_i(S)}$  for any  $j \in S \setminus T$ . Then, we have:

$$\begin{aligned} u_i(b_i^*, \mathbf{b}_{-i}) &= v_i(T) - \sum_{j \in T} b_{ij}^* \geq \sum_{j \in T \cap S} a_r(\{j\}) - \sum_{j \in T \cap S} \max_{k \neq i} b_{kj} - \delta|T \cap S| \\ &\geq \sum_{j \in T \cap S} a_r(\{j\}) \frac{\min\{v_i(S), B_i\}}{v_i(S)} - \sum_{j \in T \cap S} \max_{k \neq i} b_{kj} - \delta|S| \\ &\geq \sum_{j \in S} a_r(\{j\}) \frac{\min\{v_i(S), B_i\}}{v_i(S)} - \sum_{j \in S} \max_{k \neq i} b_{kj} - \delta|S| \\ &= \min\{v_i(S), B_i\} - \sum_{j \in S} \max_{k \neq i} b_{kj} - \delta|S|. \end{aligned}$$

□

Let  $v_1, \dots, v_n$  denote the valuations of the players, and fix a particular player  $i$ . Let OPT denote the value of the optimal solution, and let  $S_i^*$  denote the set of items that player  $i$  receives in an optimal allocation and  $S_i = x_i(\mathbf{b})$  be  $i$ 's allocation in the pure Nash equilibrium. Suppose that  $v_i(S_i^*) > 0$  (we will handle the case that  $v_i(S_i^*) = 0$  separately). Let  $a_r$  be the maximizing additive valuation for the set  $S_i^*$ , and consider the following deviating strategy for player  $i$  for some  $\delta > 0$  to be set later:  $b_{ij}^* = \min\{a_r(\{j\}) \frac{\min\{v_i(S_i^*), B_i\}}{v_i(S_i^*)}, \max_{k \neq i} b_{kj} + \delta\}$  for  $j \in S_i^*$  and  $b_{ij}^* = 0$  for  $j \notin S_i^*$ . Notice that the alternative strategy  $b_i^*$  satisfies no over-budgeting.

Then, by Lemma B.2, we have  $u_i(b_i^*, \mathbf{b}_{-i}) \geq \min\{v_i(S_i^*), B_i\} - \sum_{j \in S_i^*} \max_{k \neq i} b_{kj} - \delta|S_i^*|$ . Since  $\mathbf{b}$  is a pure Nash equilibrium, the following must hold:

$$v_i(S_i) \geq u_i(b_i, \mathbf{b}_{-i}) \geq u_i(b_i^*, \mathbf{b}_{-i}) \geq \min\{v_i(S_i^*), B_i\} - \sum_{j \in S_i^*} \max_{k \neq i} b_{kj} - \delta|S_i^*|.$$

Hence, if  $v_i(S_i^*) > 0$  and  $v_i(S_i) \leq B_i$ , then we have shown that  $\min\{v_i(S_i), B_i\} = v_i(S_i) \geq \min\{v_i(S_i^*), B_i\} - \sum_{j \in S_i^*} \max_{k \neq i} b_{kj} - \delta|S_i^*|$ . Now, consider a player  $i$  such that  $v_i(S_i^*) = 0$  and  $v_i(S_i) \leq B_i$ . Then again we have  $\min\{v_i(S_i), B_i\} \geq \min\{v_i(S_i^*), B_i\} \geq \min\{v_i(S_i^*), B_i\} -$

$\sum_{j \in S_i^*} \max_{k \neq i} b_{kj} - \delta |S_i^*|$ . Finally, consider a player  $i$  such that  $v_i(S_i) > B_i$ . In such a case, we have  $\min\{v_i(S_i), B_i\} = B_i \geq \min\{v_i(S_i^*), B_i\} \geq \min\{v_i(S_i^*), B_i\} - \sum_{j \in S_i^*} \max_{k \neq i} b_{kj} - \delta |S_i^*|$ . Hence, in each case, we have the same lower bound on  $\min\{v_i(S_i), B_i\}$ . Putting these together and summing over all bidders, we get:

$$\begin{aligned}
\text{LW}(\mathbf{b}) &= \sum_{i=1}^n \min\{v_i(S_i), B_i\} \geq \sum_{i=1}^n \min\{v_i(S_i^*), B_i\} - \sum_{i=1}^n \sum_{j \in S_i^*} \max_{k \neq i} b_{kj} - \delta \sum_{i=1}^n |S_i^*| \\
&\geq \text{OPT} - \sum_{j=1}^m \max_k b_{kj} - \delta m \\
&= \text{OPT} - \sum_{i=1}^n \sum_{j \in S_i} b_{ij} - \delta m \\
&\geq \text{OPT} - \sum_{i=1}^n \min\{v_i(S_i), B_i\} - \delta m = \text{OPT} - \text{LW}(\mathbf{b}) - \delta m,
\end{aligned}$$

where the last inequality follows since in any Nash equilibrium, the bids  $\mathbf{b}$  must satisfy  $\sum_{j \in S_i} b_{ij} \leq v_i(S_i)$  and  $\sum_{j \in S_i} b_{ij} \leq B_i$ . Noting that we can set  $\delta = \frac{2\epsilon}{m}$  gives the theorem.  $\square$

## C Lower Bounds

We give an example with additive valuations which shows that a randomized tie-breaking rule can lead to a Liquid Price of Anarchy which is  $\Omega(n)$  in the following theorem.

**Theorem C.1** (Theorem 3). *With a randomized tie-breaking rule, there are simultaneous first and second price auction games which have an  $\Omega(n)$  Liquid Price of Anarchy, even when agents play pure strategies.*

*Proof.* Consider a simultaneous first price auction in which there are  $n$  items and  $n$  bidders (the proof goes through for simultaneous second price auctions as well). Each bidder has the same valuation profile for the items, namely they all value the first item at  $n^4$  and the remaining  $n - 1$  items at 1. Moreover, each bidder has a budget of 1. Observe that the optimal solution is  $n$ , which is attained by giving each agent one item, for a total Liquid Welfare of  $n$  (since each player is budget-capped at 1).

On the other hand, consider the following randomized tie-breaking rule, where all ties among the highest bids that occur for item 1 are broken uniformly at random (the tie-breaking rule for the remaining items can be arbitrary). Since item 1 is valued so highly, the pure strategy profile in which each bidder bids 1 (i.e., their maximum possible bid) for item 1 is a pure Nash equilibrium. In particular, each player wins item 1 with probability  $\frac{1}{n}$ , and since all  $n$  agents are tied for the maximum bid, their utility is  $\frac{1}{n} \cdot (n^4 - 1)$ . If they switch, their utility will be at most  $n - 1$  (the same holds for second price auctions). This leads to a Liquid Welfare of 1, which implies the Liquid Price of Anarchy is  $\Omega(n)$ .  $\square$

In fact, the same problem also exists when the tie-breaking rule is deterministic and players can randomize their strategies.

