

Strong Equilibrium in Cost Sharing Connection Games

Amir Epstein^{*}
School of Computer Science
Tel-Aviv University
Tel-Aviv, 69978, Israel
amirep@tau.ac.il

Michal Feldman[†]
School of Computer Science
The Hebrew University of
Jerusalem
Jerusalem, 91904, Israel
mfeldman@cs.huji.ac.il

Yishay Mansour[‡]
School of Computer Science
Tel-Aviv University
Tel-Aviv, 69978, Israel
mansour@tau.ac.il

ABSTRACT

In this work we study cost sharing connection games, where each player has a source and sink he would like to connect, and the cost of the edges is either shared equally (fair connection games) or in an arbitrary way (general connection games). We study the graph topologies that guarantee the existence of a strong equilibrium (where no coalition can improve the cost of each of its members) regardless of the specific costs on the edges.

Our main existence results are the following: (1) For a single source and sink we show that there is always a strong equilibrium (both for fair and general connection games). (2) For a single source multiple sinks we show that for a series parallel graph a strong equilibrium always exists (both for fair and general connection games). (3) For multi source and sink we show that an extension parallel graph always admits a strong equilibrium in fair connection games.

As for the quality of the strong equilibrium we show that in any fair connection games the cost of a strong equilibrium is $\Theta(\log n)$ from the optimal solution, where n is the number of players. (This should be contrasted with the $\Omega(n)$ price of anarchy for the same setting.) For single source general connection games and single source single sink fair connection games, we show that a strong equilibrium is always an optimal solution.

^{*}Research supported in part by the German-Israeli Foundation and by the Israel Science Foundation.

[†]Research partially supported by a grant of the Israel Science Foundation, BSF, Lady Davis Fellowship, and an IBM faculty award.

[‡]Research partially supported by a grant of the Israel Science Foundation, BSF, Lady Davis Fellowship, and an IBM faculty award.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee.

EC'07, June 13–16, 2007, San Diego, California, USA.
Copyright 2007 ACM 978-1-59593-653-0/07/0006 ...\$5.00.

Categories and Subject Descriptors

C.2.4 [Computer-Communication Networks]: Distributed Systems; F.2.0 [Analysis of Algorithms and Problem Complexity]: General; J.4 [Social and Behavioral Sciences]: Economics; K.4.4 [Electronic Commerce]: Payment schemes

General Terms

Theory, Economics, Algorithms

Keywords

game theory, Nash equilibrium, strong equilibrium, coalitions, price of anarchy, strong price of anarchy, network design, cost sharing games

1. INTRODUCTION

Computational game theory has introduced the issue of incentives to many of the classical combinatorial optimization problems. The view that the demand side is many times not under the control of a central authority that optimizes the global performance, but rather under the control of individuals with different incentives, has led already to many important insights.

Consider classical routing and transportation problems such as multicast or multi-commodity problems, which are many times viewed as follows. We are given a graph with edge costs and connectivity demands between nodes, and our goal is to find a minimal cost solution. The classical centralized approach assumes that all the individual demands can both be completely coordinated and have no individual incentives. The game theory point of view would assume that each individual demand is controlled by a player that optimizes its own utility, and the resulting outcome could be far from the optimal solution.

When considering individual incentives one needs to discuss the appropriate solution concept. Much of the research in computational game theory has focused on the classical *Nash equilibrium* as the primary solution concept. Indeed Nash equilibrium has many benefits, and most importantly it always exists (in mixed strategies). However, the solution concept of Nash equilibrium is resilient only to unilateral deviations, while in reality, players may be able to coordinate their actions.

A *strong equilibrium* [4] is a state from which no coalition (of any size) can deviate and improve the utility of *every* member of the coalition (while possibly lowering the utility

of players outside the coalition). This resilience to deviations by coalitions of the players is highly attractive, and one can hope that once a strong equilibrium is reached it is highly likely to sustain. From a computational game theory point of view, an additional benefit of a strong equilibrium is that it has a potential to reduce the distance between the optimal solution and the solution obtained as an outcome of selfish behavior. The strong price of anarchy (SPoA), introduced in [1], is the ratio between the cost of the worst strong equilibrium and the cost of an optimal solution.

Obviously, SPoA is meaningful only in those cases where a strong equilibrium exists. A major downside of strong equilibrium is that most games do not admit any strong equilibrium. Even simple classical games like the *prisoner's dilemma* do not possess any strong equilibrium (which is also an example of a congestion game that does not possess a strong equilibrium¹). This unfortunate fact has reduced the concentration in strong equilibrium, despite its highly attractive properties. Yet, [1] have identified two broad families of games, namely job scheduling and network formation, where a strong equilibrium always exists and the SPoA is significantly lower than the price of anarchy (which is the ratio between the worst Nash equilibrium and the optimal solution [15, 18, 5, 6]).

In this work we concentrate on cost sharing connection games, introduced by [3, 2]. In such a game, there is an underlying directed graph with edge costs, and individual users have connectivity demands (between a source and a sink). We consider two models. The *fair cost connection model* [2] allows each player to select a path from the source to the sink². In this game the cost of an edge is shared equally between all the players that selected the edge, and the cost of the player is the sum of its costs on the edges it selected. The *general connection game* [3] allows each player to offer prices for edges. In this game an edge is bought if the sum of the offers at least covers its cost, and the cost of the player is the sum of its offers on the bought edges (in both games we assume that the player has to guarantee the connectivity between its source and sink).

In this work we focus on two important issues. The first one is identifying under what conditions the existence of a strong equilibrium is guaranteed, and the second one is the quality of the strong equilibria. For the existence part, we identify families of graph topologies that possess some strong equilibrium for any assignment of edge costs. One can view this separation between the graph topology and the edge costs, as a separation between the underlying infrastructure and the costs the players observe to purchase edges. While one expects the infrastructure to be stable over long periods of time, the costs the players observe can be easily modified over short time periods. Such a topological characterization of the underlying infrastructure provides a network designer topological conditions that will ensure stability in his network.

Our results are as follows. For the single commodity case (all the players have the same source and sink), there is a strong equilibrium in any graph (both for fair and general connection games). Moreover, the strong equilibrium is also

¹while any congestion game is known to admit at least one Nash equilibrium in pure strategies [16].

