Mechanism Design on Discrete Lines and Cycles

ELAD DOKOW, Bar-Ilan University and Carlos III de Madrid
MICHAL FELDMAN, Harvard School of Engineering and Applied Sciences and Hebrew University of Jerusalem
RESHEF MEIR, Hebrew University of Jerusalem
ILAN NEHAMA, Hebrew University of Jerusalem

We study strategyproof (SP) mechanisms for the location of a facility on a discrete graph. We give a full characterization of SP mechanisms on lines and on sufficiently large cycles. Interestingly, the characterization deviates from the one given by Schummer and Vohra [2004] for the continuous case. In particular, it is shown that an SP mechanism on a cycle is close to dictatorial, but all agents can affect the outcome, in contrast to the continuous case. Our characterization is also used to derive a lower bound on the approximation ratio with respect to the social cost that can be achieved by an SP mechanism on certain graphs. Finally, we show how the representation of such graphs as subsets of the binary cube reveals common properties of SP mechanisms and enables one to extend the lower bound to related domains.

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1. INTRODUCTION

In many natural settings that involve social or economic interactions, the involved parties cannot be assumed to cooperate (e.g., to report private information truthfully when asked to). Rather, participating parties act as self-interested agents, each aiming to maximize her own benefit.

A *mechanism* receives the agents' reports as input and outputs an outcome based on the received reports. A mechanism is said to be *strategyproof* (SP) if it is a dominant strategy for every agent to truthfully reveal her private information, independent of the reports of other agents. A generic SP mechanism is the Vickrey-Clarke-Groves mechanism (widely known as the *VCG* mechanism, see e.g. [Groves 1973]). VCG mechanisms are strategyproof for every preference structure, but these mechanisms require monetary transfers, assuming quasi-linear preferences. Unfortunately, in many real world situations, monetary transfers might be infeasible due to technical, ethical, or legal considerations. In such cases, one is restricted to devise mechanisms that do not involve payments. Our primary interest is in characterizing the set of SP mechanisms

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without monetary transfers in certain domains. In some cases, we require additional desired properties.

In addition, we study the social welfare that can be guaranteed by an SP mechanism. It is easy to see that in some domains, the optimal solution (i.e., the solution that maximizes the social welfare) cannot be achieved by an SP mechanism (when payments are not allowed). In these cases, we wish to find the best approximation that can be guaranteed by an SP mechanism. This agenda, often termed *approximate mechanism design without money*, was recently advocated by Procaccia and Tennenholtz [2009].

The *facility location* problem is a very natural setting, in which agents whose ideal locations are located in some metric space report their ideal locations (which is their private information), and a facility location is determined based on these reports, with some objective function in mind. For example, one might wish to locate the facility at a point that will minimize the average distance to an agent. This family of problems arises in many real-life scenarios, such as locating a library or a bus station in a street. Actually, this problem arises also in more virtual scenarios, such as deciding on the salary of a manager during a board meeting of a firm. For simple objective functions, the problem is trivial if truthfulness can be assumed. However, a naïve mechanism may fail if agents act strategically. For example, when considering where to locate a bus station in a street, it is fairly easy to see that a mechanism that picks the average point of all reported locations can be easily manipulated. In particular, an agent can bias the chosen facility location toward her ideal location by misreporting it.

1.1. Previous work

While the most studied setting for Mechanism design without money (MDw/oM) is facility location, SP mechanisms without money have been proposed and analyzed in a wide variety of domains such as matching [Schummer and Vohra 2007; Ashlagi et al. 2010; Dughmi and Ghosh 2010], resource allocation [Guo and Conitzer 2010; Guo et al. 2009; Othman et al. 2010], machine learning [Dekel et al. 2010; Meir et al. 2011], judgment aggregation [Dietrich and List 2007b; Nehring and Puppe 2007], and even auctions [Harrenstein et al. 2009].

The deterministic facility location problem on a *continuous graph* was studied by Schummer and Vohra [2004]. They characterized deterministic SP mechanisms on a line, and extended the characterization to trees. They further showed that on a circular graph, every SP mechanism (that is also onto) must be dictatorial. That is, the location of the facility always coincides with the location of the dictator.

This result was later leveraged by Alon et al. [2010] to derive a lower approximation bound of n (the number of agents) for the social welfare of any SP facility location problem on a continuous cycle. The authors then demonstrated how a constant approximation can be guaranteed with a randomized mechanism.

There have also been many other studies of the facility location problem and its variations, for example where more than one facility should be placed, or where each agent controls several locations [Lu et al. 2009, 2010]. While most of the deterministic and randomized algorithms are not tailored in particular for continuous or discrete graphs (some in fact work on any metric space), all lower bounds that we are aware of rely on the continuity of the domain quite strongly. It is therefore important to clarify the necessity of the continuity assumption.

Meir et al. [2011; 2012] studied characterizations and approximation bounds for the strategyproof classification problem. While this mat at first seem quite unrelated to the problem at hand, a variation of their model (which they call the "realizable" setting) is in fact equivalent to facility location on subgraphs of the binary cube.[1] Meir et al. showed that any deterministic and onto SP mechanism (on some specific domain) *must be dictatorial*, and proved a similar result for randomized mechanisms. Nevertheless, this strong negative result has to be shown to still hold in the realizable setting, and to the best of our knowledge, no non-trivial lower approximation bound for discrete facility location problem has been derived.

Our model is closely related to the literature on *voting with single-peaked preferences*. Strategyproof mechanisms in the general single-peaked model on the line [Black 1986] were characterized by Moulin [1980]. Single-peaked preferences on the binary cube have been considered by Barberà et al. [1991] as well as others. Note that the general single-peaked setting allows for richer preference structures, and thus strategyproofness is a more restrictive requirement in such models. Yet, in our model an agent's location implies not only her peak, but her entire preference structure. This special case of single-peakedness is by now standard in the facility location literature, rationalized by the assumption that agents' dissatisfaction is linear in the distance from the selected location. This assumption often holds in various domains whether the distance is geographical, temporal (i.e. time to wait), or virtual (e.g. number of issues with dissatisfactory outcome).

1.2. Our contribution

In this paper, we follow a setting studied by Schummer and Vohra [2004] and later by Alon et al. [2010] for facility location on graphs. We replace the continuous graph in the original model with a discrete unweighted graph, where the agents and the facility are restricted to vertices only. We feel that in some practical settings, the discrete model may be more appropriate than its continuous counterpart.

We give an exact characterization of deterministic SP (and onto) mechanisms on certain families of discrete graphs, focusing on a discrete line and discrete circular graphs (cycles). For both families, we give an embedding of the graph as a subset of the binary cube, which interestingly allows us to express sufficient and necessary properties of SP mechanisms using natural notions.

For large cycles, our characterization implies that every onto SP mechanism is nearly-dictatorial. While this result resembles that of Schummer and Vohra for continuous cycles, we show that the size of the cycle matters: for small cycles there are anonymous SP mechanisms that are very far from dictatorial. Further, even for large cycles there exist SP mechanisms in which all the agents have some level of influence. As a corollary, we get the first lower approximation bound on discrete facility location and show it to be linear in the number of agents. This result also entails a similar lower bound for realizable SP classification problems, thereby showing that the negative result of Meir et al. [2012] still applies under particular important restrictions.