**Theorem C.2.** *With a deterministic tie-breaking rule, there are some simultaneous first price and second price auction games for which the Liquid Price of Anarchy is  $\Omega(n)$  when agents mix their strategies.*

*Proof.* Consider a simultaneous first price auction in which there are  $n$  items and  $n$  bidders (the proof goes through for simultaneous second price auctions as well), each of which has a budget of 1. Items 1 and 2 have a tie-breaking rule which favors players lexicographically (i.e., player 1 is the most preferred while player  $n$  is the least preferred), while the rest of the items have an arbitrary tie-breaking rule. All players value the first two items at  $2^{2^n}$ . The first  $n - 2$  players value the remaining items  $3, \dots, n$  at 1, while players  $n - 1$  and  $n$  value items  $3, \dots, n$  at 0. Observe that the optimal solution is  $n$ , which is attained by allocating item 1 to player  $n - 1$ , item 2 to player  $n$ , and giving each of the remaining  $n - 2$  items to each of the remaining  $n - 2$  players.

Consider the following bad mixed Nash equilibrium. Players  $n$  and  $n - 2$  have pure strategies where both of them bid their full budget on item 1 (recall all players have a budget of 1). Players  $n - 1$  and  $n - 3$  have pure strategies where both of them bid their full budget on item 2. The first  $n - 4$  players randomize their strategies by bidding their full budget on item 1 with probability  $\frac{1}{2}$  and bidding their full budget on item 2 with probability  $\frac{1}{2}$ . This leads to a Liquid Welfare of 2, since only two players win an item. Hence, the Liquid Price of Anarchy is  $\Omega(n)$ .

Let us see why this mixed strategy profile forms an equilibrium. Players  $n - 1$  and  $n$  never win an item since the tie-breaking rule prefers player  $n - 3$  to player  $n - 1$  and prefers player  $n - 2$  to player  $n$ , but they cannot deviate and improve their utility. In particular, both players value items  $n - 3, \dots, n$  at 0, so they cannot bid on such items. Moreover, even if player  $n$  deviates and bids their full budget on item 2, they still lose since the tie-breaking rule prefers player  $n - 3$  (similarly, player  $n - 1$  cannot improve their utility by deviating to item 1 since the tie-breaking rule prefers player  $n - 2$ ). Player  $n - 2$  currently wins exactly when players  $1, \dots, n - 4$  all randomly choose item 2, which happens with probability  $\frac{1}{2^{n-4}}$ , and hence has an expected utility of at least  $\frac{1}{2^{n-4}}(2^{2^n} - 1) \geq 2^n$  (similarly, player  $n - 3$  wins when players  $1, \dots, n - 4$  all choose item 1, and hence player  $n - 3$  has the same expected utility as player  $n - 2$ ). Consider any other possible pure strategy for player  $n - 2$ . They cannot win item 1 since they bid strictly less than 1, and they cannot win item 2 since player  $n - 3$  is preferred by the tie-breaking rule. Their expected utility in any such deviation is at most  $n - 2 < 2^n$ , which is attained by winning each of the items valued at 1. A similar argument holds for player  $n - 3$ , the only difference being that if they deviate and bid their whole budget on item 1, they can win the item (but this results in the same expected utility). Finally, observe that each player  $i > n - 3$  wins an item of value  $2^{2^n}$  when all players  $1, \dots, i - 1$  bid on the other item of value  $2^{2^n}$ , and hence player  $i$ 's expected utility is at least  $2^n$ . Any deviation will result in the same expected utility, or an expected utility of at most  $n - 2$ . Hence, we have a mixed Nash equilibrium.  $\square$

In the following theorems, we consider the affect of shares on the Liquid Price of Anarchy. We write the proofs assuming the market clearing mechanism for shares discussed in Section 3, but the proofs can easily be extended to handle the share model where shares are treated as separate items and buyers can submit different bids for every single share.

**Theorem C.3.** *With a randomized tie-breaking rule, there are some simultaneous first price and second price auction games for which the Liquid Price of Anarchy is  $\Omega\left(\frac{n}{h}\right)$  when the number of shares is  $h$ , even when agents play pure strategies.*

*Proof.* We proceed in a manner similar to the proof of Theorem 3. In particular, we again consider a simultaneous first price auction in which there are  $n$  items and  $n$  bidders, where each of the items has  $h$  shares. Each bidder has the same valuation profile for the items, namely they all value the first item at  $n^4$ , and hence extract a value of  $\frac{n^4}{h}$  per share, while the remaining  $n - 1$  items are valued at 1, each share of which has value  $\frac{1}{h}$ . Moreover, each bidder has a budget of 1 as before.

Observe that the optimal solution is  $n$ , which is attained by giving each agent every share of a particular item, for a total Liquid Welfare of  $n$  (since each player is budget-capped at 1).

On the other hand, consider the following randomized tie-breaking rule, where all ties among the highest bids that occur for each share of item 1 are broken uniformly at random (the tie-breaking rule for the remaining shares can be arbitrary). Since item 1 is valued so highly, the pure strategy profile in which each bidder wants 1 share of the first item and bids 1 per share (i.e., their maximum possible bid) is a pure Nash equilibrium. In particular, each player wins one share of item 1 with probability  $\frac{h}{n}$ , and since all  $n$  agents are tied for the maximum bid, their utility is  $\frac{h}{n} \cdot (\frac{n^4}{h} - 1)$ . If they switch, their utility will be at most  $n - 1$  (the same holds for second price auctions). Notice that no player would ever bid on multiple shares of the first item, since this would mean they are bidding strictly less than one per share and never win a share. This leads to a Liquid Welfare of at most  $h$ , which implies the Liquid Price of Anarchy is at least  $\frac{n}{h}$ .  $\square$

In fact, we can obtain a similar negative result when players can mix their strategies, even when the tie-breaking rule is deterministic. As mentioned earlier, the proof is written in the context of the market clearing mechanism for shares, but can easily be extended to the case where shares are treated as separate items and players can submit different bids for each share.

**Theorem C.4** (Theorem 4). *With a deterministic tie-breaking rule, there are some simultaneous first price and second price auction games for which the Liquid Price of Anarchy is  $\Omega(\frac{n}{h})$  when the number of shares is  $h$  and agents mix their strategies.*

*Proof.* We proceed in a manner similar to the proof of Theorem C.2, but adapt the example for the share model. Consider a simultaneous first price auction in which there are  $n$  items, each with  $h$  shares, and  $n$  bidders, each with a budget of 1 (the proof goes through for simultaneous second price auctions as well). Items 1 and 2 have a tie-breaking rule which favors players lexicographically (i.e., player 1 is the most preferred while player  $n$  is the least preferred), while the rest of the items have an arbitrary tie-breaking rule. All players value the first two items at  $2^{2n}$ , each share of which is valued at  $\frac{2^{2n}}{h}$ . The first  $n - 2$  players value the remaining items  $3, \dots, n$  at 1, each share of which is valued at  $\frac{1}{h}$ , while players  $n - 1$  and  $n$  value items  $3, \dots, n$  at 0. Observe that the optimal solution is  $n$ , which is attained by allocating item 1 to player  $n - 1$ , item 2 to player  $n$ , and giving each of the remaining  $n - 2$  items to each of the remaining  $n - 2$  players (here, giving an item to a player means giving all shares of the item to the player).