²The fair cost sharing scheme is also attractive from a mechanism design point of view, as it is a strategyproof cost-sharing mechanism [14].

the optimal solution (namely, the players share a shortest path from the common source to the common sink). For the case of a single source and multiple sinks (for example, in a multicast tree), we show that in a fair connection game there is a strong equilibrium if the underlying graph is a series parallel graph, and we show an example of a non-series parallel graph that does not have a strong equilibrium. For the case of multi-commodity (multi sources and sinks), we show that in a fair connection game if the graph is an extension parallel graph then there is always a strong equilibrium, and we show an example of a series parallel graph that does not have a strong equilibrium. As far as we know, we are the first to provide a topological characterization for equilibrium existence in multi-commodity and single-source network games.

For any fair connection game we show that if there exists a strong equilibrium it is at most a factor of $\Theta(\log n)$ from the optimal solution, where n is the number of players. This should be contrasted with the $\Theta(n)$ bound that exists for the price of anarchy [2].

For single source general connection games, we show that any series parallel graph possesses a strong equilibrium, and we show an example of a graph that does not have a strong equilibrium. In this case we also show that any strong equilibrium is optimal.

Related work

Topological characterizations for single-commodity network games have been recently provided for various equilibrium properties, including equilibrium existence [12, 7, 8], equilibrium uniqueness [10] and equilibrium efficiency [17, 11]. The existence of pure Nash equilibrium in single-commodity network congestion games with player-specific costs or weights was studied in [12]. The existence of strong equilibrium was studied in both utility-decreasing (e.g., routing) and utility-increasing (e.g., fair cost-sharing) congestion games. [7, 8] have provided a full topological characterization for a SE existence in single-commodity utility-decreasing congestion games, and showed that a SE always exists if and only if the underlying graph is extension-parallel. [19] have shown that in single-commodity utility-increasing congestion games, the topological characterization is essentially equivalent to parallel links. In addition, they have shown that these results hold for correlated strong equilibria as well (in contrast to the decreasing setting, where correlated strong equilibria might not exist at all). While the fair cost sharing games we study are utility increasing network congestion games, we derive a different characterization than [19] due to the different assumptions regarding the players' actions.³

2. MODEL

2.1 Game Theory definitions

A game $\Lambda = \langle N, (\Sigma_i), (c_i) \rangle$ has a finite set $N = \{1, \dots, n\}$ of players. Player $i \in N$ has a set Σ_i of actions, the joint action set is $\Sigma = \Sigma_1 \times \dots \times \Sigma_n$ and a joint action $S \in \Sigma$ is also called a *profile*. The cost function of player i is

³In [19] they allow to restrict some players from using certain links, even though the links exist in the graph, while we do not allow this, and assume that the available strategies for players are fully represented by the underlying graph.

$c_i : \Sigma \rightarrow \mathbb{R}^+$, which maps the joint action $S \in \Sigma$ to a non-negative real number. Let $S = (S_1, \dots, S_n)$ denote the profile of actions taken by the players, and let $S_{-i} = (S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n)$ denote the profile of actions taken by all players other than player i . Note that $S = (S_i, S_{-i})$. The *social cost* of a game Λ is the sum of the costs of the players, and we denote by $OPT(\Lambda)$ the minimal social cost of a game Λ . i.e., $OPT(\Lambda) = \min_{S \in \Sigma} cost_\Lambda(S)$, where $cost_\Lambda(S) = \sum_{i \in N} c_i(S)$.

A joint action $S \in \Sigma$ is a *pure Nash equilibrium* if no player $i \in N$ can benefit from unilaterally deviating from his action to another action, i.e., $\forall i \in N \forall S'_i \in \Sigma_i : c_i(S_{-i}, S'_i) \geq c_i(S)$. We denote by $NE(\Lambda)$ the set of pure Nash equilibria in the game Λ .

Resilience to coalitions: A *pure deviation* of a set of players $\Gamma \subset N$ (also called *coalition*) specifies an action for each player in the coalition, i.e., $\gamma \in \times_{i \in \Gamma} \Sigma_i$. A joint action $S \in \Sigma$ is not resilient to a pure deviation of a coalition Γ if there is a pure joint action γ of Γ such that $c_i(S_{-\Gamma}, \gamma) < c_i(S)$ for every $i \in \Gamma$ (i.e., the players in the coalition can deviate in such a way that *each* player in the coalition reduces its cost). A pure Nash equilibrium $S \in \Sigma$ is a *k-strong equilibrium*, if there is no coalition Γ of size at most k , such that S is not resilient to a pure deviation by Γ . We denote by $k\text{-SE}(\Lambda)$ the set of k -strong equilibria in the game Λ . We denote by $SE(\Lambda)$ the set of n -strong equilibria, and call $S \in SE(\Lambda)$ a *strong equilibrium* (SE).

Next we define the price of anarchy (PoA) [9] and the k -strong price of anarchy ($k\text{-SPoA}$) for the game Λ . The *Price of Anarchy* (PoA) is the ratio between the *maximal* cost of a pure Nash equilibrium (assuming one exists) and the social optimum, i.e., $\max_{S \in NE(\Lambda)} cost_\Lambda(S) / OPT(\Lambda)$. Similarly, the *Price of Stability* (PoS) is the ratio between the *minimal* cost of a pure Nash equilibrium and the social optimum, i.e., $\min_{S \in NE(\Lambda)} cost_\Lambda(S) / OPT(\Lambda)$. The *k-Strong Price of Anarchy* ($k\text{-SPoA}$) is the ratio between the *maximal* cost of a k -strong equilibrium (assuming one exists) and the social optimum, i.e., $\max_{S \in k\text{-SE}(\Lambda)} cost_\Lambda(S) / OPT(\Lambda)$. The SPoA is the $n\text{-SPoA}$. Similarly, the *Strong Price of Stability* (SPoS) is the ratio between the *minimal* cost of a pure strong equilibrium and the social optimum, i.e., $\min_{S \in SE(\Lambda)} cost_\Lambda(S) / OPT(\Lambda)$. Note that both $k\text{-SPoA}$ and SPoS are defined only if some strong equilibrium exists.

2.2 Cost Sharing Connection Games

A cost sharing connection game has an underlying directed graph $G = (V, E)$ where each edge $e \in E$ has an associated cost $c_e \geq 0$ ⁴. In a connection game each player $i \in N$ has an associated source s_i and sink t_i .

In a *fair connection game* the actions Σ_i of player i include all the paths from s_i to t_i . The cost of each edge is shared equally by the set of all players whose paths contain it. Given a joint action, the cost of a player is the sum of his costs on the edges it selected. More formally, the cost function of each player on an edge e , in a joint action S , is $f_e(n_e(S)) = \frac{c_e}{n_e(S)}$, where $n_e(S)$ is the number of players that selected a path containing edge e in S . The cost of player i , when selecting path $Q_i \in \Sigma_i$ is $c_i(S) = \sum_{e \in Q_i} f_e(n_e(S))$.