Due to space constraints, some proofs are deferred to the full version of this paper. [2]

2. PRELIMINARIES

Consider a graph $G = \langle V, E \rangle$ with a set V of vertices and a set E of edges, where edges have no weights nor direction. The vertices $v \in V$ will be also referred to as the *locations*, and the two terms will be used interchangeably. The distance between two vertices $v, v' \in V$, denoted $d(v, v')$, is the length of the minimum-length path between

[1]The original, non-realizable setting of Meir at al. can be interpreted as a generalization of the facility location problem, where agents may be placed in locations that are forbidden for the facility. See more details on this mapping in Section 6.

[2]The full version is available from `http://tinyurl.com/d6493vo`.

v and v', where the length of a path is defined as the number of edges along the path.[3] Note that d is a distance metric. We extend the notion of distance between vertices to distance between sets of vertices, where the distance between two sets of vertices $A, A' \subseteq V$, denoted $d(A, A')$ is defined as $d(A, A') = \min_{v \in A, v' \in A'} d(v, v')$. In this paper we will be especially interested in two types of graphs, namely lines and cycles.

Line graphs. A line graph with $k + 1$ vertices is denoted by $L_k = \{0, 1, \ldots, k\}$. We refer to an increase of the index as a movement in the *right* direction and similarly we refer to a decrease as a movement in the *left* direction. Clearly, in line graphs, every two vertices are connected by a single path. For $v' > v$, we denote by the closed interval $[v, v']$ the set of vertices $\{v, v + 1, \ldots, v' - 1, v'\}$, and by the open interval (v, v') the set of vertices $\{v + 1, \ldots, v' - 1\}$.

Cycle graphs. A cycle graph with k vertices is denoted by $R_k = \{0, 1, \ldots, k - 1\}$. We refer to an increase (respectively, decrease) of the index as a movement in the *clockwise* (resp., *counter-clockwise*) direction. We denote the closed arc between v and v' in the clockwise direction as $[v, v']$ and the open arc in the clockwise direction as (v, v').

In our model, there are n agents that are located on vertices of the graph. Let $N = \{1, \ldots, n\}$ be the set of agents, and $\mathbf{a} = (a_1, \ldots, a_n) \in V^n$ be a *location profile*, where a_j denotes the location of agent j for every $j \in N$. The locations of all the agents excluding agent j is denoted by a_{-j}.

A *facility location mechanism* (or *mechanism* in short) for a graph $G = \langle V, E \rangle$ is a function $f : V^n \to V$, specifying the chosen facility location for every location profile. Note that we assume here that the possible facility locations are exactly the vertices of the graphs. Given an agent j's location $a_j \in V$ and a facility location $x \in V$, agent j's cost is given by $d(a_j, x)$. It is assumed that agents prefer to minimize their cost; that is, agents prefer having the facility located as close to them as possible (and are indifferent between locations of the same distance from them).

Interestingly, facility location mechanisms on the k-dimensional binary cube $\{0, 1\}^k$ (or, more accurately, on certain subsets of it) are closely related to mechanisms in other domains, such as judgment aggregation and classification. We shall elaborate on this important topic in Sections 6 and 7.

2.1. Properties of mechanisms

We start with several definitions of mechanism properties, which are independent of the graph topology. While some of these properties are standard in the literature, we provide their definitions for completeness.

Definition 2.1. A mechanism f is *onto*, if for every $x \in V$ there is $\mathbf{a} \in V^n$ s.t. $f(\mathbf{a}) = x$.

This property is a very basic requirement (sometimes referred to as society sovereignty), and as such we will restrict attention to mechanisms satisfying this condition. As can be seen below it is also the corollary of other natural properties, which we define next.

Definition 2.2. A mechanism f is *unanimous* if for every $x \in V$, $f(x, x, \ldots, x) = x$.

Definition 2.3. A location $y \in V$ is said to Pareto dominate a location $x \in V$ under a given profile if all the agents strictly prefer y over x (i.e. $d(y, a_j) < d(x, a_j)$ for every $j \in N$). A mechanism f is *Pareto* if for all $\mathbf{a} \in V^n$, there is no location $y \in V$ that Pareto dominates $f(\mathbf{a})$ w.r.t. the profile $\mathbf{a}$.

[3]If v, v' are not connected then $d(v, v') = \infty$, however we only consider connected graphs in this paper.

Note that this requirement is slightly weaker than the more common definition of Pareto, requiring that no other location can strictly benefit one of the agents without hurting any other agent. It is easy to verify that Pareto implies unanimity, which in turn implies onto.

An agent j is said to be a *dictator* in f if for every location profile $\mathbf{a} \in V^n$, it holds that $f(\mathbf{a}) = a_j$. We define the following relaxation of the dictatorship notion.

Definition 2.4. An agent j is said to be an m-*dictator* in f, if for every $\mathbf{a} \in V^n$, it holds that $d(f(\mathbf{a}), a_j) \leq m$. A mechanism f is m-dictatorial if there exists an agent j that is an m-dictator in f.

Note that a 0-dictator is essentially a dictator. It is argued that dictatorial mechanisms are "unfair" in the sense that the agent's identity plays a major role in the decision of the facility location. Completely fair mechanisms that ignore agents' identities altogether are said to be anonymous.

Definition 2.5. A mechanism f is *anonymous*, if for every profile $\mathbf{a}$ and every permutation of agents $\pi : N \to N$, it holds that $f(a_1, \dots, a_n) = f(a_{\pi(1)}, \dots, a_{\pi(n)})$.

Our main interest is in *strategyproof* mechanisms, defined as follows.

Definition 2.6. A mechanism f is said to be *strategyproof* (SP), if no agent can strictly benefit by misreporting her location; that is, for every profile $\mathbf{a} \in V^n$, every agent $j \in N$ and every alternative location $a'_j \in V$, it holds that

$$d(a_j, f(a'_j, a_{-j})) \geq d(a_j, f(\mathbf{a})).$$

The following folk lemma gives a necessary condition for a mechanism to be onto and SP. For a proof see e.g. [Barberà and Peleg 1990].

LEMMA 2.7. *Every mechanism that is both onto and SP, is unanimous.*

2.2. The social cost

In addition to the characterization of SP mechanisms, we shall be also interested in the performance of a given mechanism, as evaluated with respect to some well-defined objective function. The social cost function considered in this work is the sum of the distances of the agents' locations from the chosen facility location. That is, given a location profile $\mathbf{a}$ and a facility location x, the social cost is given by $SC(x, \mathbf{a}) = \sum_{j \in N} d(a_j, x)$. When evaluating a mechanism's performance, we use the standard worst-case approximation notion. Formally, given a profile $\mathbf{a}$, let $\text{opt}(\mathbf{a})$ be an optimal facility location (i.e., $\text{opt}(\mathbf{a}) \in \text{argmin}_{x \in V} SC(x, \mathbf{a})$). We say that a mechanism f provides an α-approximation if for every $\mathbf{a} \in V^n$, $SC(f(\mathbf{a}), \mathbf{a}) \leq \alpha \cdot SC(\text{opt}(\mathbf{a}), \mathbf{a})$.

It is well known that in some domains, strategyproofness comes at the expense of the performance. A natural challenge is to identify cases where good performance can be achieved by a strategyproof mechanism.

