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Efficient graph topologies in network routing games ☆

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Abstract

A topology is *efficient* for network games if, for any game over it, every Nash equilibrium is socially optimal. It is well known that many topologies are not efficient for network games. We characterize *efficient* topologies in network games with a finite set of players, each wishing to transmit an atomic unit of unsplittable flow.

We distinguish between two classes of atomic network routing games. In *network congestion games* a player's cost is the sum of the costs of the edges it traverses, while in *bottleneck routing games*, it is its maximum edge cost. In both classes, the social cost is the *maximum* cost among the players' costs.

We show that for symmetric network congestion games the efficient topologies are Extension Parallel networks, while for symmetric bottleneck routing games the efficient topologies are Series Parallel networks. In the *asymmetric* case the efficient topologies include only trees with parallel edges.

JEL classification: C62; C72

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1. Introduction

A very natural setting of routing includes multiple players that each would like to route traffic between some source and destination nodes in a given network, and chooses a path connecting the two nodes, possibly among multiple paths. Each edge in the network has a non-decreasing cost function (e.g., delay) that depends on the amount of flow routed through that edge, and each player would like to route its traffic as to minimize its own cost. This general setting is referred to as a *network routing game*, and is common in various network settings, such as communication, transportation and computer networks. A Nash equilibrium in these games is a collection of paths (one per player) where no single player can reduce its cost by deviating unilaterally to a different path.

In such settings, Nash equilibria may be inefficient. A dazzling example of this phenomenon is the Braess's paradox (Braess, 1968), which refers to situations in which increasing the cost of an edge may decrease the

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travel time in the network. This phenomenon has been demonstrated in various network settings, including transportation networks (Steinberg and Zangwill, 1983), communication and computer systems (Roughgarden, 2002; Cohen and Jeffries, 1997), and electrical circuits (Cohen and Horowitz, 1991). Motivated by these qualitative observations, there has been recently a lot of interest among researchers in the field of algorithmic game theory in *quantitatively* bounding the inefficiency that is incurred due to the different objective functions of the different players. In order to bound this inefficiency, one needs to define a social objective function, then to quantify how far the outcome of a Nash equilibrium is from the social optimum. Papadimitriou (2001) has coined the term "Price of Anarchy", which is the ratio between the worst Nash equilibrium and the social optimum. This measure has been widely used in a variety of network routing models (Roughgarden, 2002, 2004, 2005; Awerbuch et al., 2005; Christodoulou and Koutsoupias, 2005; Banner and Orda, 2006).

In this work, we take a different approach. Rather than quantifying the inefficiency of Nash equilibria, we provide a tight characterization of network *topologies* in which every Nash equilibrium is efficient. We concentrate on topological properties of graphs, and say that a graph topology is *efficient* if, for any assignment of non-decreasing cost functions on the edges, the resulting game has the property that the cost of any Nash equilibrium coincides with the social optimum cost. One can view this separation between the graph topology and the assignment of costs as a separation between the underlying infrastructure and the characteristics of the edges (such as their delay). While one expects the infrastructure to be stable over long period of times, the edges' characteristics, and thus the costs the players observe, can be easily modified over short time periods. Once an efficient network topology is established, it is guaranteed that, no matter how the cost functions on the edges would evolve, every Nash equilibrium of the game will achieve the optimal outcome. Thus, efficient topologies should be desired by network designers, who wish to guarantee the efficiency in their network despite the fact they do not control the actions of the individual users.

Milchtaich (2006a) claimed that in contrast to Pareto efficiency of the equilibrium, which strongly depends on the network topology (as he demonstrated), the price of anarchy is virtually independent of the network topology. This claim is based on the work of Roughgarden (2002), who showed that under the standard aggregative (or equivalently average) social cost function and additive individual cost functions, the worst price of anarchy occurs already in the simplest (non-tree) network, composed of two parallel edges. Our results demonstrate that whether network topology affects the price of anarchy or not depends on the definition of the social cost function. In our case, where the social cost is the *maximum* over the individuals' costs, the network topology strongly affects the price of anarchy.

In particular, we analyze two classes of atomic network routing games, namely *network congestion games* and *bottleneck routing games*. In both classes each of several identical players chooses a single path on which to transmit a unit of traffic, and the social cost is the *maximum* cost among the individual players' costs. This social cost function is different than the more standard social cost which is the sum, or equivalently average, of the players' costs. It resembles the max-min fairness criterion, where the goal of the network operator is to maximize the fairness among the users by minimizing the cost of the "poorest" user. This social cost (known as the *makespan*) has been widely studied in the context of job scheduling games (Koutsoupias and Papadimitriou, 1999; Czumaj and Vöcking, 2002), and has recently been considered also within the framework of network routing games (Roughgarden, 2004; Christodoulou and Koutsoupias, 2005; Busch and Magdon-Ismail, 2007; Banner and Orda, 2006).