In a *general connection game* the actions Σ_i of player i

⁴In some of the existence proofs, we assume that $c_e > 0$ for simplicity. The full version contains the complete proofs for the case $c_e \geq 0$.

is a payment vector p_i , where $p_i(e)$ is how much player i is offering to contribute to the cost of edge e .⁵ Given a profile p , any edge e such that $\sum_i p_i(e) \geq c_e$ is considered *bought*, and E_p denotes the set of bought edges. Let $G_p = (V, E_p)$ denote the graph bought by the players for profile $p = (p_1, \dots, p_n)$. Clearly, each player tries to minimize his total payment which is $c_i(p) = \sum_{e \in E_p} p_i(e)$ if s_i is connected to t_i in G_p , and infinity otherwise.⁶ We denote by $c(p) = \sum_i c_i(p)$ the total cost under the profile p . For a subgraph H of G we denote the total cost of the edges in H by $c(H)$.

A *symmetric* connection game implies that the source and sink of all the players are identical. (We also call a symmetric connection game a *single source single sink* connection game, or a *single commodity* connection game.) A *single source* connection game implies that the sources of all the players are identical. Finally, A *multi commodity* connection game implies that each player has its own source and sink.

2.3 Extension Parallel and Series Parallel Directed Graphs

Our directed graphs would be acyclic, and would have a *source* node (from which all nodes are reachable) and a *sink* node (which every node can reach). We first define the following actions for composition of directed graphs.

- **Identification:** The *identification* operation allows to collapse two nodes to one. More formally, given graph $G = (V, E)$ we define the *identification* of a node $v_1 \in V$ and $v_2 \in V$ forming a new node $v \in V$ as creating a new graph $G' = (V', E')$, where $V' = V - \{v_1, v_2\} \cup \{v\}$ and E' includes the edges of E where the edges of v_1 and v_2 are now connected to v .
- **Parallel composition:** Given two directed graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1$ and $s_2 \in V_2$ and sinks $t_1 \in V_1$ and $t_2 \in V_2$, respectively, we define a new graph $G = G_1 || G_2$ as follows. Let $G' = (V_1 \cup V_2, E_1 \cup E_2)$ be the union graph. To create $G = G_1 || G_2$ we identify the sources s_1 and s_2 , forming a new source node s , and identify the sinks t_1 and t_2 , forming a new sink t .
- **Series composition:** Given two directed graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1$ and $s_2 \in V_2$ and sinks $t_1 \in V_1$ and $t_2 \in V_2$, respectively, we define a new graph $G = G_1 \rightarrow G_2$ as follows. Let $G' = (V_1 \cup V_2, E_1 \cup E_2)$ be the union graph. To create $G = G_1 \rightarrow G_2$ we identify the vertices t_1 and s_2 , forming a new vertex u . The graph G has a source $s = s_1$ and a sink $t = t_2$.
- **Extension composition :** A series composition when one of the graphs, G_1 or G_2 , is composed of a single directed edge is an extension composition, and we denote it by $G = G_1 \rightarrow_e G_2$.

An *extension parallel graph* (EPG) is a graph G consisting of either: (1) a single directed edge (s, t) , (2) a graph $G = G_1 || G_2$ or (3) a graph $G = G_1 \rightarrow_e G_2$, where G_1 and G_2 are extension parallel graphs. A *series parallel graph* (SPG)

⁵We limit the players to select a path connecting s_i to t_i and payment only on those edges.

⁶This implies that in equilibrium every player has its sink and source connected by a path in G_p .

is a graph G consisting of either: (1) a single directed edge (s, t) , (2) a graph $G = G_1 || G_2$ or (3) a graph $G = G_1 \rightarrow G_2$, where G_1 and G_2 are series parallel graphs.

Given a path Q and two vertices u, v on Q , we denote the subpath of Q from u to v by $Q_{u,v}$. The following lemma, whose proof appears in the full version, would be the main topological tool in the case of single source graph.

LEMMA 2.1. *Let G be an SPG with source s and sink t . Given a path Q , from s to t , and a vertex t' , there exist a vertex $y \in Q$, such that for any path Q' from s to t' , the path Q' contains y and the paths $Q'_{y,t'}$ and Q are edge disjoint. (We call the vertex y the intersecting vertex of Q and t' .)*

3. FAIR CONNECTION GAMES

This section derives our results for fair connection games.

3.1 Existence of Strong Equilibrium

While it is known that every fair connection game possesses a Nash equilibrium in pure strategies [2], this is not necessarily the case for a strong equilibrium. In this section, we study the existence of strong equilibrium in fair connection games. We begin with a simple case, showing that every symmetric fair connection game possesses a strong equilibrium.

THEOREM 3.1. *In every symmetric fair connection game there exists a strong equilibrium.*

PROOF. Let s' be the source and t' be the sink of all the players. We show that a profile S in which all the players choose the same shortest path Q (from the source s' to the sink t') is a strong equilibrium. Suppose by contradiction that S is not a SE. Then there is a coalition Γ that can deviate to a new profile S' such that the cost of every player $j \in \Gamma$ decreases. Let Q'_j be a new path used by player $j \in \Gamma$. Since Q is a shortest path, it holds that $c(Q'_j \setminus (Q \cap Q'_j)) \geq c(Q \setminus (Q \cap Q'_j))$, for any path Q'_j . Therefore for every player $j \in \Gamma$ we have that $c_j(S') \geq c_j(S)$. However, this contradicts the fact that all players in Γ reduce their cost. (In fact, no player in Γ has reduced its cost.)

□

While every symmetric fair connection game admits a SE, it does not hold for every fair connection game. In what follows, we study the network topologies that admit a strong equilibrium for *any* assignment of edge costs, and give examples of topologies for which a strong equilibrium does not exist. The following lemma, whose proof appears in the full version, plays a major role in our proofs of the existence of SE.

LEMMA 3.2. *Let Λ be a fair connection game on a series parallel graph G with source s and sink t . Assume that player i has $s_i = s$ and $t_i = t$ and that Λ has some SE. Let S be a SE that minimizes the cost of player i (out of all SE), i.e., $c_i(S) = \min_{T \in SE} c_i(T)$ and let S^* be the profile that minimizes the cost of player i (out of all possible profiles), i.e., $c_i(S^*) = \min_{T \in \Sigma} c_i(T)$. Then, $c_i(S) = c_i(S^*)$.*

The next lemma considers parallel composition.