As mentioned above, we assume that possible agent locations and facility locations coincide. In the more general case, the set of allowed facility locations (i.e., the range of f) may be more restricted than the set of possible agent locations V. For example, a bus stop may need to be located on a main road, while the agents can be located anywhere in the city. Clearly, the necessary conditions for strategyproofness provided in this paper, as well as lower bounds on the approximation ratio, carry over to the general case.

3. SP MECHANISM OVER A DISCRETE LINE

In this section we provide a characterization of onto SP mechanisms on a discrete line.

Given a location $x \in L_k$, agent j's cost is $d(a_j, x) = |a_j - x|$. Given a set of vertices $S \subseteq V$, we denote by $|S|$ the cardinality of S. In the following definitions, a_j, b_j etc. are possible locations in L_k for agent j.

Definition 3.1. A mechanism f on a line is *monotone* (MON) if for every $j \in N$ and every $b_j > a_j$, $f(a_{-j}, b_j) \geq f(a_{-j}, a_j)$.

In other words, monotonicity of a mechanism means that if an agent moves in a certain direction, the outcome cannot move in the other direction as an effect. The following two properties bound the effect of an agent's movement on the outcome of the mechanism.

Definition 3.2. A mechanism f is *m-step independent* (m-SI) if the two following properties hold: **(a)** For every $j \in N, a'_j > a_j$, if $d([a_j, a'_j], f(\mathbf{a})) > m$, then $f(a'_j, a_{-j}) = f(\mathbf{a})$. **(b)** For every $j \in N, a'_j \leq a_j$, if $d([a'_j, a_j], f(a)) > m$, then $f(a'_j, a_{-j}) = f(\mathbf{a})$.

Definition 3.3. A mechanism f is *disjoint independent* (DI) if for every $j \in N, a'_j \in L_k$, if $f(\mathbf{a}) = x \neq x' = f(a'_j, a_{-j})$, then $|A \cap X| \geq 2$, where A is the segment defined by a_j and a'_j (i.e., $A = \left[\min\left(a_j, a'_j\right), \max\left(a_j, a'_j\right)\right]$) and X is the segment defined by x, x'.

Intuitively, m-SI means that a deviation that occurs in an interval sufficiently far from the original outcome does not affect it. The DI property means that an agent can affect the outcome of the mechanism only in a way in which its trajectory intersects the trajectory of the facility in at least two consecutive points.

A mechanism is said to be *strongly m-step independent* (m-SSI) if it is both m-SI and DI. For example, the median mechanism (and in fact any order statistics mechanism) is strongly 0-SSI.

Our first primary result characterizes all the mechanisms that satisfy the requirements of onto and SP on the line.

THEOREM 3.4. *An onto mechanism f on the line is SP if and only if it is MON and 1-SSI.*

In the remainder of this section we sketch the proof of Theorem 3.4. An alternative characterization is given in Section 6, using the notations of the binary cube.

LEMMA 3.5. *Every SP mechanism is monotone.*

Another fact that is used in the proof, is that a monotone mechanism f is Pareto iff it is unanimous.

Notice that the Pareto property (Def. 2.3) has a simpler form in this domain: $f(\mathbf{a}) \in [\min_{j \in N} a_j, \max_{j \in N} a_j]$. The following lemma is the main building block in the proof of Theorem 3.4.

LEMMA 3.6. *Every SP, unanimous mechanism for the line is 1-SI.*

A few remarks are in order. It is not hard to verify (see full version for details) that every 0-SI monotone mechanism on the line is SP. This fact can be seen as a particular case of Nehring and Puppe [2007] theorem. They show that for any subset of the binary cube (see Section 6), 0-SI (called IIA) and monotonicity are sufficient and necessary conditions for being an SP mechanism, for a certain definition of SP that is stronger than ours. The following example shows that 0-SI is *not* a necessary condition: Consider a setting with two players and the following mechanism f on L_2: $f(a_1, a_2) = 2$ if $a_1 = 2$ or $a_2 = 2$, $f(a_1, a_2) = 1$ if $a_1 = a_2 = 1$, and $f(a_1, a_2) = 0$ otherwise. The reader can check that this is an SP, onto and unanimous mechanism; however, it is not 0-SI, since moving from the profile $(0, 1)$ to $(0, 2)$ changes the result from 0 to 2.

We now turn to sketch the proof of the main theorem of this section.

PROOF OF THEOREM 3.4. Suppose f is an onto SP mechanism; then, by Lemmas 2.7 and 3.5, it is also monotone and unanimous, and therefore, by Lemma 3.6, it is 1-SI. Suppose that f does not satisfy 1-SSI; then, there is an agent i that violates DI (i.e., caused the violation). Therefore, there is a profile (a_i, a_{-i}) and deviation a'_i s.t. $f(a_i, a_{-i}) = x \neq x' = f(a'_i, a_{-i})$ but $|A \cap X| \in \{0, 1\}$ (A and X are the segments as used in Def. 3.3). W.l.og. assume $a_i < a'_i$. f satisfies 1-SI and hence $d([a_i, a'_i]) < 2$. It is easy to see that $x = a_i + 1$ and i benefits by this move away from the facility location (since, by monotonicity, the facility moves in the same direction).

We now prove the other direction. Suppose f is an onto, monotone and 1-SSI mechanism. We will show that f is also SP. Suppose some agent j moves from a_j to $a'_j > a_j$ and by that causes the facility to move from $x = f(\mathbf{a})$ to $x' = f(a'_j, a_{-j})$ (the proof for movement to the left is symmetric). By monotonicity, $x' \geq x$. If $x \geq a_j$, then agent j does not benefit from the deviation. Otherwise, $x < a_j$; then, by 1-SI it holds that $x = a_j - 1$ (otherwise the facility will not move). By DI, it must hold that $|[a_j, a'_j] \cap [x, x']| \geq 2$, which means that $x' \geq a_j + 1$. Here again, agent j does not benefit from the deviation. □

3.1. Descriptive and axiomatic characterizations

For continuous lines, the set of SP and onto mechanisms has been characterized as all *generalized median voting schemes* (g.m.v.s) [Border and Jordan 1983; Schummer and Vohra 2004]. This basically means that $f(\mathbf{a})$ is the median selection from some subset of agents. By slightly modifying our definitions above (informally, by replacing the 1-SSI requirement with a 0-SI requirement), we get an alternative, axiomatic characterization that is similar to the one we give for the discrete case. While this definition seems very different from the definition of a g.m.v.s., the two definitions coincide by Theorem 3.4 (as both are equivalent to requiring onto and SP). Similarly, it is possible to give a descriptive characterization in the spirit of g.m.v.s. in the discrete case.

4. SP MECHANISMS ON A DISCRETE CYCLE

Schummer and Vohra [2004] proved that any onto SP mechanism on the continuous cycle must be a dictatorship. However, this is not true for discrete cycles. Clearly, any dictator mechanism is both unanimous and SP, but the converse does not hold.

Consider some cycle R_k of even length k, with any number of agents. The following is an example of an SP mechanism: The cycle is partitioned to $k/2$ pairs of neighboring points. First, the pair in which agent 1 resides is chosen. The location within this pair of points is decided by a majority vote of all other $n - 1$ agents. This is not a dictatorial mechanism, and in fact every agent has some small effect on the outcome in some profiles.