Consider the following bad mixed Nash equilibrium. Players  $n, n - 2, \dots, n - 2 \cdot h$  have pure strategies where all  $h + 1$  of them bid their full budget on one share of item 1 (recall all players have a budget of 1). Players  $n - 1, n - 3, \dots, n - 2 \cdot h - 1$  have pure strategies where all  $h + 1$  of them bid their full budget on one share of item 2. The first  $n - 2 \cdot h - 2$  players randomize their strategies by bidding their full budget on one share of item 1 with probability  $\frac{1}{2}$  and bidding their full budget on one share of item 2 with probability  $\frac{1}{2}$ . This leads to a Liquid Welfare of  $2 \cdot h$ , since only  $2 \cdot h$  players win a share. Hence, the Liquid Price of Anarchy is  $\Omega(\frac{n}{h})$ .

Let us see why this mixed strategy profile forms an equilibrium. Observe that, no matter how the first  $n - 2 \cdot h - 2$  players randomly choose which item to bid on, there are always at least  $h + 1$  players bidding on item 1 and at least  $h + 1$  players bidding on item 2. Hence, player  $n$  never wins a share since the tie-breaking rule prefers all other  $h$  agents that purely bid on item 1, and similarly player  $n - 1$  never wins a share since the tie-breaking rule prefers all other  $h$  agents playing purely on item 2. These two players cannot deviate and improve their utility, since they value items  $3, \dots, n$  at 0 and hence cannot bid on such items. Moreover, even if player  $n$  deviates and bids their full budget on one share of item 2, they still lose due to the tie-breaking rule (similarly, player  $n - 1$  cannot improve their utility by deviating to item 1 due to the tie-breaking rule). In addition,

player  $n$  is indifferent between bidding 1 on item 1 and bidding less than 1, since their utility is 0 either way (similarly, player  $n - 1$  is indifferent between bidding 1 on item 2 and bidding less than 1). In fact, every player who bids less than 1 on item 1 or less than one on item 2 loses the item with certainty.

Consider any other player  $n - 2 \cdot i$  purely bidding on item 1 (so  $1 \leq i \leq h$ ). There are  $h - i$  players who also purely bid on item 1 and are preferred by the tie-breaking rule, but if player  $n - 2 \cdot i$  deviates and purely bids their full budget on one share of item 2, the probability of winning a share decreases since there are  $h - i + 1$  players who purely bid on item 2 and are preferred by the tie-breaking rule. If player  $n - 2 \cdot i$  bids less than 1 on item 1 and less than 1 on item 2, then their utility is at most  $n - 2$  (attained by winning all shares of items  $3, \dots, n$ ). However, in the Nash equilibrium, the probability that player  $n - 2 \cdot i$  wins a share is at least the probability that all players who mix their strategies randomly choose item 2, which is given by  $\frac{1}{2^{n-2h-2}}$ . Hence, their expected utility is at least  $\frac{1}{2^{n-2h-2}} \left( \frac{2^{2n}}{h} - 1 \right) \geq 2^n \geq n - 2$ , so they cannot improve their expected utility by deviating. The argument that players purely bidding on item 2 cannot improve their expected utility by deviating is similar. The only difference is that, for any such player  $n - 2 \cdot i - 1$  for  $1 \leq i \leq h$ , the number of players that purely bid on item 2 and are preferred by the tie-breaking rule is  $h - i$ , which also holds if player  $n - 2 \cdot i - 1$  were to purely bid on item 1. Hence, such players would win a share of item 1 after deviating with the same probability of winning a share in the mixed Nash equilibrium, so they are indifferent. Finally, observe that each player  $i \geq n - 2h - 2$  is indifferent between item 1 and item 2, and they have an expected utility at least that of any player who bids purely and wins a share with positive probability, which is at least  $2^n$ . This is at least as much as their expected utility in any deviation, and hence we have a mixed Nash equilibrium.  $\square$

## D Bayesian Nash Equilibrium

We assume agents have additive valuations and submit bids on shares, and if they receive an  $x_{ij}$  fraction of shares of item  $j$ , then their value is given by  $v_i(x_{ij}) = v_{ij} \cdot x_{ij}$  ( $v_i$  is technically defined over sets, but for singleton sets we will often write  $v_{ij}$  for ease of notation). We consider a *Bayesian* setting, where the bidders' valuations are drawn independently from distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$  and write  $\mathcal{D} = \mathcal{D}_1 \times \dots \times \mathcal{D}_n$ , so that  $\mathbf{v}$  is drawn from  $\mathcal{D}$ . We assume that each valuation  $v_i$  is realized together with an associated budget  $B_i$  which we usually will omit in our notations for brevity. In fact, we will use the notation  $B_i(v_i)$  to emphasize that each player's budget can be correlated with their valuation. We think of  $\mathcal{D}$  as being public knowledge, whereas the realization  $v_i$  is known only to agent  $i$ .

We further assume that the buyers bid according to a Bayesian Nash equilibrium  $\mathbf{b} \sim \mathbf{s}(\mathbf{v})$ , where  $\mathbf{v} \sim \mathcal{D}$ . When buyers bid in simultaneous auctions, this induces a distribution of prices over all shares of items  $\mathbf{p} \sim \mathcal{F}$  from a distribution  $\mathcal{F}$  (e.g., winning bids in first price auctions, namely  $p_j^\ell = \max_i b_{ij}^\ell$ , where  $b_{ij}^\ell$  is player  $i$ 's bid for share  $\ell$  of item  $j$ ). In particular, for all items we can define an ‘‘expected price per item’’ at equilibrium or just a ‘‘price per item’’ as  $\bar{\mathbf{p}} = (\bar{p}_1, \dots, \bar{p}_m)$ , where  $\bar{p}_j = \alpha \sum_{\ell=1}^h \mathbb{E}_{\mathcal{F}}[p_j^\ell]$ , for some  $\alpha > 1$  ( $\alpha = 2$  will be sufficient for us). This induces a natural ‘‘expected price per share,’’ namely  $\frac{\bar{p}_i}{h}$ . One simple observation about  $\bar{\mathbf{p}}$  is the following:

**Observation D.1.** *Revenue is related to prices:  $\text{REV}(\mathbf{s}) = \frac{1}{\alpha} \sum_{j=1}^m \bar{p}_j$ , where  $\text{REV}(\mathbf{s})$  denotes the expected revenue at the equilibrium profile  $\mathbf{s}$ .*

We next show that if players bid on some fraction of shares of item  $j$  uniformly at random according to  $\bar{p}_j$ , then they win a large number of shares in expectation.

**Claim D.1.** For any item  $j$ , if a player bids on a  $\delta$ -fraction of shares chosen uniformly at random of item  $j$  at a given price  $\frac{\bar{p}_j}{h}$  per share, then the player receives in expectation at least  $h \cdot \delta \cdot \left(1 - \frac{1}{\alpha}\right)$  shares of the item (i.e., at least a  $\delta \cdot \left(1 - \frac{1}{\alpha}\right)$ -fraction of item  $j$ ).