LEMMA 3.3. *Let Λ be a fair connection game on graph $G = G_1 || G_2$, where G_1 and G_2 are series parallel graphs. If*

every fair connection game on the graphs G_1 and G_2 possesses a strong equilibrium, then the game Λ possesses a strong equilibrium.

PROOF. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have sources s_1 and s_2 and sinks t_1 and t_2 respectively. Let T_i be the set of players with an endpoint in $V_i \setminus \{s, t\}$, for $i \in \{1, 2\}$. (An endpoint is either a source or a sink of a player). Let T_3 be the set of players j such that $s_j = s$ and $t_j = t$. Let Λ_1 and Λ_2 be the original game on the respective graphs G_1 and G_2 with players $T_1 \cup T_3$ and $T_2 \cup T_3$, respectively.

Let S' and S'' be the SE in Λ_1 and Λ_2 that minimizes the cost of players in T_3 , respectively. Assume w.l.o.g. that $c_i(S') \leq c_i(S'')$ where player $i \in T_3$. In addition, let \bar{S} be the game on the graph G_2 with players T_2 and let \bar{S} be a SE in Λ_2' .

We will show that the profile $S = S' \cup \bar{S}$ is a SE in Λ . Suppose by contradiction that S is not a SE. Then, there is a coalition Γ that can deviate such that the cost of every player $j \in \Gamma$ decreases. By Lemma 3.2 and the assumption that $c_i(S') \leq c_i(S'')$, a player $j \in T_3$ cannot improve his cost. Therefore, $\Gamma \subseteq T_1 \cup T_2$. But this is a contradiction to S' being a SE in Λ_1 or \bar{S} being a SE in Λ_2' . □

The following theorem considers the case of single source fair connection games.

THEOREM 3.4. *Every single source fair connection game on a series-parallel graph possesses a strong equilibrium.*

PROOF. We prove the theorem by induction on the network size $|V|$. The claim obviously holds if $|V| = 2$. We show the claim for a series composition, i.e., $G = G_1 \rightarrow G_2$, and for a parallel composition, i.e., $G = G_1 || G_2$, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are SPG's with sources s_1, s_2 , and sinks t_1, t_2 , respectively.

series composition. Let $G = G_1 \rightarrow G_2$. Let T_1 be the set of players j such that $t_j \in V_1$, and T_2 be the set of players j such that $t_j \in V_2 \setminus \{s_2\}$.

Let Λ_1 and Λ_2 be the original game on the respective graphs G_1 and G_2 with players $T_1 \cup T_2$ and T_2 , respectively. For every player $i \in T_2$ with action S_i in the game Λ let $S_i \cap E_1$ be his induced action in the game Λ_1 , and let $S_i \cap E_2$ be his induced action in the game Λ_2 .

Let S' be a SE in Λ_1 that minimizes the cost of players in T_2 (such a SE exists by the induction hypothesis and Lemma 3.2). Let S'' be any SE in Λ_2 . We will show that the profile $S = S' \cup S''$ is a SE in the game Λ , i.e., for player $j \in T_2$ we use the profile $S_j = S'_j \cup S''_j$.

Suppose by contradiction that S is not a SE. Then, there is a coalition Γ that can deviate such that the cost of every player $j \in \Gamma$ decreases. Now, there are two cases:

Case 1: $\Gamma \subseteq T_1$. This is a contradiction to S' being a SE.

Case 2: There exists a player $j \in \Gamma \cap T_2$. By Lemma 3.2, player j cannot improve his cost in Λ_1 so the improvement is due to Λ_2 . Consider the coalition $\Gamma \cap T_2$, it would still improve its cost. However, this contradicts the fact that S'' is a SE in Λ_2 .

parallel composition. Follows from Lemma 3.3. □

While multi-commodity fair connection games on series parallel graphs do not necessarily possess a SE (see Theorem 3.6), fair connection games on extension parallel graphs always possess a strong equilibrium.

THEOREM 3.5. *Every fair connection game on an extension parallel graph possesses a strong equilibrium.*

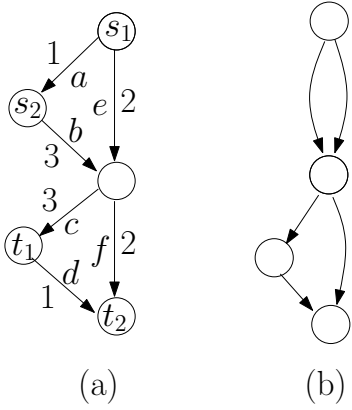


Figure 1: Graph topologies.

PROOF. We prove the theorem by induction on the network size $|V|$. Let Λ be a fair connection game on an EPG $G = (V, E)$. The claim obviously holds if $|V| = 2$. If the graph G is a parallel composition of two EPG graphs G_1 and G_2 , then the claim follows from Lemma 3.3. It remains to prove the claim for extension composition. Suppose the graph G is an extension composition of the graph G_1 consisting of a single edge $e = (s_1, t_1)$ and an EPG $G_2 = (V_2, E_2)$ with terminals s_2, t_2 , such that $s = s_1$ and $t = t_2$. (The case that G_2 is a single edge is similar.)

Let T_1 be the set of players with source s_1 and sink t_1 (i.e., their path is in G_1). Let T_2 be the set of players with source and sink in G_2 . Let T_3 be the set of players with source s_1 and sink in $V_2 \setminus t_1$.

Let Λ_1 and Λ_2 be the original game on the respective graphs G_1 and G_2 with players $T_1 \cup T_3$ and $T_2 \cup T_3$, respectively. Let S', S'' be SE in Λ_1 and Λ_2 respectively. We will show that the profile $S = S' \cup S''$ is a SE in the game Λ . Suppose by contradiction that S is not a SE. Then, there is a coalition Γ of minimal size that can deviate such that the cost of any player $j \in \Gamma$ decreases. Clearly, $T_1 \cap \Gamma = \emptyset$, since players in T_1 have a single strategy. Hence, $\Gamma \subseteq T_2 \cup T_3$. Any player $j \in T_2 \cup T_3$ cannot improve his cost in Λ_1 . Therefore, any player $j \in T_2 \cup T_3$ improves his cost in Λ_2 . However, this contradicts the fact that S'' is a SE in Λ_2 . \square

In the following theorem we provide a few examples of topologies in which a strong equilibrium does not exist, showing that our characterization is almost tight.