Moreover, if the cycle contains only few vertices, then there exist some completely anonymous mechanisms (i.e., very far from dictatorships) that are SP. See Section 4.4 for detailed examples.

We still want to claim that when k is large enough, then any onto SP mechanism on the cycle R_k is "close" to a dictator. Note that even in the example above, the facility is always next to agent 1, which makes this agent a 1-dictator. The main result of this section shows that this is always the case (see formal statement in Theorem 4.10).

Main theorem. *For sufficiently large cycles, any onto SP mechanism is* 1*-dictatorial.*

In Section 6 we complete the characterization (for even k) by considering the embedding of the cycle in the binary cube.

As a proof outline of the main theorem, we go through the following steps. We first consider the case of two agents, proving that any SP mechanism must be Pareto and deduce that the facility must be located next to one of the agents. It then follows that a 1-dictator must exist. The next step extends this last claim for three agents, using a reduction to the $n = 2$ case. Finally, we extend the result to any number of agents using an inductive argument close to the one used by Schummer and Vohra [2004] (and to similar ideas in [Kalai and Muller 1977; Svensson 1999]).

Before diving into the case of 2 agents, we prove two general lemmas for onto SP mechanisms. Let $\mathbf{a}$, $\mathbf{a}'$ be two profiles that differ only in the location of one agent, w.l.o.g. agent 1, and denote $x = f(\mathbf{a}), x' = f(\mathbf{a}')$. We refer to it as if agent 1 *moves* from a_1 to a'_1.

LEMMA 4.1. *If agent 1 moves* closer *to x along the shorter arc between them, then $x' = x$. I.e., if $|(a, x]| \leq \lfloor k/2 \rfloor$ and $a' \in (a, x]$ then $x' = x$.*

PROOF. W.l.o.g. we assume that a moves clockwise. First assume that she moves one step $a' = a + 1$. Assume toward a contradiction that $y = f(a', a_{-1}) \neq x$. Then either $y \in [a, x)$ (in which case $a \to a'$ is a manipulation), or $y \in (x, a)$. If $|[a, y]| \leq \lfloor k/2 \rfloor$, then since $x \in (a', y)$ it is closer to $a' = a + 1$ than y, meaning that $a' \to a$ is a manipulation.

Therefore, the shorter arc between a and y is $[y, a]$, of length $\leq \lfloor k/2 \rfloor$. Of course, $d(y, a) \geq d(x, a)$ (otherwise $a \to a'$ is a manipulation). However, this means that

$$d(a', x) = d(a, x) - 1 \leq d(a, y) - 1 = (d(a', y) - 1) - 1 < d(a', y),$$

i.e.,that $a' \to a$ is a manipulation.

Therefore, one step toward x does not affect the facility location. The assertion of the lemma is established by induction on the number of steps needed to move from a to a'. □

LEMMA 4.2. *Suppose that agent 1 moves* one step away *from x (along the longer arc between x and a). Let y be the point on the longer arc s.t. $d(a', y) = d(a, x)$. Then either $x' = x$ (i.e., no change); or $d(x', y) \leq 1$.*
(If x is antipodal to a, then any movement is toward x.)

PROOF. W.l.o.g. $a' = a + 1$. If $x' \in (x, y - 2]$ then $a \to a'$ is a manipulation. If $x' \in [y + 2, x)$ then $d(a', x') > d(a', x)$, and thus $a \to a'$ is a manipulation. □

For the case of the cycle we define a specific property of *cycle-Pareto*. For the exact definition see the full version. However, for our proof sketch it is sufficient to note that cycle-Pareto is very similar to Def. 2.3 (in fact for even size cycles the definitions coincide).

We prove below that any SP mechanism for 2 agents on large enough cycles must satisfy cycle-Pareto. In the full version we show that this result extends to any number of agents.

4.1. Two agents on the cycle

LEMMA 4.3. *If $a, b, f(a, b)$ are on the same semi-cycle[4], then $f(a, b)$ is between a and b.*

PROOF. Assume otherwise, w.l.o.g. $a \in (b, f(a, b))$. Then b can manipulate by reporting a, since $f(a, a)$ is closer to b than $f(a, b)$. □

LEMMA 4.4. *Let $k \geq 13$, $n = 2$. If f is SP and onto on R_k, then f is cycle-Pareto.*

[4] I.e., there is a segment of length at most $\frac{k}{2}$ that includes the three points.

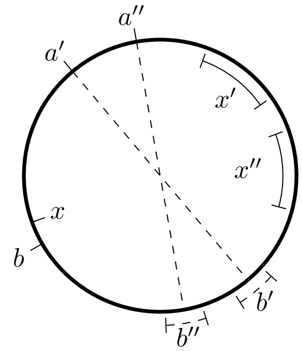

Fig. 1. (Proof Sketch of Lemma 4.4) Agents' locations (a, b, etc.) appear outside the cycle, and facility locations (x, x', etc.) appear inside the cycle.

PROOF SKETCH FOR $k \geq 100$. We start from some profile where $x = f(a, b)$ is violating cycle-Pareto. We use the notation $A \pm B$ as a shorthand for "in the range $[A - B, A + B]$".

(1) We show that cycle-Pareto can only be violated when the facility is at distance (exactly) 2 from some agent (w.l.o.g. $d(b, x) = 2$), and agents are almost antipodal.
(2) We set two locations for agent 1, as $a' = x + 20 = b + 22$ and $a'' = x + 30 = b + 32$. Observe that $f(a', b) = f(a'', b) = x$. For each of the profiles a', b and a'', b, we move agent 2 counterclockwise (away from x), and denote by b' [respectively, b''] the first step s.t. $f(a', b') \neq x$ [resp., $f(a'', b'') \neq x$]. Denote $x' = f(a', b'), x'' = f(a'', b'')$. See Figure 1 for an illustration.
(3) We show that b' must be roughly antipodal to a', as otherwise we get a violating profile that is contradicting (1). Then by Lemma 4.2, x' is a reflection of x along the axis $a' \leftrightarrow b'$. It follows that $|[x', x)| = |[x', b')| + |[b', x)| = 2|[b', x)| \pm 3 = k - 40 \pm 5$.
(4) Following a similar argument, we get that $|[x'', x)| = k - 60 \pm 5$, which means $x' \neq x''$.

Finally, denote $z = f(a'', b')$. By Lemma 4.1, $z = f(a'', b'') = x''$, as agent 2 approaches x''. On the other hand, by the same argument $z = f(a', b') = x'$, as agent 1 approaches x'. Then we get a contradiction as $x' = z = x''$. □

LEMMA 4.5. *Let $k \geq 13$. For all $a, b \in R_k$, $x = f(a, b)$, $d(a, x) \leq 1$ or $d(b, x) \leq 1$.*

PROOF. Assume that there is some violating profile, then w.l.o.g. it is $x = f(a, b)$, where $x = b + 2$, and $a > x$. Also, by cycle-Pareto and Lemma 4.1, $a' = b + 5$ and $a'' = b + 7$ have the same outcome $x = f(a', b) = f(a'', b)$.

However, we show in the (full) proof of Lemma 4.4 that exactly this pair of profiles leads to a contradiction. □

We can now use the results above to prove the main result for two agents.