*Proof.* Suppose towards a contradiction that the expected number of shares won by bidder  $i$  is less than  $\delta \cdot h \cdot \left(1 - \frac{1}{\alpha}\right)$ . In particular, it means that

$$\delta \cdot h \cdot \left(1 - \frac{1}{\alpha}\right) > \sum_{\ell=1}^h \Pr [i \text{ bids on share } \ell] \cdot \Pr_{\mathcal{F}} \left[ p_j^\ell < \frac{\bar{p}_j}{h} \right] \geq \sum_{\ell=1}^h \delta \cdot \Pr_{\mathcal{F}} \left[ p_j^\ell < \frac{\bar{p}_j}{h} \right].$$

We further use the definition of  $\bar{p}_j$  and Markov's inequality to obtain a contradiction as follows:

$$\begin{aligned} \frac{\bar{p}_j}{\alpha} &= \sum_{\ell=1}^h \mathbb{E}_{\mathcal{F}} \left[ p_j^\ell \right] \geq \sum_{\ell=1}^h \frac{\bar{p}_j}{h} \cdot \Pr_{\mathcal{F}} \left[ p_j^\ell \geq \frac{\bar{p}_j}{h} \right] \\ &= \sum_{\ell=1}^h \frac{\bar{p}_j}{h} \left( 1 - \Pr_{\mathcal{F}} \left[ p_j^\ell < \frac{\bar{p}_j}{h} \right] \right) > \bar{p}_j - \frac{\bar{p}_j}{h} \cdot h \cdot \left( 1 - \frac{1}{\alpha} \right). \end{aligned}$$

□

When relating prices to Liquid Welfare we notice that

**Observation D.2.** Revenue is bounded by the Liquid Welfare:  $\text{REV}(\mathbf{s}) \leq \text{LW}(\mathbf{s})$ , where  $\text{LW}(\mathbf{s})$  denotes the expected Liquid Welfare at the equilibrium profile  $\mathbf{s}$ .

For each fixed profile  $\mathbf{v}$  of additive valuations we consider the following fractional relaxation (LP) of the allocation problem with the goal of optimizing Liquid Welfare:

$$\begin{aligned} &\text{Maximize} && \sum_{i=1}^n \sum_{j=1}^m v_{ij} z_{ij} \\ &\text{Subject to} && \sum_j v_{ij} z_{ij} \leq B_i(v_i) \quad \forall i \\ &&& \sum_i z_{ij} \leq 1 \quad \forall j \\ &&& z_{ij} \geq 0 \quad \forall i, j \end{aligned}$$

We denote by  $\mathbf{y}(\mathbf{v}) = (y_{ij})$  the optimal solution to the above LP. Notice that the optimal fractional solution for the Liquid Welfare would never benefit from allocating a set of items to a player such that their value for the set exceeds their budget.

We slightly abuse notations by defining  $y_{ij}(v_i) \stackrel{\text{def}}{=} \mathbb{E}_{\mathbf{v}_{-i} \sim \mathcal{D}_{-i}} [y_{ij}(v_i, \mathbf{v}_{-i})]$  for each bidder  $i$  with fixed valuation  $v_i$ . We observe that the vector of allocations  $\mathbf{y}_i(v_i)$  is budget feasible, i.e.,  $\sum_j v_{ij} y_{ij}(v_i) \leq B_i(v_i)$ , as it is an average of budget feasible allocations for the fixed valuation  $v_i$  and budget  $B_i(v_i)$ . Furthermore, solutions to this LP give an upper bound on the optimal Liquid Welfare OPT.

**Observation D.3.** The optimal fractional solution to LP is better than the optimal allocation:

$$\sum_{i=1}^n \mathbb{E}_{v_i \sim \mathcal{D}_i} \left[ \sum_{j=1}^m v_{ij} \cdot y_{ij}(v_i) \right] \geq \text{OPT}.$$



We now define some notation that will be useful in order to obtain our result. We let  $q_{ij}(\mathbf{v})$  and  $q_{ij}(v_i)$  be the expected fraction of shares that player  $i$  receives from item  $j$  at an equilibrium strategy  $\mathbf{s}$  for a fixed valuation profile  $\mathbf{v}$  and  $\mathbf{v}_{-i} \sim \mathcal{D}_{-i}$ , respectively. In addition, for each agent  $i$ , we consider a set of high value items  $J_i(v_i) \stackrel{\text{def}}{=} \{j \mid v_{ij} \geq \bar{p}_j\}$ . We further define  $Q_i(\mathbf{v})$  to be the probability that  $v_i(x_i) \geq B_i(v_i)$  at equilibrium for a fixed valuation profile  $\mathbf{v}$  (recall that  $x_i$  denotes the random set that player  $i$  receives in the Bayesian Nash equilibrium). We similarly define  $Q_i(v_i) = \mathbb{E}_{\mathbf{v}_{-i}}[Q_i(\mathbf{v})]$  for a fixed valuation  $v_i$ . We also define three sets of bidders, the first two of which are for budget feasibility reasons and the last of which is for bidders that often fall under their budget in equilibrium (these sets need not be disjoint). In particular, for a fixed parameter  $\gamma > 1$  to be determined later, we define sets  $\mathcal{I}_1$ ,  $\mathcal{I}_2$ , and  $\mathcal{I}_3$ :

$$\mathcal{I}_1(\mathbf{v}) \stackrel{\text{def}}{=} \left\{ i \mid \gamma \sum_{j \in J_i(v_i)} \bar{p}_j \cdot q_{ij}(v_i) \leq B_i(v_i) \right\}, \quad \mathcal{I}_2(\mathbf{v}) \stackrel{\text{def}}{=} \left\{ i \mid \sum_{j \in [m]} \frac{\bar{p}_j}{h} \leq B_i(v_i) \right\}, \quad \text{and}$$

$$\mathcal{I}_3(\mathbf{v}) \stackrel{\text{def}}{=} \left\{ i \mid Q_i(v_i) \leq \frac{1}{2\gamma} \right\}.$$

Throughout our proof, we focus on bidders in the set  $\mathcal{I}(\mathbf{v}) \stackrel{\text{def}}{=} \mathcal{I}_1(\mathbf{v}) \cap \mathcal{I}_2(\mathbf{v}) \cap \mathcal{I}_3(\mathbf{v})$ . We define sets  $\bar{\mathcal{I}}_1(\mathbf{v}) \stackrel{\text{def}}{=} [n] \setminus \mathcal{I}_1(\mathbf{v})$ ,  $\bar{\mathcal{I}}_2(\mathbf{v}) \stackrel{\text{def}}{=} [n] \setminus \mathcal{I}_2(\mathbf{v})$ ,  $\bar{\mathcal{I}}_3(\mathbf{v}) \stackrel{\text{def}}{=} [n] \setminus \mathcal{I}_3(\mathbf{v})$ , and  $\bar{\mathcal{I}}(\mathbf{v}) \stackrel{\text{def}}{=} [n] \setminus \mathcal{I}(\mathbf{v})$ . We note that although the set  $\mathcal{I}$  depends on the entire valuation profile  $\mathbf{v}$ , whether  $i \in \mathcal{I}(\mathbf{v})$  or  $i \in \bar{\mathcal{I}}(\mathbf{v})$  is determined only by  $v_i$  alone. To this end, we need to argue that bidders outside of the set  $\mathcal{I}$  do not contribute a lot to the Liquid Welfare at equilibrium  $\mathbf{s}$ .