THEOREM 3.6. *The following connection games exist: (1) There exists a multi-commodity fair connection game on a series parallel graph that does not possess a strong equilibrium. (2) There exists a single source fair connection game that does not possess a strong equilibrium.*

PROOF. For claim (1) consider the graph depicted in Figure 1(a). This game has a unique NE where $S_1 = \{e, c\}$, $S_2 = \{b, f\}$, and each player has a cost of 5.⁷ However, consider the following coordinated deviation S' . $S'_1 = \{a, b, c\}$,

⁷In any NE of the game, player 1 will buy the edge e and player 2 will buy the edge f . This is since the alternate path, in the respective part, will cost the player 2.5. Thus, player 1 (player 2) will buy the edge c (edge b) alone, and each player will have a cost of 5.

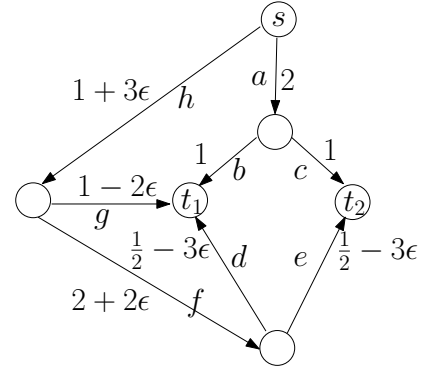


Figure 2: Example of a single source connection game that does not admit SE.

and $S'_2 = \{b, c, d\}$. In this profile, each player pays a cost of 4, and thus improves its cost.

For claim (2) consider a single source fair connection game on the graph G depicted in Figure 2. There are two players. Player $i = 1, 2$ wishes to connect the source s to its sink t_i and the unique NE is $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, and each player has a cost of 2.⁸ Then, both players can deviate to $S'_1 = \{h, f, d\}$ and $S'_2 = \{h, f, e\}$, and decrease their costs to $2 - \epsilon/2$. \square

Unfortunately, our characterization is not completely tight. The graph in Figure 1(b) is an example of a non-extension parallel graph which always admits a strong equilibrium.

3.2 Strong Price of Anarchy

While the price of anarchy in fair connection games can be as bad as n , the following theorem shows that the strong price of anarchy is bounded by $H(n) = \sum_{i=1}^n \frac{1}{i} = \Theta(\log n)$.

THEOREM 3.7. *The strong price of anarchy of a fair connection game with n players is at most $H(n)$.*

PROOF. Let Λ be a fair connection game on the graph G . We denote by $\Lambda(\Gamma)$ the game played on the graph G by a set of players Γ , where the action of player $i \in \Gamma$ remains Σ_i (the same as in Λ). Let $S = (S_1, \dots, S_n)$ be a profile in the game Λ . We denote by $S(\Gamma) = S_\Gamma$ the induced profile of players in Γ in the game $\Lambda(\Gamma)$. Let $n_e(S(\Gamma))$ denote the load of edge e under the profile $S(\Gamma)$ in the game $\Lambda(\Gamma)$, i.e., $n_e(S(\Gamma)) = |\{j | j \in \Gamma, e \in S_j\}|$. Similar to congestion games [16, 13] we denote by $\Phi(S(\Gamma))$ the potential function of the profile $S(\Gamma)$ in the game $\Lambda(\Gamma)$, where $\Phi(S(\Gamma)) = \sum_{e \in E} \sum_{j=1}^{n_e(S(\Gamma))} f_e(j)$, and define $\Phi(S(\emptyset)) = 0$. In our case, it holds that

$$\Phi(S) = \sum_{e \in E} c_e \cdot H(n_e(S)). \quad (1)$$

Let S be a SE, and let S^* be the profile of the optimal solution. We define an order on the players as follows. Let $\Gamma_n = \{1, \dots, n\}$ be the set of all the players. For each $k =$

⁸We can show that this is the unique NE by a simple case analysis: (i) If $S_1 = \{h, f, d\}$ and $S_2 = \{h, f, e\}$, then player 1 can deviate to $S'_1 = \{h, g\}$ and decrease his cost. (ii) If $S_1 = \{h, g\}$ and $S_2 = \{h, f, e\}$, then player 2 can deviate to $S'_2 = \{a, c\}$ and decrease his cost. (iii) If $S_1 = \{h, g\}$ and $S_2 = \{a, c\}$, then player 1 can deviate to $S'_1 = \{a, b\}$ and decrease his cost.

$n, \dots, 1$, since S is a SE, there exists a player in Γ_k , w.l.o.g. call it player k , such that,

$$c_k(S) \leq c_k(S_{-\Gamma_k}, S_{\Gamma_k}^*). \quad (2)$$

In this way, Γ_k is defined recursively, such that for every $k = n, \dots, 2$ it holds that $\Gamma_{k-1} = \Gamma_k \setminus \{k\}$. (I.e., after the renaming, $\Gamma_k = \{1, \dots, k\}$.)

Let $c_k(S(\Gamma_k))$ denote the cost of player k in the game $\Lambda(\Gamma_k)$ under the induced profile $S(\Gamma_k)$. It is easy to see that $c_k(S(\Gamma_k)) = \Phi(S(\Gamma_k)) - \Phi(S(\Gamma_{k-1}))$.⁹ Therefore,

$$\begin{aligned} c_k(S) &\leq c_k(S_{-\Gamma_k}, S_{\Gamma_k}^*) \\ &\leq c_k(S^*(\Gamma_k)) = \Phi(S^*(\Gamma_k)) - \Phi(S^*(\Gamma_{k-1})). \end{aligned} \quad (3)$$

Summing over all players, we obtain:

$$\begin{aligned} \sum_{i \in N} c_i(S) &\leq \Phi(S^*(\Gamma_n)) - \Phi(S^*(\phi)) \\ &= \Phi(S^*(\Gamma_n)) = \sum_{e \in S^*} c_e \cdot H(n_e(S^*)) \\ &\leq \sum_{e \in S^*} c_e \cdot H(n) = H(n) \cdot OPT(\Lambda), \end{aligned}$$

where the first inequality follows since the sum of the right hand side of equation (3) telescopes, and the second equality follows from equation (1). \square

Next we bound the SPoA when coalitions of size at most k are allowed.