THEOREM 4.6. *Assume $k \geq 13$, $n = 2$. Let f be an onto SP mechanism on R_k, then f is a 1-dictator.*

PROOF. Take some profile a, b where $x = f(a, b)$, $d(x, b) > 1$. By Lemma 4.5, x must be adjacent a, i.e., $d(a, x) \leq 1$. We will show that agent 1 is a 1-dictator (in the analogous case where $d(x, a) > 1$, agent 2 will be a 1-dictator). Assume, toward a contradiction, that there is some location b' for agent 2 s.t. $d(y, a) > 1$, where $y = f(a, b')$ (and by Lemma 4.5, $d(y, b') \leq 1$). We can gradually move agent 2 from b to b' until the change occurs, and thus, w.l.o.g., $b' = b+1$. By Lemma 4.1, moving agent 2 toward x cannot change the outcome, thus the order on the arc $[x, b']$ must be $x < x+1 < b < b'$.

We must have $d(b,x) \leq d(b,y)$, as otherwise there is a manipulation $b \to b'$. Thus, $d(b,a) - 1 \leq d(b,b') + 1 = 2$, i.e., $d(a,b) \leq 3$. Also, $d(a,b) \geq 1$ since otherwise $d(y,a) = d(y,b) \leq 1$ in contradiction to our assumption. Thus there are three possible cases, and we will show that each leads to a contradiction.

(I) If $d(a,b) = 1$, then since $d(x,b) > 1$ we have $x = a-1$ (in contradiction to Lemma 4.3).

(II) If $d(a,b) = 2$, then $x = a$ (since $x = a-1$ contradicts lemma 4.3). Thus $d(y,b) \geq d(x,b) = d(a,b) = 2$, which means $y = b'+1 = b+2$. This induces a manipulation for agent 1 $a \to b'$ (by unanimity).

(III) If $d(a,b) = 3$, then since $k > 8$, all of the points are on a semi-cycle and thus $x \in \{a, a+1\}$, $y \in \{b', b'-1\}$ (again, by lemma 4.3). However this clearly means that $d(y,b) \leq 1 < d(x,b)$, and there is a manipulation $b \to b'$ for agent 2. □

4.2. Three agents on the cycle

LEMMA 4.7. *Assume $k \geq 13$, $n = 3$. Let f be a unanimous SP mechanism on R_k. Then either f has a 1-dictator, or any pair is a 1-dictator. That is if there are two agents j, j' s.t. $a_j = a_{j'}$, then $d(f(\mathbf{a}), a_j) \leq 1$.*

PROOF. Let f be an SP unanimous rule for $n = 3$ agents. We define a two agent mechanism for every pair $j, j' \in N$ by letting j be a duplicate of j' (For ease of notation we will refer to the agents of $g^{j,j'}$ by agent I and agent II, the third agent by j'', and the original agents by agent 1, agent 2, and agent 3),

$$g^{12}(a,b) = f(a,a,b) \quad ; \quad g^{23}(a,b) = f(b,a,a) \quad ; \quad g^{31}(a,b) = f(a,b,a).$$

Clearly, the mechanism $g^{j,j'}$ is unanimous, since $g^{j,j'}(a,a) = f(a,a,a) = a$ for all j, j'.

We argue that $g^{j,j'}$ is SP. Indeed, otherwise there is a manipulation either for agent II (which is also a manipulation in f, which is a contradiction to SP) or for agent I (say, $a \to a'$). In the latter case we can construct a manipulation in f by iteratively switching agents j, j' from a to a'. Either j or j' strictly gains by this move and thus has a manipulation.

Since $g^{j,j'}$ is a unanimous and SP, by Theorem 4.6 it has a 1-dictator. If the dictator is agent II then j'' is also a 1-dictator of f. Otherwise, suppose that $f(a_j, a_{j'}, a_{j''}) = x$, and $d(x, a_{j''}) > 1$. However it then follows by Lemma 4.1 that $f(x, x, a_{j''}) = x$ as well, which is a contradiction.

If agent I is a 1-dictator of g, then whenever $a_j = a_{j'}$, $d(f(\mathbf{a}), a_j) \leq 1$. □

LEMMA 4.8. *Let f be an SP, unanimous rule for 3 agents on R_k for $k \geq 13$. For all $a, b, c \in R_k$, $x = f(a,b,c)$, $d(a,x) \leq 1$ or $d(b,x) \leq 1$ or $d(c,x) \leq 1$.*

PROOF. By Lemma 4.7, either there is a 1-dictator (in which case we are done), or every pair of agents standing together serve as a 1-dictator.

Let $u_1, u_2, u_3, x = f(u_1, u_2, u_3)$ s.t. x is at least 2 steps from all agents. We have that there is a semi-cycle in which x and two other points are consequent, and thus x must be between them (otherwise the more distant agent of the two has a manipulation, similarly to Lemma 4.3). W.l.o.g. $u_1 + 1 < x < u_2 - 1$ (i.e., ordered that way on an arc). Now suppose that agent 3 moves to u_1 or u_2, whichever closer to her (assume w.l.o.g. u_1 is closer). Then $y = f(u_1, u_2, u_1)$ is close to u_1. We thus have

$$d(u_3, y) \leq d(u_3, u_1) + d(u_1, y) \leq (d(u_3, x) - 2) + 1 < d(u_3, x),$$

i.e., there is a manipulation for agent 3. □

THEOREM 4.9. *Assume $k \geq 22$, $n = 3$. Let f be an onto SP mechanism on R_k, then f is a 1-dictator.*

We give a simpler proof for large cycles. The proof for $k \geq 22$ requires some more details, and appears in the full version of this paper.

PROOF FOR $k \geq 100$. Assume, toward a contradiction, that f has no 1-dictator. By Lemmas 4.8,4.7 we know that $f(\mathbf{a})$ is always close to at least one agent, and if there is a pair in the same place a^* then $d(f(\mathbf{a}), a^*) \leq 1$.

Let $\mathbf{a}$ be a profile where $a_2 = a_1 - 20; a_3 = a_1 + 20$. Thus all three agents and $x = f(\mathbf{a})$ are on the same semi-cycle and x is near a_1 (otherwise there is a manipulation for agent 2 or agent 3 by joining agent 1). Let $a'_2 = a_2 + 8$, then by Lemma 4.1, $f(a_1, a'_2, a_3) = x$. From each profile $\mathbf{a}, \mathbf{a}'$ we move agent 3 toward a_2 (or a'_2) along the *longer arc* between them, until the facility "jumps" to agent 2. This must occur at some point by Lemma 4.7. Denote by b_3 [resp., b'_3] the first point s.t. $f(a_1, a_2, b_3) \neq f(\mathbf{a})$ [resp., $f(a_1, a'_2, b'_3) \neq f(\mathbf{a}')$]. It must hold that b_3 is in the middle of the long arc between a_1, a_2 (plus or minus 1), since otherwise there would be a manipulation $b_3 \to b_3 - 1$ or vice versa. Thus $b_3 \in [a_1 + k/2 - 11, a_1 + k/2 - 9]$. From the same argument, $b'_3 \in [a_1 + k/2 - 7, a_1 + k/2 - 5]$ and therefore $b'_3 > b_3$. Finally, consider the two profiles $z = f(a_1, a'_2, b_3); w = f(a_1, a_2, b_3)$. Since $b_3 < b'_3$, z is next to a_1, and thus $d(z, a'_2) \geq d(a_1, a'_2) - 1 = 11$. On the other hand, w is next to a_2 (by the definition of b_3), thus $d(w, a'_2) \leq d(a_2, a'_2) + 1 = 9 < 11 \leq d(z, a'_2)$, which means that $a'_2 \to a_2$ is a manipulation for agent 2, in contradiction to SP. □

4.3. n agents on the cycle

Finally, we leverage the results of the previous sections to obtain a necessary condition for mechanisms on the discrete cycle for the general case of n agents.