The two games we study differ in the cost of the individual players. In the former class, network congestion games, a player's cost is the *aggregate cost*, i.e., the sum of the costs of the edges in its path. This cost function is applicable when the cost function resembles, for example, delay, where the user's total delay is the sum of the delays on the various links it traverses. In the second class, bottleneck routing games, a player's cost is the *maximum cost*, i.e., the maximum edge cost among the edges in its path. This cost function emerges in various scenarios, such as bandwidth allocation problems, where the user's bandwidth is limited by the most loaded link in its path, and sensor networks, where the battery lifetime is limited by the lifetime of the battery with the shortest lifetime in the path.

1.1. Our contribution

We show that the price of anarchy may be very much related to the topology of the underlying network, depending on the criterion for social efficiency. For the maximum social cost the network topology greatly affects the price of anarchy. This is in contrast to the average social cost (studied by other researchers), where the price of anarchy is independent of the network topology.

In particular, we identify interesting relations between the functional form of the individual and the social costs and the network topology. We show that the interesting topological characterizations arise in the symmetric (single-commodity) case, where all the players share the same source and destination nodes. In network congestion games, the efficient topologies are exactly extension-parallel networks, while for bottleneck routing games, the efficient topologies are exactly series-parallel networks. Our analysis shows an even stronger property in such topologies. It shows that each player can guarantee, by its own action solely, that its cost would be no greater than the social optimum cost. That is, every player, for any joint action (i.e., selection of paths) of the other players, has a best response whose cost is at most the social optimum cost (recall that the social cost is the maximum among the players' costs).

For the multi-commodity case (i.e., multiple source and/or destinations), we show that the efficient topologies are very limited. In network congestion games, the only efficient topologies are either trees or two nodes with parallel edges. In bottleneck routing games, the only efficient topologies are trees with parallel edges.

1.2. Related work

Topological characterizations for symmetric network games (i.e., where all players share the same source and destination nodes) have been recently provided for various equilibrium properties, including (Nash and strong) equilibrium existence (Milchtaich, 2006b; Epstein et al., 2007; Holzman and Law-Yone, 1997, 2003), equilibrium uniqueness (Milchtaich, 2005) and equilibrium efficiency (Roughgarden, 2002; Milchtaich, 2006a). Holzman and Law-Yone (2003) provided a characterization of directed networks that admit a strong equilibrium in symmetric congestion routing games for any non-decreasing cost functions on the edges. Interestingly, they showed that a network is guaranteed to admit a strong equilibrium if and only if it is an extension-parallel network.

Milchtaich (2006a) studied a routing model with a continuum of players, each with an almost negligible effect on the others (in contrast to our model, which assumes a finite set of users, each having a non-negligible effect on the others). He showed that networks in which every Nash equilibrium is weakly Pareto efficient are exactly extension parallel graphs, and networks in which the Braess's paradox cannot occur are exactly series parallel graphs.

Most of the work on the price of anarchy in network congestion games assumes that the social cost is the sum of the players' costs. The non-atomic case of this setting was studied by Roughgarden and Tardos (2004) while the atomic case was studied by Christodoulou and Koutsoupias (2005) and Awerbuch et al. (2005). The maximum social cost was studied by Roughgarden (2004) (in the non-atomic setting), who showed that the price of anarchy is $k-1$ (where k is the number of vertices in the network). Christodoulou and Koutsoupias (2005) provided price of anarchy bounds for the atomic case. They showed that for linear edge cost functions the price of anarchy is 2.5 for symmetric games and $\Theta(\sqrt{n})$ for asymmetric games,[1] where n is the number of players. They also showed that for polynomials of degree d cost functions the price of anarchy is $d^{\Theta(d)}$ for symmetric games and for asymmetric games they showed a lower bound of $\Omega(n^{d/(d+1)})$ and an upper bound of $O(n)$.

The bottleneck routing model has been first studied by Banner and Orda (2006), who considered atomic bottleneck routing games with weighted user demands. They considered the unsplittable flow model, where each user routes its traffic along a single path, and the splittable flow model, where each user can split its traffic among multiple paths. They established the existence of pure Nash equilibrium in these models and also studied the price of anarchy under the maximum social cost. They showed that in both the splittable and the unsplittable flow models the price of anarchy is unbounded. For the unsplittable flow model they showed that the price of anarchy is unbounded even in a simple network of two nodes connected by two parallel edges. Busch and Magdon-Ismail (2007) provided some price of anarchy results for undirected graphs. They considered unsplittable atomic bottleneck routing games, where the cost of each edge equals its congestion, which is the number of players that selected a path that uses this edge. They showed price of anarchy bounds in terms of the topological properties of the network. Specifically, they showed that the price of anarchy is $O(\ell + \log k)$, where ℓ is the longest path of any player. Furthermore, they showed that the price of anarchy is $O(h^2 + \log^2 k)$ and they proved a lower bound of $h-1$, where h is the length of the longest cycle in the network.