THEOREM 3.8. *The k -SPoA of a fair connection game with n players is at most $\frac{n}{k} \cdot H(k)$.*

PROOF. Let S be a SE of Λ , and S^* be the profile of the optimal solution of Λ . To simplify the proof, we assume that n/k is an integer. We partition the players to n/k groups $T_1, \dots, T_{n/k}$ each of size k . Let Λ_j be the game on the graph G played by the set of players T_j . Let $S(T_j)$ denote the profile of the k players in T_j in the game Λ_j induced by the profile S of the game Λ . By Theorem 3.7, it holds that for each game Λ_j , $j = 1, \dots, n/k$,

$$\begin{aligned} cost_{\Lambda_j}(S(T_j)) &= \sum_{i \in T_j} c_i(S(T_j)) \\ &\leq H(k) \cdot OPT(\Lambda_j) \leq H(k) \cdot OPT(\Lambda). \end{aligned}$$

Summing over all games Λ_j , $j = 1, \dots, n/k$,

$$cost_{\Lambda}(S) \leq \sum_{j=1}^{n/k} cost_{\Lambda_j}(S(T_j)) \leq \frac{n}{k} \cdot H(k) \cdot OPT(\Lambda),$$

where the first inequality follows since for each group T_j and player $i \in T_j$, it holds that $c_i(S) \leq c_i(S(T_j))$. \square

Next we show an almost matching lower bound. (The lower bound is at most $H(n) = O(\log n)$ from the upper bound and both for $k = O(1)$ and $k = \Omega(n)$ the difference is only a constant.)

THEOREM 3.9. *For fair connection games with n players, k -SPoA $\geq \max\{\frac{n}{k}, H(n)\}$.*

⁹This follows since for any strategy profile S , if a single player k deviates to strategy S'_k , then the change in the potential value $\Phi(S) - \Phi(S'_k, S_{-k})$ is exactly the change in the cost to player k .

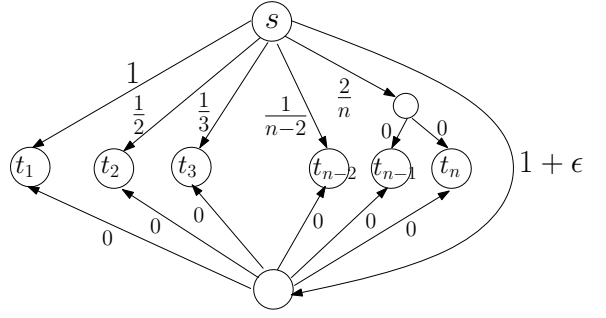


Figure 3: Example of a network topology in which $SPoS > PoS$.

PROOF. For the lower bound of $H(n)$ we observe that in the example presented in [2], the unique Nash equilibrium is also a strong equilibrium, and therefore k -SPoA = $H(n)$ for any $1 \leq k \leq n$. For the lower bound of n/k , consider a graph composed of two parallel links of costs 1 and n/k . Consider the profile S in which all n players use the link of cost n/k . The cost of each player is $1/k$, while if any coalition of size at most k deviates to the link of cost 1, the cost of each player is at least $1/k$. Therefore, the profile S is a k -SE, and k -SPoA = n/k . \square

The results of Theorems 3.7 and 3.8 can be extended to concave cost functions. Consider the extended fair connection game, where each edge has a cost which depends on the number of players using that edge, $c_e(n_e)$. We assume that the cost function $c_e(n_e)$ is a nondecreasing, concave function. Note that the cost of an edge $c_e(n_e)$ might increase with the number of players using it, but the cost per player $f_e(n_e) = c_e(n_e)/n_e$ decreases when $c_e(n_e)$ is concave.

THEOREM 3.10. *The strong price of anarchy of a fair connection game with nondecreasing concave edge cost functions and n players is at most $H(n)$.*

PROOF. The proof is analogous to the proof of Theorem 3.7. For the proof we show that $cost(S) \leq \Phi(S^*) \leq H(n) \cdot cost(S^*)$. We first show the first inequality. Since the function $c_e(x)$ is concave, the cost per player $c_e(x)/x$ is a nonincreasing function. Therefore inequality (3) in the proof of Theorem 3.7 holds. Summing inequality (3) over all players we obtain $cost(S) = \sum_i c_i(S) \leq \Phi(S^*(\Gamma_n)) - \Phi(S^*(\phi)) = \Phi(S^*)$. The second inequality follows since $c_e(x)$ is nondecreasing and therefore $\sum_{x=1}^{n_e} (c_e(x)/x) \leq H(n_e) \cdot c_e(n_e)$. \square

Using the arguments in the proof of Theorem 3.10 and the proof of Theorem 3.8 we derive,

THEOREM 3.11. *The k -SPoA of a fair connection game with nondecreasing concave edge cost functions and n players is at most $\frac{n}{k} \cdot H(k)$.*

Since the set of strong equilibria is contained in the set of Nash equilibria, it must hold that $SPoA \leq PoA$, meaning that the SPoA can only be improved compared to the PoA. However, with respect to the price of stability the opposite direction holds, that is, $SPoS \geq PoS$. We next show that there exists a fair connection game in which the inequality is strict.

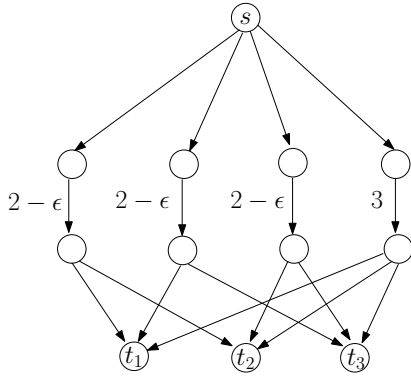


Figure 4: Example of a single source general connection game that does not admit a strong equilibrium. The edges that are not labeled with costs have a cost of zero.

THEOREM 3.12. *There exists a fair connection game in which $SPoS > PoS$.*

PROOF. Consider a single source fair connection game on the graph G depicted in Figure 3.¹⁰ Player $i = 1, \dots, n$ wishes to connect the source s to his sink t_i . Assume that each player $i = 1, \dots, n-2$ has his own path of cost $1/i$ from s to t_i and players $i = n-1, n$ have a joint path of cost $2/n$ from s to t_i . Additionally, all players can share a common path of cost $1 + \epsilon$ for some small $\epsilon > 0$. The optimal solution connects all players through the common path of cost $1 + \epsilon$, and this is also a Nash equilibrium with total cost $1 + \epsilon$. It is easy to verify that the solution where each player $i = 1, \dots, n-2$ uses his own path and users $i = n-1, n$ use their joint path is the unique strong equilibrium of this game with total cost $\sum_{i=1}^{n-2} \frac{1}{i} + \frac{2}{n} = \Theta(\log n)$ \square

While the example above shows that the SPoS may be greater than the PoS, the upper bound of $H(n) = \Theta(\log n)$, proven for the PoS [2], serves as an upper bound for the SPoS as well. This is a direct corollary from theorem 3.7, as $SPoS \leq SPoS$ by definition.