THEOREM 4.10. *Let f be an onto and SP mechanism on R_k, where $k \geq 22$, then f is 1-dictatorial.*

PROOF. We prove the theorem by induction on the number of agents n. Assume the assertion holds for $n - 1$ (we know it holds for $n \leq 3$). Let f be an SP unanimous rule for $n \geq 4$ agents. We define two mechanisms for $n - 1$ agents:

$$g(a_{-1}) = f(a_1 = a_2, a_{-1}) \quad ; \quad h(a_{-3}) = f(a_3 = a_4, a_{-3}).$$

Now, similarly to the proof of Lemma 4.7, both g and h are unanimous and SP and therefore both are 1-dictator mechanisms by the induction hypothesis. If we have that some $j \neq 2$ is the dictator of g we are done (since then j is a 1-dictator of f), and similarly for any $j' \neq 4$ in h.

Assume toward a contradiction that agents 2 and 4 are the 1-dictators of g and h, respectively. Then take any profile where $a_1 = a_2$, $a_3 = a_4$ and $d(a_2, a_4) > 2$ (this is always possible for $k > 4$). We then have that $x = f(\mathbf{a})$ satisfies both $d(x, a_2) \leq 1$ and $d(x, a_4) \leq 1$, i.e.,$d(a_2, a_4) \leq 2$ in contradiction to the way we defined the profile. □

4.4. Small cycles

A natural question is the critical size of a cycle, for which there still exist SP mechanisms that are not 1-dictatorial. The proofs above show that the critical size for $n = 2$ is at most 12, and for $n \geq 3$ it is at most 21. We want to know whether these bounds are tight.

PROPOSITION 4.11. *There are onto and anonymous SP mechanisms for two agents on R_k, for all $k \leq 12$.*

PROPOSITION 4.12. *There are onto and anonymous SP mechanisms for three agents on R_k, for all $k \leq 14$ and $k = 16$.*

For $k \leq 7$, the following "median-like" mechanism will work for $n = 3$: let $(a_3, a_1]$ be the longest clockwise arc between agents, then $f(\mathbf{a}) = a_2$. Break ties clockwise, if needed. For two agents we simply fix the location of one virtual agent.

For higher values of k the "median" mechanism is no longer SP, but we have been able to construct anonymous SP mechanisms using a computer search for all the specified values. A tabular description of these mechanisms is available online [A].

Proposition 4.11 settles the question of the maximal size for which non-1-dictatorial mechanisms for two agents exist. For three agents, we close the gap between Proposition 4.12 and Theorem 4.9 by performing an exhaustive search on all mechanisms with three agents for $k \in \{15, 17, 18, 19, 20, 21\}$.[5] Indeed, it turns out that every mechanism in this range must be 1-dictatorial. Thus we have a full characterization of the cycle sizes for which non-1-dictatorial SP mechanisms exist.

As a direct corollary of Proposition 4.12 we get the following result (by adding any number of agents, which the mechanism ignores).

PROPOSITION 4.13. *For all $n \geq 3$, $k \leq 14$ and $k = 16$, there are onto SP mechanisms for n agents on R_k that treat the first 3 agents symmetrically. In particular, these mechanisms are not 1-dictatorial.*

Note however that the resulting mechanism is not an anonymous one.

5. IMPLICATIONS OF THE MAIN THEOREM

In this section we cover some strong implications of the result that any SP mechanism on a large cycle must be almost-dictatorial.

5.1. Cyclic graphs

The first implication is that this result extends to a much larger family of graphs. A natural conjecture is that any SP (and onto) mechanism on any graph containing a cycle that matches the conditions of the theorem, must be 1-dictatorial on a subdomain. However, we need to be careful. In the continuous case studied by Schummer and Vohra [2004], *any cyclic graph* contains a continuous cycle and thus their negative result automatically applies.

In the discrete case, this is only guaranteed to be true if we add edges *outside* the cycle. We define a *minimal cycle* as a cycle that is not cut by any string. Equivalently, the shortest path between every two vertices on the cycle is going through the edges of the cycles. The extension of our main theorem is as follows.

COROLLARY 5.1. *Let $G = (V, E)$ be graph that contains some* minimal *cycle $R \subseteq V$ that is sufficiently large (according to Table I). Then any SP onto mechanism on G has a "cycle 1-dictator" $i \in N$. That is, if all agents lie on R then $d(f(\mathbf{a}), a_i) \leq 1$.*

PROOF SKETCH. Let f be an onto SP mechanism on G. We argue that whenever $\mathbf{a} \in (R)^n$ (i.e. all agents are on the cycle R), then $f(\mathbf{a}) \in R$ as well. Assume otherwise, then by iteratively and gradually moving all agents to closest point on the cycle, the facility eventually moves to the cycle. Since moving the facility to a point between the agent and the original location is a manipulation, the facility must "jump" to a distant location (from the agent) on the cycle. Moreover, there must be at least two

[5] While the number of mechanisms for 3 agents and bounded k is finite, the size of the search space is huge ($k^{\Theta(k^n)}$). Thus any naïve search would be infeasible. However, by using the lemmas from Section 4 we can significantly reduce the search space so that the search completes in several minutes.

such distinct locations, induced by agents moving clockwise (a set $S \subseteq N$) and agents moving counterclockwise ($T \subseteq N$). Moving from one such profile to the other requires a single step by two different agents, one from each set. One of these steps must move the facility from a point closer to S to a point closer to T, and therefore it is a manipulation for one of the agents. $\square$

If we take a large cycle and add internal edges (so that it is no longer minimal), then there may be non-dictatorial mechanisms that are SP. As a simple example, the main theorem applies on R_{14} with $n = 2$. However if we add the edge $(0, 7)$, this forms two cycles of length 8. The following mechanism is SP and onto: if the two agents are on different cycles, then $f(\mathbf{a}) = 0$. If they are on the same cycle, then we apply a "median-like" mechanism for R_8 (fully described in the full version and proved to be SP), where the point 0 serves as the dummy agent for both cycles.

5.2. The social cost

Dictatorial mechanisms typically have poor performance in terms of social welfare (or cost). While for a low number of agents a dictatorial facility location mechanism may still be reasonable in terms of the the cost (in fact, for $n = 2$ the dictator mechanism is optimal w.r.t. the social cost), for more agents the main theorem provides us with a lower bound that linearly increases with the number of agents (the corollary still holds for lower values of k, as appear on Table I).