[1] In what follows, we use the Big-Theta, Big-O and Big-Omega notations, which are used to describe the asymptotic behavior of functions. We denote $g(n) = O(f(n))$ if $\exists c > 0$ such that $g(n) \leqslant c \cdot f(n)\ \forall n \geqslant 1$. We denote $g(n) = \Omega(f(n))$ if $\exists c > 0$ such that $g(n) \geqslant c \cdot f(n)\ \forall n \geqslant 1$. Finally, we denote $g(n) = \Theta(f(n))$ if both $g(n) = O(f(n))$ and $g(n) = \Omega(f(n))$, i.e., $\exists c_1, c_2 > 0$ such that $c_1 \cdot f(n) \leqslant g(n) \leqslant c_2 \cdot f(n)\ \forall n \geqslant 1$.

2. Model

2.1. Game theoretic preliminaries

A game is a tuple $\Lambda = (N, \{\Sigma_i\}_{i \in N}, \{c_i(\cdot)\}_{i \in N})$, where $N = \{1, \ldots, n\}$ is a finite set of players, Σ_i is a set of actions (also called strategies) for player i, and $c_i(\cdot)$ is a cost function of player i, mapping from $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ to the reals. We denote by $S = (S_1, \ldots, S_n) \in \Sigma$ the joint action taken by the players, and by $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ the joint action taken by players other than player i. (We denote $S = (S_i, S_{-i})$.) Finally, we denote by S_T the joint action taken by the players in T.

Pure Nash Equilibrium: A joint action $S \in \Sigma$ is a *pure Nash Equilibrium* if no player $i \in N$ can benefit from unilaterally deviating from its action to another action, i.e., $\forall i \in N, \forall S_i' \in \Sigma_i$: $c_i(S_{-i}, S_i') \geqslant c_i(S)$.

The network routing games we consider have an underlying undirected graph $G = (V, E)$, and a finite set of players N. Each player $i \in N$ wishes to transmit an atomic unit of flow from a source node $s_i \in V$ to a sink (destination) node $t_i \in V$, where the flow is unsplittable. The action set Σ_i of player i is the set of all simple paths in G connecting s_i to t_i.

Each edge $e \in E$ is associated with a non-decreasing cost function $\ell_e : \{1, 2, \ldots, n\} \to \Re$, mapping the number of players for which $e \in S_i$ to a real cost. Given a joint action $S = (S_1, \ldots, S_n)$, we denote by $n_e(S) = |\{i | e \in S_i\}|$ the number of players in the joint action S that route using edge e.

We distinguish between two classes of games, which differ in the cost function of the individual players. In a *network congestion game* the cost of a player is the *sum* of the costs of the edges in its path, i.e., $c_i(S) = \sum_{e \in S_i} \ell_e(n_e(S))$, and in a *bottleneck routing game* the cost of a player is the *maximum* cost among the edges in its path, i.e., $c_i(S) = \max_{e \in S_i} \ell_e(n_e(S))$.

It has been shown by Rosenthal (1973) that every congestion game possesses a pure Nash equilibrium. Banner and Orda (2006) have shown that every bottleneck routing game possesses a pure Nash equilibrium as well. Thus, we can restrict attention to pure Nash equilibria.

Proposition 2.1. *(See Rosenthal, 1973; Banner and Orda, 2006.) Every network congestion game and every bottleneck routing game possesses at least one pure Nash equilibrium.*

In addition to the individual cost functions, there is also a well-defined optimization problem, in which we wish to minimize the *social cost* of a game. The social cost of a joint action $S \in \Sigma$ is denoted by $cost_\Lambda(S)$. The social optimum is $OPT(\Lambda) = \min_{S \in \Sigma} cost_\Lambda(S)$. There are two natural social cost functions, namely the aggregate or the maximum cost among all the players. In this paper we concentrate on the latter case; i.e., $cost_\Lambda(S) = \max_{i \in N} c_i(S)$.

Most of the focus of this paper is on symmetric network routing games in which the underlying network is symmetric.

A *symmetric* network is an undirected graph G along with two distinguished nodes, a *source* s and a *sink* t. When clear in the context, we refer to G as the symmetric network.

A network routing game is *symmetric* (also called single-commodity) if its underlying network is symmetric with source s and sink t, and nodes s and t are the respective source and sink of all the players. Otherwise (i.e., under multiple sources and/or destinations for the players), it is called an *asymmetric* (or multi-commodity) network routing game.

The goal of this paper is to provide a characterization of the network topologies in which pure Nash equilibria are guaranteed to achieve the social optimum. We use the following definition of an efficient topology.

Definition 2.2. A graph topology $G = (V, E)$ is *efficient* for a family of network routing games $\mathcal{F}$ if for every network routing game $\Lambda \in \mathcal{F}$ on the graph G, $\max_{S \in NE(\Lambda)} cost_\Lambda(S) = OPT(\Lambda)$, where $NE(\Lambda)$ is the set of pure Nash equilibria of the game Λ.