**Claim D.2.** *The total budget of players in  $\bar{\mathcal{I}}$  is small:  $\mathbb{E}_{\mathbf{v}}[\sum_{i \in \bar{\mathcal{I}}(\mathbf{v})} B_i(v_i)] < \alpha \cdot \left(\gamma + \frac{n}{h}\right) \cdot \text{REV}(\mathbf{s}) + \mathbb{E}_{\mathbf{v}}[\sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i)]$ .*

*Proof.* We first consider agents in  $\bar{\mathcal{I}}_1$  and obtain

$$\begin{aligned} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_1(\mathbf{v})} B_i(v_i) \right] &< \mathbb{E}_{\mathbf{v}} \left[ \gamma \sum_{i \in \bar{\mathcal{I}}_1(\mathbf{v})} \sum_{j \in J_i(v_i)} \bar{p}_j \cdot q_{ij}(v_i) \right] \leq \mathbb{E}_{\mathbf{v}} \left[ \gamma \sum_i \sum_{j \in J_i(v_i)} \bar{p}_j \cdot q_{ij}(v_i) \right] \\ &\leq \mathbb{E}_{\mathbf{v}} \left[ \gamma \sum_j \bar{p}_j \sum_{i: j \in J_i(v_i)} q_{ij}(\mathbf{v}) \right] \leq \gamma \sum_j \bar{p}_j \leq \gamma \cdot \alpha \cdot \text{REV}(\mathbf{s}), \end{aligned}$$

where the second to last inequality follows from the fact that  $\sum_i q_{ij}(\mathbf{v}) \leq 1$  since we have a valid fractional allocation. Next, we consider agents in  $\bar{\mathcal{I}}_2$  and obtain

$$\mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_2(\mathbf{v})} B_i(v_i) \right] < \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_2(\mathbf{v})} \sum_j \frac{\bar{p}_j}{h} \right] \leq \frac{n}{h} \sum_j \bar{p}_j \leq \frac{n}{h} \cdot \alpha \cdot \text{REV}(\mathbf{s}).$$

Combining these bounds for agents in  $\bar{\mathcal{I}}_1$  and  $\bar{\mathcal{I}}_2$  we have

$$\begin{aligned} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}(\mathbf{v})} B_i(v_i) \right] &\leq \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_1(\mathbf{v})} B_i(v_i) + \sum_{i \in \bar{\mathcal{I}}_2(\mathbf{v})} B_i(v_i) + \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i) \right] \\ &\leq \alpha \cdot \left(\gamma + \frac{n}{h}\right) \cdot \text{REV}(\mathbf{s}) + \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i) \right]. \end{aligned}$$

□

To achieve our result, we consider two main ideas for player deviations in set  $\mathcal{I}(\mathbf{v})$  (each idea actually consists of two parts). The first idea is to use the fractional solution to the LP as guidance to claim that players can extract a large amount of value relative to the optimal solution. However, for players to deviate, we must first round the fractional solution  $\mathbf{y}$  into an integral solution (here, by integral, we mean a multiple of  $\frac{1}{h}$  since this represents the fraction of shares a player receives out of  $h$  copies). Define the first LP deviation (integral part) to be  $\mathbf{b}_1^{[i]} = (b'_i, \mathbf{b}_{-i})$ , where in  $b'_i$  buyer  $i$  bids on a random  $\lfloor y_{ij} \rfloor_h$ -fraction of each item  $j \in J_i(v_i)$  with price  $\bar{p}_j$  (here,  $\lfloor y \rfloor_h = \frac{\lfloor y \cdot h \rfloor}{h}$ ). Define the second LP deviation (fractional part) to be  $\mathbf{b}_1^{\{i\}} = (b'_i, \mathbf{b}_{-i})$ , where in  $b'_i$  buyer  $i$  bids on a random  $\{y_{ij}\}_h$ -fraction of each item  $j \in J_i(v_i)$  with price  $\bar{p}_j$  (here,  $\{y\}_h = \frac{1}{h}$  if  $y > 0$ , and  $\{y\}_h = 0$  otherwise). We note that both LP deviations  $\mathbf{b}_1^{[i]}$  and  $\mathbf{b}_1^{\{i\}}$  are feasible, since  $v_{ij} \geq \bar{p}_j$  for every  $j \in J_i(v_i)$ , and  $\sum_j v_{ij} \cdot y_{ij} \leq B_i(v_i)$  as  $\mathbf{y}$  is a solution to LP along with  $\mathcal{I}(\mathbf{v}) \subseteq \mathcal{I}_2(\mathbf{v})$ . Moreover, for any  $y_{ij}(\mathbf{v})$ , we have  $\lfloor y_{ij}(\mathbf{v}) \rfloor_h + \{y_{ij}(\mathbf{v})\}_h \geq y_{ij}(\mathbf{v})$ .

**Lemma D.1** (LP deviations). *Buyers in  $\mathcal{I}$  at equilibrium  $\mathbf{s}$  derive large value:*

$$\mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] \geq \left( \frac{1}{2} - \frac{1}{2\alpha} \right) \left( \text{OPT} - \alpha \left( 1 + \gamma + \frac{n}{h} \right) \text{REV}(\mathbf{s}) - \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}_3(\mathbf{v})} B_i(v_i) \right] \right).$$

*Proof.* For the integral part of the LP deviation, since  $\mathbb{E}_{\mathbf{b} \sim \mathbf{s}}[v_i(\mathbf{b})] \geq \mathbb{E}_{\mathbf{b} \sim \mathbf{s}}[u_i(\mathbf{b})]$  and  $\mathbf{s}$  is a Bayesian Nash equilibrium, we have:

$$\begin{aligned} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \mathbb{E}_{\mathbf{b} \sim \mathbf{s}(\mathbf{v})} [u_i(\mathbf{b})] \right] \\ &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \mathbb{E}_{\mathbf{b}_{-i} \sim \mathbf{s}_{-i}(\mathbf{v}_{-i})} \left[ u_i \left( \mathbf{b}_1^{[i]} \right) \right] \right] \\ &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \in J_i(v_i)} \left( 1 - \frac{1}{\alpha} \right) \cdot \lfloor y_{ij}(v_i) \rfloor_h (v_{ij} - \bar{p}_j) \right], \quad (7) \end{aligned}$$

where to derive the last inequality we use Claim D.1. Similarly, for the fractional part of the LP deviation we have:

$$\begin{aligned} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \mathbb{E}_{\mathbf{b} \sim \mathbf{s}(\mathbf{v})} [u_i(\mathbf{b})] \right] \\ &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \mathbb{E}_{\mathbf{b}_{-i} \sim \mathbf{s}_{-i}(\mathbf{v}_{-i})} \left[ u_i \left( \mathbf{b}_1^{\{i\}} \right) \right] \right] \\ &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \in J_i(v_i)} \left( 1 - \frac{1}{\alpha} \right) \cdot \{y_{ij}(v_i)\}_h (v_{ij} - \bar{p}_j) \right], \quad (8) \end{aligned}$$