COROLLARY 3.13. *The strong price of stability of a fair connection game with n players is at most $H(n) = O(\log n)$.*

4. GENERAL CONNECTION GAMES

In this section, we derive our results for general connection games.

4.1 Existence of Strong Equilibrium

We begin with a characterization of the existence of a strong equilibrium in symmetric general connection games. Similar to Theorem 3.1 (using a similar proof) we establish,

THEOREM 4.1. *In every symmetric fair connection game there exists a strong equilibrium.*

While every single source general connection game possesses a pure Nash equilibrium [3], it does not necessarily admit some strong equilibrium.¹¹

¹⁰This is a variation on the example given in [2].

¹¹We thank Elliot Anshelevich, whose similar topology for the fair-connection game inspired this example.

THEOREM 4.2. *There exists a single source general connection game that does not admit any strong equilibrium.*

PROOF. Consider single source general connection game with 3 players on the graph depicted in Figure 4. Player i wishes to connect the source s with its sink t_i . We need to consider only the NE profiles: (i) if all three players use the link of cost 3, then there must be two agents whose total sum exceeds 2, thus they can both reduce cost by deviating to an edge of cost $2 - \epsilon$. (ii) if two of the players use an edge of cost $2 - \epsilon$ jointly, and the third player uses a different edge of cost $2 - \epsilon$, then, the players with non-zero payments can deviate to the path with the edge of cost 3 and reduce their costs (since before the deviation the total payments of the players is $4 - 2\epsilon$). We showed that none of the NE are SE, and thus the game does not possess any SE. \square

Next we show that for the class of series parallel graphs, there is always a strong equilibrium in the case of a single source.

THEOREM 4.3. *In every single source general connection game on a series-parallel graph, there exists a strong equilibrium.*

PROOF. Let Λ be a single source general connection game on a SPG $G = (V, E)$ with source s and sink t . We present an algorithm that constructs a specific SE. We first consider the following partial order between the players. For players i and j , we have that $i \mapsto j$ if there is a directed path from t_i to t_j . We complete the partial order to a full order (in an arbitrary way), and w.l.o.g. we assume that $1 \mapsto 2 \mapsto \dots \mapsto n$.

The algorithm **COMPUTE-SE**, considers the players in an increasing order, starting with player 1. Each player i will fully buy a subset of the edges, and any player $j > i$ will consider the cost of those (bought) edges as zero. When **COMPUTE-SE** considers player j , the cost of the edges that players 1 to $j-1$ have bought is set to zero, and player j fully buys a shortest path Q^j from s to t_j . Namely, for every edges $e \in Q^j \setminus \cup_{i < j} Q^i$ we have $p_j(e) = c_e$ and otherwise $p_j(e) = 0$. We next show that the algorithm **COMPUTE-SE** computes a SE.

Assume by way of contradiction that the profile p is not a SE. Then, there exists a coalition that can improve the costs of all its players by a deviation. Let Γ be such a coalition of minimal size and let player $i = \max\{j \in \Gamma\}$. For a player $j \in \Gamma$ let \bar{Q}^j and \bar{p}^j be the path and payment of player j after the deviation, respectively. Let Q' be a path from the sink of player i , i.e. t_i , to the sink of G , i.e. t . Then $Q = \bar{Q}^i \cup Q'$ is a path from the source s to the sink t . For any player $j < i$, let y_j be the intersecting vertex of Q and t_j (by Lemma 2.1 one is guarantee to exist). Let y be the furthest vertex on the path Q such that $y = y_j$ for some $j < i$. The path from the source s to node y was fully paid for by players $j < i$ in p (before the deviation). There are two cases we consider.

case a: After the deviation player i does not pay for edges in $\bigcup_{j \in \Gamma \setminus \{i\}} \bar{Q}^j$. This is a contradiction to the minimality of the coalition Γ size, since the players in $\Gamma \setminus \{i\}$ can form a smaller coalition with payments \bar{p} .

case b: Otherwise, we show that player i cost after the deviation, i.e. $c_i(\bar{p})$, is at least his cost before the deviation, i.e. $c_i(p)$, contradicting the fact that player i improved his cost. Recall that given two vertices u, v on path \bar{Q} we denote by $\bar{Q}_{u,v}$ the subpath of \bar{Q} from u to v .

Before the deviation of the coalition Γ , a path from s to y was fully paid for by the players $j < i$. Next we show that no player $k > i$ pays for any edge on any path from s to t_i . Consider a player $k > i$ and let $Q'_k = Q^k \cup Q''_k$, where Q''_k is a path connecting t_k to t . Let y_k be the intersecting vertex of Q'_k and t_i . Since there exists a path from s to y_k that was fully paid for by players $j < k$ before the deviation, in particular the path Q^i_{s,y_k} , player k will not pay for any edge on any path connecting s and y_k . Therefore player i fully pays for all edges on the path Q^i_{y,t_i} , i.e., $\bar{p}_i(e) = c_e$ for all edges $e \in Q^i_{y,t_i}$. Now consider the algorithm **COMPUTE-SE** at the step when player i selects a shortest path from the source s to its sink t_i and determines his payment p_i . At this point, player i could buy the path Q^i_{y,t_i} , since a path from s to y was already paid for by players $j < i$. Hence, $c_i(\bar{p}) \geq c_i(p)$. This contradicts the fact that player i improved its cost and therefore not all the players in Γ reduce their cost. This implies that p is a strong equilibrium. \square

4.2 Strong Price of Anarchy

While for every single source general connection game, it holds that $PoS = 1$ [3], the price of anarchy can be as large as n , even for two parallel edges. Here, we show that any strong equilibrium in single source general connection games yields the optimal cost.

THEOREM 4.4. *In single source general connection game, if there exists a strong equilibrium, then the strong price of anarchy is 1.*

PROOF. Let $p = (p_1, \dots, p_n)$ be a strong equilibrium, and let T^* be the minimum cost Steiner tree on all players, rooted at the (single) source s . Let T_e^* be the subtree of T^* disconnected from s when edge e is removed. Let $\Gamma(T_e)$ be the set of players which have sinks in T_e . For a set of edges E , let $c(E) = \sum_{e \in E} c_e$. Let $P(T_e) = \sum_{i \in \Gamma(T_e)} c_i(p)$.

Assume by way of contradiction that $c(p) > c(T^*)$. We will show that there exists a sub-tree T' of T^* , that connects a subset of players $\Gamma \subseteq N$, and a new set of payments \bar{p} , such that for each $i \in \Gamma$, $c_i(\bar{p}) < c_i(p)$. This will contradict the assumption that p is a strong equilibrium.