For symmetric network routing games, we use the following definition of an efficient symmetric network.

Definition 2.3. A symmetric network $G = (V, E)$ with source s and sink t is *efficient* for a family of symmetric network routing games $\mathcal{F}$ if for every symmetric network routing game $\Lambda \in \mathcal{F}$ on the symmetric network G, $\max_{S \in NE(\Lambda)} cost_\Lambda(S) = OPT(\Lambda)$, where $NE(\Lambda)$ is the set of pure Nash equilibria of the game Λ.

2.2. Graph theoretic preliminaries

In this section, we provide some definitions and properties of symmetric networks. A symmetric network G is *embedded* in a symmetric network G' if G' is isomorphic[2] to G or to a network derived from G by applying the following operations any number of times in any order:

(i) *Subdivision* of an edge (i.e., its replacement by a path of two edges).
(ii) *Addition* of a new edge joining two existing nodes.
(iii) *Extension* of the source or the sink (i.e., addition of a new edge joining s or t with a new node, which becomes the new source or sink, respectively).

We next define the following operations on symmetric networks:

- Identification: The *identification* operation is the collapse of two nodes into one. More formally, given graph $G = (V, E)$ we define the *identification* of a node $v_1 \in V$ and $v_2 \in V$ forming a new node $v \in V$ as creating a new graph $G' = (V', E')$, where $V' = V \setminus \{v_1, v_2\} \cup \{v\}$ and E' includes the edges of E where the edges of v_1 and v_2 are now connected to v.
- Parallel composition: Given two symmetric networks, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1$ and $s_2 \in V_2$ and sinks $t_1 \in V_1$ and $t_2 \in V_2$, respectively, we define a new symmetric network $G = G_1 \| G_2$ as follows. Let $G' = (V_1 \cup V_2, E_1 \cup E_2)$ be the union network. To generate $G = G_1 \| G_2$ from G' we identify the sources s_1 and s_2, forming a new source node s, and identify the sinks t_1 and t_2, forming a new sink t.
- Series composition: Given two symmetric networks, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1$ and $s_2 \in V_2$ and sinks $t_1 \in V_1$ and $t_2 \in V_2$, respectively, we define a new symmetric network $G = G_1 \to G_2$ as follows. Let $G' = (V_1 \cup V_2, E_1 \cup E_2)$ be the union network. To generate $G = G_1 \to G_2$ from G' we identify the vertices t_1 and s_2, forming a new vertex u. The network G has a source $s = s_1$ and a sink $t = t_2$.
- Extension composition: A series composition when one of the symmetric networks, G_1 or G_2, is composed of a single edge. We denote it by $G = G_1 \to_e G_2$.

With this, we are ready to define series-parallel and extension-parallel networks.

Series-parallel (SP) networks: Every SP network is constructed inductively from two simpler SP networks by either a series composition or by a parallel composition. Formally, a symmetric network consisting of a single edge is an SP network. In addition, given two SP networks, G_1 and G_2, the network $G = G_1 \| G_2$ is an SP network, and the network $G = G_1 \to G_2$ is an SP network.

Extension-parallel (EP) networks: Every EP network is constructed inductively either by a parallel composition of two simpler EP networks, or by an extension composition of a simpler EP network and a single edge. Formally, a symmetric network consisting of a single edge is an EP network. In addition, given two EP networks, G_1 and G_2, the network $G = G_1 \| G_2$ is an EP network, and if one of the networks is composed of a single edge, the network $G = G_1 \to_e G_2$ is an EP network.

The following lemmas, which are established in Milchtaich (2006a), serve us in our analysis in Sections 3 and 4.

Lemma 2.4. *A symmetric network G is an* EP *network if and only if none of the three symmetric networks in Fig.* 1 *is embedded in G.*

[2] Two symmetric networks G' and G'' are said to be *isomorphic* if there is a one-to-one correspondence between their node sets and between the edge sets such that (i) the incidence relation (an edge $e \in E$ and a node $v \in V$ are said to be incident with each other if v is an end node of e) is preserved and (ii) the source and sink in G' are paired with the source and sink in G'', respectively.

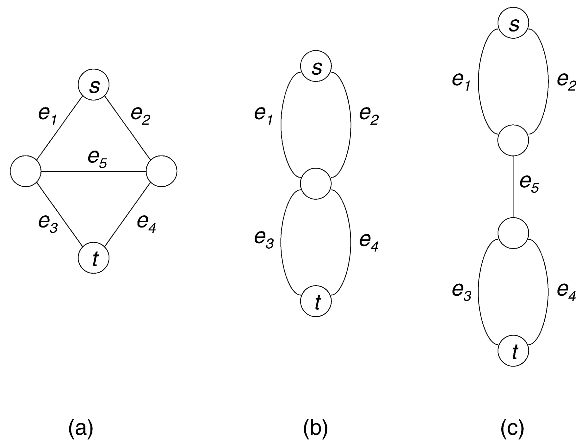

Fig. 1. Network (a) is embedded in every network which is not an SP network, and one of the networks (a), (b) or (c) is embedded in every network which is not an EP network.