Combining Equation (7) and Equation (8) we get

$$\begin{aligned}
2\mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \in J_i(v_i)} \left(1 - \frac{1}{\alpha}\right) \cdot ([y_{ij}(v_i)]_h + \{y_{ij}(v_i)\}_h) (v_{ij} - \bar{p}_j) \right] \\
&\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \in J_i(v_i)} \left(1 - \frac{1}{\alpha}\right) y_{ij}(v_i) (v_{ij} - \bar{p}_j) \right] \\
&= \left(1 - \frac{1}{\alpha}\right) \left( \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \in J_i(v_i)} v_{ij} \cdot y_{ij}(\mathbf{v}) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \in J_i(v_i)} \bar{p}_j \cdot y_{ij}(\mathbf{v}) \right] \right). \tag{9}
\end{aligned}$$

We further estimate

$$\begin{aligned}
\mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \in J_i(v_i)} v_{ij} \cdot y_{ij}(\mathbf{v}) \right] &= \mathbb{E}_{\mathbf{v}} \left[ \sum_{i,j} v_{ij} \cdot y_{ij}(\mathbf{v}) - \sum_{i \notin \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot y_{ij}(\mathbf{v}) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \notin J_i(v_i)} v_{ij} \cdot y_{ij}(\mathbf{v}) \right] \\
&\geq \mathbb{E}_{\mathbf{v}} \left[ \sum_{i,j} v_{ij} \cdot y_{ij}(\mathbf{v}) - \sum_{i \notin \mathcal{I}(\mathbf{v})} B_i(v_i) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \notin J_i(v_i)} \bar{p}_j \cdot y_{ij}(\mathbf{v}) \right],
\end{aligned}$$

where in the last inequality we used the condition from the LP that  $\sum_j v_{ij} \cdot y_{ij}(\mathbf{v}) \leq B_i(v_i)$  and that  $v_{ij} \leq \bar{p}_j$  for each  $j \notin J_i(v_i)$ . We substitute the last estimate into Equation (9) and obtain a lower bound on  $2\mathbb{E}_{\mathbf{v}}[\sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i)]$ :

$$\begin{aligned}
&\left(1 - \frac{1}{\alpha}\right) \mathbb{E}_{\mathbf{v}} \left[ \sum_{i,j} v_{ij} \cdot y_{ij}(\mathbf{v}) - \sum_{i \notin \mathcal{I}(\mathbf{v})} B_i(v_i) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \notin J_i(v_i)} \bar{p}_j \cdot y_{ij}(\mathbf{v}) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \in J_i(v_i)} \bar{p}_j \cdot y_{ij}(\mathbf{v}) \right] \\
&= \left(1 - \frac{1}{\alpha}\right) \mathbb{E}_{\mathbf{v}} \left[ \sum_{i,j} v_{ij} \cdot y_{ij}(\mathbf{v}) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j \bar{p}_j \cdot y_{ij}(\mathbf{v}) - \sum_{i \notin \mathcal{I}(\mathbf{v})} B_i(v_i) \right] \\
&\geq \left(1 - \frac{1}{\alpha}\right) \left( \text{OPT} - \alpha \left(1 + \gamma + \frac{n}{h}\right) \text{REV}(\mathbf{s}) - \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}_3(\mathbf{v})} B_i(v_i) \right] \right),
\end{aligned}$$

where the last inequality follows from the LP constraint that  $\sum_i y_{ij}(\mathbf{v}) \leq 1$  for each  $j$ , the observation  $\sum_j \bar{p}_j = \alpha \cdot \text{REV}(\mathbf{s})$ , and Claim D.2.  $\square$

We now turn to our second type of deviation, but we need to further restrict the set of items that players bid on. In particular, we let  $\Gamma_i(v_i) = \left\{ j \mid q_{ij}(v_i) \leq \frac{1}{\gamma} \right\}$ , and define  $G_i(v_i) = J_i(v_i) \cap \Gamma_i(v_i)$ . We now define the  $\gamma$ -boosting deviation (integral part) as  $\mathbf{b}_2^{\lfloor i \rfloor} = (b'_i, \mathbf{b}_i)$ , where in  $b'_i$  buyer  $i$  bids on a random  $\lfloor \gamma \cdot q_{ij}(v_i) \rfloor_h$ -fraction of each item  $j \in G_i(v_i)$  with price  $\bar{p}_j$ , where  $\gamma > 1$  is a constant to be determined later. Note that each  $\mathbf{b}_2^{\lfloor i \rfloor}$  deviation for every  $i \in \mathcal{I}(\mathbf{v})$  is feasible since  $\mathcal{I}(\mathbf{v}) \subseteq \mathcal{I}_1(\mathbf{v})$ . Similarly, we define the fractional part of the  $\gamma$ -boosting deviation as  $\mathbf{b}_2^{\{i\}}$ , which is also a feasible deviation since  $\mathcal{I}(\mathbf{v}) \subseteq \mathcal{I}_2(\mathbf{v})$ . Also, since players bid on items in  $G_i(\mathbf{v}) \subseteq \Gamma_i(\mathbf{v})$ , we have  $\lfloor \gamma \cdot q_{ij}(v_i) \rfloor_h \leq 1$  (we also have  $\{\gamma \cdot q_{ij}(v_i)\}_h \leq 1$ , which holds for all items by definition).

**Lemma D.2** ( $\gamma$ -boosting deviation). *The value derived by buyers in  $\mathcal{I}$  is comparable to the Liquid Welfare obtained at equilibrium:*

$$\left(1 - \frac{2\alpha}{\gamma(\alpha - 1)}\right) \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] \leq \alpha \cdot \text{REV}(\mathbf{s}) + 2 \cdot \text{LW}(\mathbf{s}) - \frac{1}{\gamma} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i) \right].$$

*Proof.* For the integral part of the  $\gamma$ -boosting deviation, we can now again obtain bounds via the Bayesian Nash equilibrium condition and Claim D.1:

$$\begin{aligned} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \mathbb{E}_{\mathbf{b} \sim \mathbf{s}(\mathbf{v})} [u_i(\mathbf{b})] \right] \\ &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \mathbb{E}_{\mathbf{b}_{-i} \sim \mathbf{s}_{-i}(\mathbf{v}_{-i})} \left[ u_i(\mathbf{b}_2^{[i]}) \right] \right] \\ &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \in G_i(v_i)} \left(1 - \frac{1}{\alpha}\right) \lfloor \gamma \cdot q_{ij}(v_i) \rfloor_h (v_{ij} - \bar{p}_j) \right]. \end{aligned}$$

Similarly, for the fractional part of the  $\gamma$ -boosting deviation we get:

$$\begin{aligned} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \mathbb{E}_{\mathbf{b}_{-i} \sim \mathbf{s}_{-i}(\mathbf{v}_{-i})} \left[ u_i(\mathbf{b}_2^{[i]}) \right] \right] \\ &\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \in G_i(v_i)} \left(1 - \frac{1}{\alpha}\right) \{\gamma \cdot q_{ij}(v_i)\}_h (v_{ij} - \bar{p}_j) \right]. \end{aligned}$$

Together these two deviations give us

$$2 \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] \geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \in G_i(v_i)} \left(1 - \frac{1}{\alpha}\right) \gamma \cdot q_{ij}(v_i) (v_{ij} - \bar{p}_j) \right]. \quad (10)$$