First we show how to find a sub-tree T' of T^* , such that for any edge e , the payments of players with sinks in T_e^* is more than the cost of $T_e^* \cup \{e\}$. To build T' , define an edge e to be *bad* if the cost of $T_e^* \cup \{e\}$ is at least the payments of the players with sinks in T_e^* , i.e., $c(T_e^* \cup \{e\}) \geq P(T_e^*)$. Let B be the set of bad edges. We define T' to be $T^* - \cup_{e \in B} (T_e^* \cup \{e\})$. Note that we can find a subset B' of B such that $\cup_{e \in B} (T_e^* \cup \{e\})$ is equal to $\cup_{e \in B'} (T_e^* \cup \{e\})$ and for any $e_1, e_2 \in B'$ we have $T_{e_1}^* \cap T_{e_2}^* = \emptyset$. (The set B' will include any edge $e \in B$ for which there is no other edge $e' \in B$ on the path from e to the source s .) Considering the edges in $e \in B'$ we can see that any subtree T_e^* we delete from T can not decrease the difference between the payments and the cost of the remaining tree. Therefore, in T' for every edge e , we have that $c(T_e' \cup \{e\}) < P(T_e')$.

Now we have a tree T' and our coalition will be $\Gamma(T')$. What remain is to find payments \bar{p} for the players in $\Gamma(T')$ such that they will buy the tree T' and every player in $\Gamma(T')$ will lower its cost, i.e. $c_i(p) > c_i(\bar{p})$ for $i \in \Gamma(T')$. (Recall that the payments have the restriction that player i can only pay for edges on the path from s to t_i .)

We will now define the coalition payments \bar{p} . Let $c_i(\bar{p}, T_e') = \sum_{e \in T_e'} \bar{p}_i(e)$ be the payments of player i for the sub-

tree T_e' . We will show that for every subtree T_e' , $c_i(\bar{p}, T_e' \cup \{e\}) < c_i(p)$, and hence $c_i(\bar{p}) < c_i(p)$. Consider the following bottom up process that defines \bar{p} . We assign the payments of edge e in T' , after we assign payments to all the edges in T_e' . This implies that when we assign payments for e , we have that the sum of the payments in T_e' is equal to $c(T_e') = \sum_{i \in \Gamma(T_e')} c_i(\bar{p}, T_e')$. Since e was not a bad edge, we know that $c(T_e' \cup \{e\}) = c(T_e') + c_e < P(T_e')$. Therefore, we can update the payments \bar{p} of players $i \in \Gamma(T_e')$, by setting $\bar{p}_i(e) = c_e \Delta_i / (\sum_{j \in \Gamma(T_e')} \Delta_j)$, where $\Delta_j = c_j(p) - c_j(\bar{p}, T_e')$. After the update we have for player $i \in \Gamma(T_e')$,

$$\begin{aligned} c_i(\bar{p}, T_e' \cup \{e\}) &= c_i(\bar{p}, T_e') + \bar{p}_i(e) \\ &= c_i(\bar{p}, T_e') + \Delta_i \frac{c_e}{\sum_{j \in \Gamma(T_e')} \Delta_j} \\ &= c_i(p) - \Delta_i \left(1 - \frac{c_e}{P(\Gamma(T_e')) - c(T_e')}\right), \end{aligned}$$

where we used the fact that $\sum_{j \in \Gamma(T_e')} \Delta_j = P(\Gamma(T_e')) - c(T_e')$. Since $c_e < P(\Gamma(T_e')) - c(T_e')$ it follows that $c_i(\bar{p}, T_e' \cup \{e\}) < c_i(p)$. \square

5. REFERENCES

- [1] N. Andelman, M. Feldman, and Y. Mansour. Strong Price of Anarchy. In *SODA '07*, 2007.
- [2] E. Anshelevich, A. Dasgupta, J. M. Kleinberg, É. Tardos, T. Wexler, and T. Roughgarden. The price of stability for network design with fair cost allocation. In *FOCS*, pages 295–304, 2004.
- [3] E. Anshelevich, A. Dasgupta, E. Tardos, and T. Wexler. Near-Optimal Network Design with Selfish Agents. In *STOC'03*, 2003.
- [4] R. Aumann. Acceptable Points in General Cooperative n-Person Games. In *Contributions to the Theory of Games*, volume 4, 1959.
- [5] A. Czumaj and B. Vöcking. Tight bounds for worst-case equilibria. In *SODA*, pages 413–420, 2002.
- [6] A. Fabrikant, A. Luthra, E. Maneva, C. Papadimitriou, and S. Shenker. On a network creation game. In *ACM Symposium on Principles of Distributed Computing (PODC)*, 2003.
- [7] R. Holzman and N. Law-Yone. Strong equilibrium in congestion games. *Games and Economic Behavior*, 21:85–101, 1997.
- [8] R. Holzman and N. L.-Y. (Lev-tov). Network structure and strong equilibrium in route selection games. *Mathematical Social Sciences*, 46:193–205, 2003.
- [9] E. Koutsoupias and C. H. Papadimitriou. Worst-case equilibria. In *STACS*, pages 404–413, 1999.
- [10] I. Milchtaich. Topological conditions for uniqueness of equilibrium in networks. *Mathematics of Operations Research*, 30:225244, 2005.
- [11] I. Milchtaich. Network topology and the efficiency of equilibrium. *Games and Economic Behavior*, 57:321346, 2006.
- [12] I. Milchtaich. The equilibrium existence problem in finite network congestion games. *Forthcoming in Lecture Notes in Computer Science*, 2007.
- [13] D. Monderer and L. S. Shapley. Potential Games. *Games and Economic Behavior*, 14:124–143, 1996.
- [14] H. Moulin and S. Shenker. Strategyproof sharing of

submodular costs: Budget balance versus efficiency.
Economic Theory, 18(3):511–533, 2001.

- [15] C. Papadimitriou. Algorithms, Games, and the Internet. In *Proceedings of 33rd STOC*, pages 749–753, 2001.
- [16] R. W. Rosenthal. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.
- [17] T. Roughgarden. The Price of Anarchy is Independent of the Network Topology. In *STOC'02*, pages 428–437, 2002.
- [18] T. Roughgarden and E. Tardos. How bad is selfish routing? *Journal of the ACM*, 49(2):236 – 259, 2002.
- [19] O. Rozenfeld and M. Tennenholtz. Strong and correlated strong equilibria in monotone congestion games. In *Workshop on Internet and Network Economics*, 2006.