Lemma 2.5. *A symmetric network G is an* SP *network if and only if the symmetric network in Fig.* 1(a) *is not embedded in G.*

The following simple observation serves us in our characterization.

Observation 2.6. Let G be a symmetric network that is not efficient for a family of symmetric network routing games $\mathcal{F}$, and suppose G is embedded in a symmetric network G'. Then, G' is not efficient for the family $\mathcal{F}$ either.

This observation can be easily established as follows. For every symmetric network routing game on the network G, there exists an equivalent symmetric network routing game on the network G', which is obtained by setting the edge cost functions in the following manner: For a subdivision operation of an edge e with cost $\ell_e(x)$ into two edges e_1 and e_2, we assign the cost functions $\ell_{e_1}(x) = 0$ and $\ell_{e_2}(x) = \ell_e(x)$; for an addition operation of an edge e, we assign it the cost function $\ell_e(x) = +\infty$, and for an extension operation of an edge e we assign it the cost function $\ell_e(x) = 0$. The original game (played on G) can be simulated on the network G' using the appropriate latency functions as specified above. One can verify (by induction) that if the embedded network G is not efficient for a given network routing game, then any network G', such that G is embedded in G', is not efficient either.

3. Efficient topologies in network congestion games

In network congestion games the individual cost function is given by $c_i(S) = \sum_{e \in S_i} \ell_e(n_e(S))$ (where $\ell_e(x)$ is non-decreasing in x), and the social cost is given by $cost(S) = \max_i c_i(S)$. As noted above, for the aggregate social cost, the worst possible price of anarchy is obtained already for a simple topology of two parallel links. Here, we characterize efficient network topologies under the maximum social cost. Our results indicate that under the maximum social cost the network topology does affect the price of anarchy. In our analysis, we make use of some observations established in Milchtaich (2006a), who characterized network topologies that guarantee that every Nash equilibrium is weakly Pareto optimal (but, in contrast to our model, analyzed the non-atomic case, in which there is a continuum of users, each with a negligible effect on the others). Our main focus is on symmetric network congestion games, which are defined on symmetric networks. As we show, for the asymmetric case, any efficient graph is a forest or a graph with two vertices.

The following lemma serves us in both symmetric network congestion games and symmetric bottleneck routing games.

Lemma 3.1. *Let Λ be a symmetric network routing game on the symmetric network $G = G_1 \| G_2$, with source s and sink t, where G_1 and G_2 are* SP *networks. Assume that for any symmetric network routing game on the networks G_1 or G_2, for any joint action of the players, the cost of the best response strategy of any player is at most the optimal* *social cost. Then, in the game Λ for any joint action of the players, the cost of the best response strategy of any player is at most the optimal social cost.*

Proof. Let S^* be the optimal joint action, i.e., $OPT(\Lambda) = cost_\Lambda(S^*)$, and let T_j^* be the set of players using paths in G_j according to S^* and $x_j^* = |T_j^*|$. Let S be a joint action of the players in Λ, and let T_j be the set of players using paths in G_j according to S and $x_j = |T_j|$. There are two cases:

Case 1. $x_1 = x_1^*$ and $x_2 = x_2^*$. Let Λ_1 (resp., Λ_2) be a symmetric network routing game on the symmetric network G_1 (resp., G_2) with players T_1 (resp., T_2), with the original edge cost functions. Given S, let S' (resp., S'') be the induced joint action of players in T_1 (resp., T_2). (Observe that actions in S' and S'' are paths in G_1 and G_2 respectively, therefore joint actions of T_1 and T_2 in Λ_1 and Λ_2, respectively.)

By the assumption on the networks G_1 and G_2, for every player i in the game Λ_1 with best response strategy (to S'_{-i}) P'_i, $c_i(P'_i, S'_{-i}) \leqslant OPT(\Lambda_1)$ and for every player i in the game Λ_2 with best response strategy (to S''_{-i}) P''_i, $c_i(P''_i, S''_{-i}) \leqslant OPT(\Lambda_2)$. For $j = 1, 2$, since $x_j = x_j^*$, it follows that $OPT(\Lambda_j) \leqslant OPT(\Lambda)$. Let P_i be player i's best response to S_{-i} in Λ. Thus, for any player $i \in T_1$, $c_i(P_i, S_{-i}) \leqslant c_i(P'_i, S_{-i})$. Since $c_i(P'_i, S_{-i})$ in Λ is equal to $c_i(P'_i, S'_{-i})$ in Λ_1, and by the inequalities obtained above, it follows that for every player $i \in T_1$, $c_i(P_i, S_{-i}) \leqslant OPT(\Lambda)$. The claim holds for every player $i \in T_2$ analogously.