We further estimate the term  $\sum_i \mathbb{E}_{v_i} [\mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \in G_i(v_i)} q_{ij}(v_i) (v_{ij} - \bar{p}_j)]$  on the RHS of Equation (10), which can be rewritten as:

$$\begin{aligned} &\mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \in G_i(v_i)} (v_{ij} - \bar{p}_j) \cdot q_{ij}(v_i) \right] \\ &= \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \notin G_i(v_i)} v_{ij} \cdot q_{ij}(v_i) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \in G_i(v_i)} \bar{p}_j \cdot q_{ij}(v_i) \right] \\ &\geq \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \notin \Gamma_i(v_i)} v_{ij} \cdot q_{ij}(v_i) - \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j \bar{p}_j \cdot q_{ij}(v_i) \right] \\ &\geq \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] - \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \notin \Gamma_i(v_i)} v_{ij} \cdot q_{ij}(v_i) \right] - \alpha \cdot \text{REV}(\mathbf{s}), \end{aligned} \quad (11)$$

where the first inequality holds as  $\sum_{j \notin G_i(v_i)} v_{ij} q_{ij}(v_i) \leq \sum_{j \notin J_i(v_i)} v_{ij} q_{ij}(v_i) + \sum_{j \notin \Gamma_i(v_i)} v_{ij} q_{ij}(v_i)$  and  $v_{ij} < \bar{p}_j$  for every  $j \notin J_i(v_i)$ , and the last inequality holds as  $\mathbb{E}_{\mathbf{v}}[\sum_i q_{ij}(v_i)] = \mathbb{E}_{\mathbf{v}}[\sum_i q_{ij}(\mathbf{v})] \leq 1$ . Our next goal will be to bound the term  $\mathbb{E}_{\mathbf{v}}[\sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \notin \Gamma_i(v_i)} v_{ij} \cdot q_{ij}(v_i)]$  on the RHS of Equation (11). Before that we need to do some preparations. To ease the notations we denote by  $j^{\{\ell\}}$  the  $\ell^{\text{th}}$  share of item  $j$ . We observe that the expected Liquid Welfare at equilibrium can be written as  $\text{LW}(\mathbf{s}) = \sum_i \mathbb{E}_{\mathbf{v}}[\mathbb{E}_{\mathbf{b} \sim \mathbf{s}(\mathbf{v})}[\min\{v_i(x_i), B_i(v_i)\}]]$ . We let  $\text{LW}(\mathbf{s}(\mathbf{v})) = \sum_i \mathbb{E}_{\mathbf{b} \sim \mathbf{s}(\mathbf{v})}[\min v_i(x_i), B_i(v_i)]$ , so that we have  $\text{LW}(\mathbf{s}) = \mathbb{E}_{\mathbf{v}}[\text{LW}(\mathbf{s}(\mathbf{v}))]$ . For any fixed valuation profile  $\mathbf{v}$ ,  $\text{LW}(\mathbf{s}(\mathbf{v}))$  is given by:

$$\begin{aligned}
& \sum_i \Pr_{\mathbf{b} \sim \mathbf{s}(\mathbf{v})} [v_i(x_i) > B_i(v_i)] \cdot B_i(v_i) + \sum_{i,j} \sum_{\ell=1}^h \Pr_{\mathbf{b} \sim \mathbf{s}(\mathbf{v})} [\{v_i(x_i) \leq B_i(v_i)\} \wedge \{i \text{ wins } j^{\{\ell\}}\}] \cdot \frac{v_{ij}}{h} \\
&= \sum_i Q_i(\mathbf{v}) \cdot B_i(v_i) + \sum_{i,j} \frac{v_{ij}}{h} \left( \sum_{\ell=1}^h \Pr [i \text{ wins } j^{\{\ell\}}] - \sum_{\ell=1}^h \Pr [\{v_i(x_i) > B_i(v_i)\} \wedge \{i \text{ wins } j^{\{\ell\}}\}] \right) \\
&= \sum_i Q_i(\mathbf{v}) \cdot B_i(v_i) + \sum_{i,j} v_{ij} \cdot \max \left\{ 0, q_{ij}(\mathbf{v}) - \frac{1}{h} \sum_{\ell=1}^h \Pr [\{v_i(x_i) > B_i(v_i)\} \wedge \{i \text{ wins } j^{\{\ell\}}\}] \right\} \\
&\geq \sum_i Q_i(\mathbf{v}) \cdot B_i(v_i) + \sum_{i,j} v_{ij} \cdot \max \{0, q_{ij}(\mathbf{v}) - \Pr [v_i(x_i) > B_i(v_i)]\} \\
&= \sum_i Q_i(\mathbf{v}) \cdot B_i(v_i) + \sum_{i,j} \max \{0, q_{ij}(\mathbf{v}) - Q_i(\mathbf{v})\} \cdot v_{ij},
\end{aligned}$$

where the second equality holds true as the expression inside the max cannot be negative and  $q_{ij}(\mathbf{v}) = \frac{1}{h} \sum_{\ell=1}^h \Pr [i \text{ wins } j^{\{\ell\}}]$  by definition of  $q_{ij}(\mathbf{v})$ , the first inequality holds since  $\Pr [v_i(x_i) > B_i(v_i)] \geq \Pr [\{v_i(x_i) > B_i(v_i)\} \wedge \{i \text{ wins } j^{\{\ell\}}\}]$ , and the last equality holds by definition of  $Q_i(\mathbf{v})$ . Taking expectation over both sides, we have:

$$\begin{aligned}
\text{LW}(\mathbf{s}) &\geq \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} Q_i(\mathbf{v}) \cdot B_i(v_i) \right] + \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j \max \{0, q_{ij}(\mathbf{v}) - Q_i(\mathbf{v})\} \cdot v_{ij} \right] \\
&\geq \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \bar{\mathcal{I}}_3]}(v_i) Q_i(v_i) \cdot B_i(v_i) \right] + \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \notin \Gamma_i(v_i)} (q_{ij}(v_i) - Q_i(v_i)) \cdot v_{ij} \right] \\
&\geq \frac{1}{2\gamma} \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \bar{\mathcal{I}}_3]}(v_i) B_i(v_i) \right] + \frac{1}{2} \sum_i \mathbb{E}_{v_i} \left[ \mathbf{1}_{[i \in \mathcal{I}]}(v_i) \sum_{j \notin \Gamma_i(v_i)} v_{ij} \cdot q_{ij}(v_i) \right] \\
&= \frac{1}{2\gamma} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i) \right] + \frac{1}{2} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \notin \Gamma_i(v_i)} v_{ij} \cdot q_{ij}(v_i) \right], \tag{12}
\end{aligned}$$