COROLLARY 5.2. *Every SP mechanism on R_k for $k \geq 22$ has an approximation ratio of at least $\frac{3}{5}n$. The ratio converges to $n - 1$ as k tends to infinity.*

PROOF. If the mechanism is not unanimous, it has an infinite approximation ratio. Otherwise it is a 1-dictator, w.l.o.g. agent n is the 1-dictator. Let $a_1 = a_2 = \ldots = a_{n-1} = k$, and $a_n = \lfloor \frac{k}{2} \rfloor$. Clearly, the optimal location is opt$=a_1$, and the optimal total distance from all agents is $\lfloor \frac{k}{2} \rfloor$. However, $f(\mathbf{a}) = \lfloor \frac{k}{2} \rfloor \pm 1$, and the total distance from the agents is at least $(n-1)\left(\lfloor \frac{k}{2} \rfloor - 1\right)$ (in fact $\min\{(n-1)\lfloor \frac{k}{2} \rfloor, n\left(\lfloor \frac{k}{2} \rfloor - 1\right)\}$). Thus the approximation ratio for $k \geq 22$ is

$$\frac{SC(f(\mathbf{a}))}{SC(\mathbf{opt}(\mathbf{a}))} \geq (n-1)\frac{\lfloor \frac{k}{2} \rfloor - 1}{\lfloor \frac{k}{2} \rfloor} \geq \frac{2n}{3}\frac{9}{10} = \frac{3}{5}n,$$

thereby proving the assertion. $\square$

5.3. The continuous case

As we mentioned in Section 3.1 for continuous lines, one can repeat the steps of our proof, with some adjustments, when the underlying graph is continuous. This results in an alternative proof that every onto SP mechanism on a continuous cycle is *dictatorial*. In fact, some steps of our proof are greatly simplified in the continuous case, leaving us with a relatively short and intuitive proof for Theorem 2 in [Schummer and Vohra 2004] (p.22). We do not include the full details here.

6. THE BINARY CUBE

A binary cube of dimension k is denoted by C_k. The set of vertices in C_k is the set of binary vectors of size k. Two vertices $v, v' \in C_k$ are *connected* if their hamming distance (i.e., the number of coordinates in which they differ) is 1. Given a vertex v, we denote by $v[i] \in \{0, 1\}$ the i'th coordinate of v. Therefore, $d(v, v') = |\{i : v[i] \neq v'[i]\}|$.

We next define several properties of mechanisms for the binary cube C_k. These definitions will serve several purposes: first, by considering a natural embedding of R_k in C_k we can provide a full characterization of SP mechanisms on the cycle in terms of the cube dimensions. Interestingly, we give an alternative characterization for mechanisms on the line using the same properties. Second, we consider some implications of our results on other domains, which correspond to the binary cube.

Suppose that V is some subset of C_k. Since every location can be thought of as having k coordinates (or attributes), the cube structure calls for some new definitions.

Definition 6.1. A mechanism f is *Cube-monotone*, if changing coordinate i of an agent can only change coordinate i in the same direction. That is, if $a_j[i] \neq a'_j[i]$ and $f(\mathbf{a})[i] \neq f(a_{-j}, a'_j)[i]$, then $f(\mathbf{a})[i] = a_j[i]$.

Another property often considered in a multi-attribute setting is *independence in irrelevant attributes*. This means that coordinate i of the facility is only determined by the values of coordinate i of the agents' locations. While this property seems unnatural in the general case of aggregating agent location on a subset of the cube, it is reasonable in a lot of related aggregation problems. For example, in preference aggregation the IIA property means pair-wise aggregation and is accepted as a desired property. As was shown by Dietrich and List [2007a] preference aggregation can be seen as aggregation on the cube. We relax this notion as follows.

Definition 6.2. A mechanism f is *m-independent of irrelevant attributes* (m-IIA) if $f(\mathbf{a})[i]$ is determined by coordinates $i-m, \ldots, i+m$ of the voters in $\mathbf{a}$.

Note that the m-IIA property depends on coordinates order, and is not preserved under a permutation of coordinates' names. 0-IIA is just IIA. The following property is also quite natural.

Definition 6.3. A mechanism f is *independent of disjoint attributes* (IDA), if the coordinates changed by the agent and the coordinates changed in the facility (if it moved) always intersect. Formally, if a_j, a'_j differ by coordinates $S \subseteq K$, and $f(a_j, a_{-j}), f(a'_j, a_{-j})$ differ by coordinates $T \subseteq K$, then either $T = \emptyset$ (i.e. no change in outcome) or $S \cap T \neq \emptyset$.

A similar property was suggested by Dietrich [2007] as *independence in irrelevant information* (in our case a coordinate is relevant to its neighborhood, and irrelevant to all other coordinates).

Definition 6.4. We say that a mechanism f is *Cube-Pareto*, if whenever all the agents agree on the same coordinate (vote the same), then this is the aggregated coordinate as well.

6.1. Embedding the line in the binary cube

We give a natural embedding of L_k in C_k. Map every $x \in L_k = \{0, 1, \ldots, k\}$ to a vector $\varphi(x) \in \{0,1\}^k$, whose first x entries are 1. Thus $\varphi(x)[i] = 1$ iff $i \leq x$. It is easy to verify that φ is distance-preserving, i.e., that $d(\varphi(x), \varphi(x')) = |x - x'| = d(x, x')$.

Every mechanism f on L_k induces a mechanism f_φ on the embedded space $\varphi(L_k) \subseteq C_k$. The following correspondences of properties follow directly from distance preserving of the mapping φ.

LEMMA 6.5. *Let f be a mechanism on L_k.*

(*1*) *f is monotone iff f_φ is Cube-monotone.*
(*2*) *f is m-SI iff f_φ is m-IIA.*
(*3*) *f is DI iff f_φ is IDA.*
(*4*) *f is Pareto iff f_φ is Cube-Pareto.*

By Lemma 6.5 we get the following theorem, which is equivalent to Theorem 3.4. It demonstrates that the same set of properties (defined w.r.t. the binary cube) is useful for characterizing mechanisms for both lines and cycles.

THEOREM 6.6. *An onto mechanism on $\varphi(L_k)$ (The line embedded in C_k) is SP if and only if it is* 1*-IIA, Cube-monotone, and IDA.*

6.2. Full characterization of SP mechanism on the cycle

Every cycle of even length can be thought of as "two lines attached in their ends". Indeed, R_{2k} can be embedded in the binary cube C_k in a very similar way to the embedding of the line. This is by mapping the first k points on the cycle (setting order and orientation on the cycle. We later show that these can be arbitrarily chosen) to vectors of the form $0^{k_1}1^{k_2}$ (as with L_k), and the remaining k points to vectors of the form $1^{k_1}0^{k_2}$. In particular, $\varphi(0) = 0^k$, and $\varphi(k) = 1^k$. As with L_k, it is not hard to verify that our mapping preserves distances, as

$$d(\varphi(x), \varphi(x')) = d(x, x') = |x - x'| \pmod{2k}.$$

We can now turn to completing the characterization of SP mechanisms on the cycle, extending Theorem 4.10.

THEOREM 6.7. *Let $2k \geq 18$ (or $2k \geq 14$ for $n = 2$). An onto mechanism on the cycle R_{2k} is SP if and only if it is* 1*-dictatorial, Cube-monotone, and IDA.*

Notice that all these properties do not depend on the choice of embedding (from the $2k$ ways to choose starting point and direction). The coordinates can be seen as a geometric property telling us where is the facility w.r.t. the agents and their antipodal points, hence the properties are independent of the embedding. For instance, Cube-Pareto can be interpreted as - "for any semi-cycle s.t. all the agents are in this semi-cycle, the facility should lie in it as well".