Case 2. There exists a network G_j for which $x_j^* > x_j$. W.l.o.g., suppose $x_1^* > x_1$. Consider an arbitrary player i. Let $x'_1 = |T_1 \cup \{i\}|$. Then, $x_1^* \geqslant x_1 + 1 \geqslant x'_1$. Let Λ_1 be a symmetric network routing game on the symmetric network G_1 with players $T_1 \cup \{i\}$. Given S, Let S' be the induced joint action of the players in $T_1 \cup \{i\}$. Let P'_i be the best response strategy of player i in Λ_1 for the joint actions S'_{-i}. It follows from the assumption on the network G_1 that in Λ_1 we have $c_i(P'_i, S'_{-i}) \leqslant OPT(\Lambda_1)$. Since $x_1^* \geqslant x'_1$, we have that $OPT(\Lambda_1) \leqslant OPT(\Lambda)$, and therefore $c_i(P'_i, S'_{-i}) \leqslant OPT(\Lambda)$.

Let P_i be the best response strategy of player i in Λ for the joint actions S_{-i}. Thus, $c_i(P_i, S_{-i}) \leqslant c_i(P'_i, S_{-i})$. Since $c_i(P'_i, S_{-i})$ in Λ equals $c_i(P'_i, S'_{-i})$ in Λ_1, and from the above inequalities, it follows that $c_i(P_i, S_{-i}) \leqslant OPT(\Lambda)$. □

Consider a symmetric network congestion game on an EP network. We show that in any joint action of the players the cost of the best response strategy of any player is at most the optimal social cost.

Lemma 3.2. *Let Λ be a symmetric network congestion game on an EP network G with source s and sink t. Consider any joint action $S \in \Sigma$. Let P_i be a best response of any player i. Then $c_i(P_i, S_{-i}) \leqslant OPT(\Lambda)$.*

Proof. We prove the lemma by induction on the network size $|E|$. Let Λ be a symmetric network congestion game on an EP network $G = (V, E)$. For $|E| = 1$ the claim holds trivially. In what follows we show that this property is preserved under extension and series compositions.

Extension composition. Suppose the network $G = G_1 \to_e G_2$ is an extension composition of the network G_1 consisting of a single edge $e = (s_1, t_1)$ and an EP network $G_2 = (V_2, E_2)$ with terminals s_2, t_2, such that $s = s_1$ and $t = t_2$ (the case that G_2 is a single edge is analogous). Let Λ' be a symmetric network congestion game on the symmetric network G_2 with the original players and the original edge cost functions. Let S' be the induced joint action of the players in the game Λ' according to S (i.e., S' is obtained from S by removing the edge e from the strategy S_j of each player j). Let P'_i be a best response of player i to S'_{-i} in the game Λ'. By the inductive hypothesis $c_i(P'_i, S'_{-i}) \leqslant OPT(\Lambda')$ in the game Λ'. It is easy to see that for any player i and for any joint action $\hat{S}$ in Λ and the induced joint action $\hat{S}'$ in Λ', the difference between player i's costs in Λ and Λ' under $\hat{S}$ and $\hat{S}'$ respectively is $\ell_e(n)$. Therefore, $P_i = \{e\} \cup P'_i$ is a best response of player i to S_{-i} in the game Λ and $c_i(P_i, S_{-i}) \leqslant OPT(\Lambda)$.

Parallel composition. Follows from Lemma 3.1. □

The following corollary follows directly from Lemma 3.2.

Corollary 3.3. *Every EP network is efficient for symmetric network congestion games.*

We next show that EP networks are the unique efficient symmetric networks for symmetric network congestion games.

Theorem 3.4. *Let G be an efficient symmetric network for symmetric network congestion games. Then, G is an* EP *network.*

Proof. By Lemma 2.4, one of the symmetric networks in Fig. 1 is embedded in every symmetric network that is not an EP network. By Observation 2.6, it is sufficient to show that each of these three networks is not efficient. Since any game played on network (b) in the figure can be simulated on networks (a) and (c) by assigning $\ell_{e_5}(x) = 0$, it is sufficient to show that network (b) is not efficient.

Consider the network given in Fig. 1(b) with the following delay functions: $\ell_{e_1}(x) = 2$, $\ell_{e_2}(x) = x$, $\ell_{e_3}(x) = x$, $\ell_{e_4}(x) = 2$. Consider a symmetric network congestion game with two players played on the symmetric network G with source s and sink t. It is easy to verify that this game admits a pure Nash equilibrium in which $S_1 = S_2 = \{e_2, e_3\}$, resulting in $cost(S) = 4$. Consider the joint action S' in which $S'_1 = \{e_1, e_3\}$ and $S'_2 = \{e_2, e_4\}$. It holds that $cost(S') = 3 < 4 = cost(S)$. Therefore, G is not efficient. □

Corollary 3.3 and Theorem 3.4 establish the following characterization.