where to obtain the first inequality we restrict the set of players that we sum over; the second inequality follows from the facts that  $q_{ij}(v_i) = \mathbb{E}_{\mathbf{v}_{-i}}[q_{ij}(\mathbf{v})]$  and  $Q_i(v_i) = \mathbb{E}_{\mathbf{v}_{-i}}[Q_i(\mathbf{v})]$ , along with the facts that  $\max\{0, q_{ij}(\mathbf{v}) - Q_i(\mathbf{v})\} \geq 0$  (which we apply for  $j \in \Gamma_i(v_i)$ ) and  $\max\{0, q_{ij}(\mathbf{v}) - Q_i(\mathbf{v})\} \geq q_{ij}(\mathbf{v}) - Q_i(\mathbf{v})$  (which we apply for  $j \notin \Gamma_i(v_i)$ ); the third inequality holds since players  $i \in \bar{\mathcal{I}}_3(\mathbf{v})$  have  $Q_i(v_i) > \frac{1}{2\gamma}$ , while for players  $i \in \mathcal{I}(\mathbf{v}) \subseteq \mathcal{I}_3(\mathbf{v})$  and items  $j \notin \Gamma_i(v_i)$  we have  $Q_i(v_i) \leq \frac{1}{2} \cdot \frac{1}{\gamma} \leq \frac{q_{ij}(v_i)}{2}$ . Now we rearrange terms from Equation (12) to get  $\mathbb{E}_{\mathbf{v}}[\sum_{i \in \mathcal{I}(\mathbf{v})} \sum_{j \notin \Gamma_i(v_i)} v_{ij} q_{ij}(v_i)] \leq 2 \cdot \text{LW}(\mathbf{s}) - \frac{1}{\gamma} \mathbb{E}_{\mathbf{v}}[\sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i)]$ . Combining Equation (10) and Equation (11), we can substitute this upper

bound to get:

$$\begin{aligned}
& 2 \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] \\
& \geq \left( 1 - \frac{1}{\alpha} \right) \gamma \left( \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] - \alpha \cdot \text{REV}(\mathbf{s}) - 2 \cdot \text{LW}(\mathbf{s}) + \frac{1}{\gamma} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i) \right] \right).
\end{aligned}$$

Dividing both sides by  $(1 - \frac{1}{\alpha}) \gamma$  and rearranging terms gives the lemma:

$$\left( 1 - \frac{2\alpha}{\gamma(\alpha - 1)} \right) \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \mathcal{I}(\mathbf{v})} \sum_j v_{ij} \cdot q_{ij}(v_i) \right] \leq \alpha \cdot \text{REV}(\mathbf{s}) + 2 \cdot \text{LW}(\mathbf{s}) - \frac{1}{\gamma} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i) \right].$$

□

Finally, we show that the Liquid Price of Anarchy of any Bayesian Nash equilibrium is bounded.

**Theorem** (Theorem 5). *In simultaneous first-price auctions with  $n$  additive bidders and budgets where every item has  $h$  equal shares (copies), the Liquid Price of Anarchy of Bayesian Nash equilibria is  $O(1 + \frac{n}{h})$  (at most 51.5, when  $h \geq n$ ).*

*Proof.* We combine the bounds from Lemma D.1 and Lemma D.2 and obtain

$$\begin{aligned}
& \alpha \cdot \text{REV}(\mathbf{s}) + 2 \cdot \text{LW}(\mathbf{s}) - \frac{1}{\gamma} \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i) \right] \geq \\
& \left( 1 - \frac{2\alpha}{\gamma(\alpha - 1)} \right) \left( \frac{1}{2} - \frac{1}{2\alpha} \right) \left( \text{OPT} - \alpha \left( 1 + \gamma + \frac{n}{h} \right) \text{REV}(\mathbf{s}) - \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i) \right] \right).
\end{aligned}$$

Since  $\text{LW}(\mathbf{s}) \geq \text{REV}(\mathbf{s})$  we further derive that

$$\begin{aligned}
& \left( \alpha + 2 + \frac{1}{2} \left( 1 - \frac{1}{\alpha} - \frac{2}{\gamma} \right) \alpha \left( 1 + \gamma + \frac{n}{h} \right) \right) \text{LW}(\mathbf{s}) \geq \\
& \frac{1}{2} \left( 1 - \frac{1}{\alpha} - \frac{2}{\gamma} \right) \text{OPT} + \left( \frac{1}{\gamma} - \frac{1}{2} \left( 1 - \frac{1}{\alpha} - \frac{2}{\gamma} \right) \right) \mathbb{E}_{\mathbf{v}} \left[ \sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i) \right].
\end{aligned}$$

As long as the factor in front of  $\mathbb{E}_{\mathbf{v}}[\sum_{i \in \bar{\mathcal{I}}_3(\mathbf{v})} B_i(v_i)]$  is nonnegative, we have  $\text{OPT} \leq O(\frac{n}{h}) \cdot \text{LW}(\mathbf{s})$  for any  $1 \leq h \leq n$  for a particular choice of parameters (e.g.,  $\alpha = 2.26, \gamma = 7.16$ ). In particular, when  $h \geq n$ , we have that the LPOA is at most 51.5. □

## E Tightness Results for Simple Auctions

In this section, we show that Theorem B.2 and Theorem B.1 are essentially tight by giving an explicit game for which the Liquid Price of Anarchy of both first price and second price auctions is arbitrarily close to 2. In fact, agents in our lower bound only have additive valuation functions.

**Theorem.** *There is a simultaneous second price auction game and a simultaneous first price auction game which have a Nash equilibrium  $\mathbf{b}$  such that the Liquid Price of Anarchy is arbitrarily close to 2.*

*Proof.* Consider a simultaneous second price auction game with  $m = 2$  items and  $n = 2$  players, and fix any  $\epsilon > 0$ . Player 1 has a budget of  $B_1 = 10 - \epsilon$ , and a value of 10 for item 1 and a value of 0 for item 2. Player 2 has a budget of  $B_2 = 10$ , and a value of 10 for both items. The player valuations are additive, so their value for a bundle is simply the sum of the values of each item. The optimal solution splits the two items between the two players, giving item 1 to player 1 and item 2 to player 2. The Liquid Welfare of this solution is  $\text{OPT} = \min\{v_1(\{1\}), B_1\} + \min\{v_2(\{2\}), B_2\} = 10 - \epsilon + 10 = 20 - \epsilon$ .

On the other hand, there exists the following Nash equilibrium  $\mathbf{b}$ . Suppose player 1 bids 0 for both items, while player 2 bids  $10 - \frac{\epsilon}{2}$  for item 1 and  $\frac{\epsilon}{2}$  for item 2. For these bids, player 2 wins both items, which results in a Liquid Welfare of  $\text{LW}(\mathbf{b}) = \min\{v_2(\{1, 2\}), B_2\} = 10$ . To see why  $\mathbf{b}$  is a Nash equilibrium, observe that player 1 is not interested in bidding for item 2 since their value is 0 for the item. Moreover, player 1's budget is not high enough to outbid player 2 for item 1, and hence player 1's utility cannot be improved (even though it is 0). Player 2's utility is as high as it can be, since player 2 gets both items and pays 0 for a utility of 20. Hence, the Liquid Price of Anarchy is given by  $\frac{\text{OPT}}{\text{LW}(\mathbf{b})} = \frac{20 - \epsilon}{10}$ .

The same setup shows that the Liquid Price of Anarchy of simultaneous first price auctions is arbitrarily close to 2. In fact, for the same game, essentially the same Nash equilibrium  $\mathbf{b}$  exists for the first price auction setting. The only difference is that player 2 can improve their utility by bidding less for the two items. If we assume that players' bids must be multiples of some fixed value, then player 2 must bid slightly above 0 for item 1 and slightly above  $10 - \epsilon$  for item 2, and now player 2 cannot improve their utility by bidding less, since doing so may result in losing one or more items.  $\square$