6.3. Beyond facility location - Implications to other domains

As mentioned in the introduction, there is a mapping between facility location mechanisms in our model, and binary classification mechanisms operating on realizable datasets. A natural question is whether we can derive characterization and approximation results that will apply to the classification setting, and in particular to natural concept classes that are in use in the machine learning literature.

We exemplify such a derivation for *linear classifiers* in $\mathbb{R}^d$, which is one of the most prominent concept classes used in the machine learning framework. The connections between Facility location, Classification, and other mechanism design problems are studied in detail by Meir et al. [2012].

Linear Classifiers. A linear classifier is composed of a unit vector $\mathbf{w} \in \mathbb{R}^d$ and a scalar u. It classifies every data point $\mathbf{x} \in \mathbb{R}^d$, to $\{+,-\}$, according to $sign(\langle \mathbf{w}, \mathbf{x} \rangle - u)$ (see Figure 2). A linear classifier in $\mathbb{R}^1$ is just a scalar u and a direction $w \in \{-,+\}$.

In a *binary classification problem*, we are given a set of labeled data points S, and are required to return a classifier $c = \langle \mathbf{w}, u \rangle$. The quality of the classifier c is measured according to the number of errors that c makes on S, which we want to minimize. Formally, $err(c, S) = |\{\langle \mathbf{x}, y \rangle \in S : c(x) \neq y\}|$. The classifier with the lowest number of errors on S is denoted by $opt(S)$.

A *classification mechanism* is a function M mapping every labeled dataset to a classifier. Typically, it is assumed that labels are acquired via some objective process (which may be noisy), in which case classification is just an optimization problem. The approx-

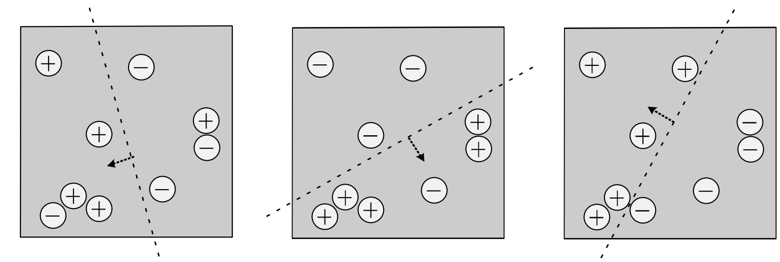

Fig. 2. A classification dataset in $\mathbb{R}^2$. The k data points of all three agents are identical, but the labels, i.e., their types, are different. The best linear classifier with respect to each agent is also shown (the arrow marks the positive half-space of the separator). Only the rightmost dataset is realizable.

imation ratio of a classification mechanism is the maximal ratio $\frac{err(M(S),S)}{err(opt(S),S)}$, taken over all possible datasets S.

However, in certain situations labels are reported by several self-interested agents, which may have different opinions on the appropriate label for each data point. In such cases, every classification mechanism induces a game, in which agents may lie in order to bias the outcome classifier closer to their own view, similarly to false reports in the facility location problem. Therefore, we are interested in *strategyproof mechanisms* (i.e. mechanisms in which no agent can gain by reporting false labels). Since such mechanisms are typically suboptimal, we study the best approximation ratio that such a (deterministic) mechanism can guarantee. For example, it is easy to see that labeling the entire dataset according to the opinion of a single arbitrary agent (a dictator) is SP, but does not guarantee any finite approximation ratio. We next show that this holds for any SP mechanism.

Reducing Linear classification to Facility location. Suppose that the dataset contains k samples in generic state on the real line $\mathbb{R}^1$. There are exactly $2k$ ways to classify the points, as all negative labels must be on one side of the classifier and likewise for the positive labels. Requiring *realizability* simply means that the opinion of each agent on the correct classification can be described by one of these $2k$ classifiers.

The cycle R_{2k} (embedded in C_k) contains the $2k$ vectors $0^{k_1}1^{k_2}$ and $1^{k_1}0^{k_2}$, where $k_1 + k_2 = k$. Note that these are exactly all the possibilities to classify the dataset with a 1-dimensional linear classifier. We can map the classification problem to a facility location problem by mapping each data point to a particular dimension of the cube C_k. The opinion of each expert can be then naturally mapped to a vertex of C_k. Moreover, due to realizability, this vertex lies on R_{2k}. We get that any SP mechanism for classification is in fact an SP facility location mechanism on the cycle R_{2k} (the facility is the vertex representing the selected classifier). The following corollary then follows from our main theorem (through Corollary 5.2).

COROLLARY 6.8. *Every SP classification mechanism for linear classifiers in $\mathbb{R}^d$ (for any $d \geq 1$) has an approximation ratio of $\Omega(n)$,* even when datasets are individually realizable.

This result is stronger than the results of Meir et al. [2012; 2010], which only hold if non-realizable datasets are allowed. For a more detailed discussion on SP linear classifiers (including a simple reduction from $\mathbb{R}^d$ to $\mathbb{R}^1$), see [Meir et al. 2010].

Table I. SP mechanisms on R_k

	$k \leq 12$	$k \in \{13, 14, 16\}$	$k \in \{15, 17, 18, 19, 20, 21\}$	$k \geq 22$
$n = 2$	**A** (Prop.4.11)	**D** (Th. 4.6)		
$n = 3$	**A** (Prop.4.12)		**D** (Search)	**D** (Th. 4.9)
$n > 3$	**ND** ($\Downarrow$ by Prop. 4.13)		**D** ($\Downarrow$ by Th. 4.10)	

Summary of results for SP mechanisms on R_k, with n agents. **D** means that every SP mechanism is 1-dictatorial. **ND** means there exists an SP non-1-dictatorial mechanism. **A** means there exists an SP anonymous mechanism.

7. DISCUSSION

Our two primary results are the complete characterization of onto SP mechanisms on the discrete line, and proving that on sufficiently large cycles, every onto SP mechanism must be close to a dictatorship. We believe that the outline of our proofs demonstrates general ideas and can perhaps assist in the construction of similar characterization results in other domains.

For cycles, we further studied how the dictatorial limitation is affected by the cycle size and the number of agents (see Table I), specifying exact size of the cycle required to allow mechanisms that are non-1-dictatorial. Interestingly, this effect of the cycle size is not completely monotone, presumably due to the additional effect of parity. Finally, we completed the characterization of onto SP mechanisms for even-sized cycles in these cases where the dictatorial condition holds.

Future directions. We conjecture that the characterization of line mechanisms can be extended to trees, similarly to the result of Schummer and Vohra [2004]. Note that the properties used for the characterization should be suitably extended first. Other directions include the characterization of SP mechanisms (both deterministic and randomized) and the study of their approximation bounds for a variety of topologies and optimization criteria. For example, one can think of extensions of this work to weighted line and cycle graphs (or weighted graphs in general), e.g. the case in which the locations on the line are $1, 2, 4, \ldots, 2^k$. Our methods in this work relied on a certain 'symmetry' in the underlying distances. Thus studying such graphs with non-uniform distances may yield a better understanding of SP mechanisms.

An intriguing open question is whether randomized SP mechanisms (on a particular structure) must also be close to a random dictatorship, as we already know to be true in other domains [Gibbard 1977; Meir et al. 2011].

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