Corollary 3.5. *For symmetric network congestion games with the maximum social cost function, a symmetric network topology G is efficient if and only if G is an* EP *network.*

Finally, we characterize efficient topologies for asymmetric network congestion games. We begin with the following useful lemma.

Lemma 3.6. *Any connected graph with at least* 3 *vertices containing a cycle is not efficient for asymmetric network congestion games.*

Proof. We prove the claim by showing that every graph containing a cycle of length 2 or a cycle of length 3 is not efficient for asymmetric network congestion games.

Suppose the graph G' contains a cycle of length 2. It is sufficient to show an example of a game played on the graph G in Fig. 2(a) indicating that G is not efficient. This is justified since this game can be simulated on any connected graph (with at least 3 vertices) containing a cycle of length 2 by assigning $\ell_e(x) = +\infty$ for any edge $e \notin G$. Consider the graph G given in Fig. 2(a) with the following delay functions: $\ell_{e_1}(x) = x$, $\ell_{e_2}(x) = 2$, $\ell_{e_3}(x) = 2$. Consider an asymmetric network congestion game with two players played on the graph G. The two players share a common source s and the sinks of players 1 and 2 are t_1 and t_2 respectively. One can verify that this game admits a pure Nash equilibrium in which $S_1 = \{e_1\}$ and $S_2 = \{e_1, e_3\}$, resulting in $cost(S) = 4$. However, the joint action S' in which $S'_1 = \{e_2\}$ and $S'_2 = \{e_1, e_3\}$ yields $cost(S') = 3 < 4 = cost(S)$. Therefore, G is not efficient.

Now suppose the graph G' contains a cycle of length 3. As above, it is sufficient to prove that the graph G depicted in Fig. 2(b) is not efficient. Consider the graph G with the following delay functions: $\ell_{e_1}(x) = 2x$, $\ell_{e_2}(x) = 2x$, $\ell_{e_3}(x) = x$. Consider an asymmetric network congestion game with two players played on the graph G. The two players share a common source s and the sinks of players 1 and 2 are t_1 and t_2 respectively. One can verify that this game admits a pure Nash equilibrium in which $S_1 = \{e_2, e_3\}$, $S_2 = \{e_1, e_3\}$, resulting in $cost(S) = 4$. However, the joint action S' in which $S'_1 = \{e_1\}$ and $S'_2 = \{e_2\}$ yields $cost(S') = 2 < 4 = cost(S)$. Therefore, G is not efficient. This completes the proof. □

The following theorem follows directly from Lemma 3.6, and the trivial observation that every tree and every graph with two vertices is efficient.

Theorem 3.7. *For asymmetric network congestion games, every efficient connected graph is a tree or a graph with two vertices.*

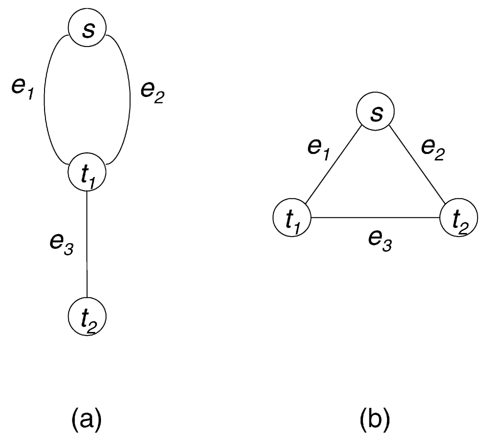

Fig. 2. Asymmetric network routing games.

4. Efficient topologies in bottleneck routing games

In the bottleneck routing games considered in this section, the individual cost function is $c_i(S) = \max_{e \in S_i} \ell_e(n_e(S))$ (where $\ell_e(x)$ is non-decreasing in x), and the social cost is given by $cost(S) = \max_i c_i(S)$. We note that for the aggregate social cost function, i.e., $cost(S) = \sum_i c_i(S)$, the price of anarchy is unbounded even for a simple topology of two parallel edges. This can be shown by the same example given in Roughgarden (2002) indicating the unbounded price of anarchy in network congestion games with the aggregate latency social cost. In this section we characterize efficient network topologies in bottleneck routing games under the maximum social cost. Like in Section 3, we focus on symmetric games since for the asymmetric case, every efficient connected graph is a tree with possibly multiple parallel edges.

Consider a symmetric bottleneck routing game on an SP network. As in symmetric network congestion games on EP networks, the following lemma shows that in any joint action of the players the cost of the best response strategy of any player is at most the optimal social cost.

Lemma 4.1. *Let Λ be a symmetric bottleneck routing game on an SP network G with source s and sink t. Consider any joint action $S \in \Sigma$. Let P_i be a best response of any player i. Then $c_i(P_i, S_{-i}) \leqslant OPT(\Lambda)$.*

Proof. We prove the lemma by induction on the network size $|E|$. For $|E| = 1$ the claim holds trivially. We show the claim for a series composition, i.e., $G = G_1 \to G_2$, and for a parallel composition, i.e., $G = G_1 \| G_2$, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are SP networks with sources s_1, s_2, and sinks t_1, t_2, respectively. Let Λ be a symmetric bottleneck routing game on an SP network $G = (V, E)$.

Series composition. Let $G = G_1 \to G_2$. Let Λ_1 (resp., Λ_2) be a symmetric bottleneck routing game played on the network G_1 (resp., G_2) with all the players with the original edge cost functions. Let S be a joint action of the game Λ and let S' and S'' be the induced joint actions of the players in the games Λ_1 and Λ_2 respectively. Consider any player i. Let P'_i and P''_i be the best-response strategies of player i to S'_{-i} and S''_{-i} in the games Λ_1 and Λ_2 respectively. In addition, let $P_i = P'_i \cup P''_i$ be a strategy of player i in the original game Λ. By the inductive hypothesis, $c_i(P'_i, S'_{-i}) \leqslant OPT(\Lambda_1)$ and $c_i(P''_i, S''_{-i}) \leqslant OPT(\Lambda_2)$. Since $OPT(\Lambda) = \max(OPT(\Lambda_1), OPT(\Lambda_2))$ and $c_i(P_i, S_{-i}) = \max(c_i(P'_i, S'_{-i}), c_i(P''_i, S''_{-i}))$, we obtain $c_i(P_i, S_{-i}) \leqslant OPT(\Lambda)$ as required.

Parallel composition. Follows from Lemma 3.1. □

The following corollary follows directly from Lemma 4.1.

Corollary 4.2. *Every SP network is efficient for symmetric bottleneck routing games.*

We next show that SP networks are the unique efficient symmetric networks for symmetric bottleneck routing games.

Theorem 4.3. *Let G be an efficient symmetric network for symmetric bottleneck routing games. Then, G is an* SP *network.*

Proof. By Lemma 2.5, the symmetric network in Fig. 1(a) is embedded in every symmetric network that is not an SP network. By Observation 2.6, it is sufficient to show that the symmetric network in Fig. 1(a) is not efficient.

Consider the network given in Fig. 1(a) with the following delay functions: $\ell_{e_1}(x) = \ell_{e_3}(x) = x$, $\ell_{e_2}(x) = \ell_{e_4}(x) = \ell_{e_5}(x) = 2x$. Consider a symmetric bottleneck routing game with six players played on the symmetric network G with source s and sink t. One can verify that this game admits a pure Nash equilibrium in which $S_1 = S_2 = S_3 = \{e_2, e_5, e_3\}$ and $S_4 = S_5 = S_6 = \{e_1, e_3\}$, resulting in $cost(S) = 6$. However, the joint action S' in which $S'_1 = S'_2 = \{e_2, e_4\}$ and $S'_3 = S'_4 = S'_5 = S'_6 = \{e_1, e_3\}$ yields $cost(S') = 4 < 6 = cost(S)$. Therefore, G is not efficient. □

Colrollary 4.2 and Theorem 4.3 establish the following characterization.

Corollary 4.4. *For symmetric bottleneck routing games under the maximum social cost function, a symmetric network topology G is efficient if and only if G is an* SP *network.*

Finally, we characterize efficient topologies for asymmetric bottleneck routing games.

Lemma 4.5. *Any graph containing a cycle of length* 3 *is not efficient for asymmetric bottleneck routing games.*

Proof. It is sufficient to prove that the graph G given in Fig. 2(b) is not efficient. (This is justified as in Lemma 3.6.) Consider the graph G with the following delay functions: $\ell_{e_1}(x) = \ell_{e_2}(x) = \ell_{e_3}(x) = x$. Consider an asymmetric bottleneck routing game with two players played on the graph G. The two players share a common source s and the sinks of players 1 and 2 are t_1 and t_2 respectively. One can verify that this game admits a pure Nash equilibrium in which $S_1 = \{e_2, e_3\}$, $S_2 = \{e_1, e_3\}$, resulting in $cost(S) = 2$. However, the joint action S' with $S'_1 = \{e_1\}$ and $S'_2 = \{e_2\}$ yields $cost(S') = 1 < 2 = cost(S)$. Therefore, G is not efficient. □

The following theorem follows from Lemma 4.5 and the simple fact that in every asymmetric bottleneck routing game on a tree (possibly with multiple parallel edges) and for every joint action of the players, the cost of the best response strategy of every player is at most the optimal social cost.

Theorem 4.6. *For asymmetric bottleneck routing games, every efficient connected graph is a tree with possibly multiple parallel edges.*

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