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Variations on the Hotelling-Downs Model

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Abstract

In this paper we expand the standard Hotelling-Downs model (Hotelling 1929; Downs 1957) of spatial competition to a setting where clients do not necessarily choose their closest candidate (retail product or political). Specifically, we consider a setting where clients may disavow all candidates if there is no candidate that is sufficiently close to the client preferences. Moreover, if there are multiple candidates that are sufficiently close, the client may choose amongst them at random. We show the existence of Nash Equilibria for some such models, and study the price of anarchy and stability in such scenarios.

1 Introduction

A toy problem illustrating the Hotelling-Downs model is the strategic positioning of two ice cream vendors along a beach front (Hotelling 1929). The model was later extended to ideological positioning in a bi-partisan democracy (Downs 1957) (See also Enelow and Hinich 1984). The model has gained a significant following, since it agrees with a key aspect of such competitions: the median placement policy. In the case of two vendors this explains why they've clumped together on a stretch of a beach, and why opposing political parties frequently agree on terms that ultimately express the interests of neither.

Unfortunately, the model introduced several assumptions that have resulted in its failure to explain the behavior of higher numbers of competitors, and limited the model's applicability. Osborne (1993) shows that the model does not, in general, admit Nash Equilibria. Variants of the model, such as allowing candidates to quit (see e.g. Sengupta and Sengupta 2008) and runoff voting protocols (see e.g. Brusco et al 2012) do admit pure Nash equilibria for more than 2 competitors.

In the original model and in the variants above, some of the assumptions may be problematic. Consider the ice cream vendor problem. The Hotelling-Downs model would have us assume that — no matter what the distance of the vendor from a client — the latter would travel this distance given that this is the closest vendor. This may hold in some cases, for lack of other alternatives, but this may be a significant problem in voting. A voter may compromise his beliefs, but only to a limit. If no competing candidate is close enough, the voter may simply abstain.

Similarly, the assumption that voters always choose the candidate closest to their political views may be questionable. Should two or more competing candidates be close enough in their political stance, a voter may see little difference between the candidates. This is also true for the proverbial sunbather seeking ice cream. If both vendors are close enough, the choice of vendor may be random.

In this paper we study a variant of the Hotelling-Downs model, modifying the core assumptions:

- All competitors (called *agents* hereinafter) have a limited attraction interval. Only clients that lie within this interval may support the agent (buy the product, cast a vote, etc.);
- The support of clients that fall in the attraction interval of several agents is randomly shared among the latter.

We consider two utility functions for the agents: *"winner-takes-all"* — this is reminiscent of a political voting situation (e.g. the winner forms the new government); and *"support maximizers"* — this is a model for commercial competition (*i.e.*, agents seek to maximize their number of customers).

One might suspect that these new variants will also suffer from lack of Nash equilibria. However, this is not the case. Our existence theorem shows that a pure Nash Equilibrium always exists.

With respect to these pure Nash Equilibria, we study client participation. Different equilibria may have a different number of clients that take part. Surprisingly, there are equilibria that do not cover the population completely, even if the number of agents grows to infinity.

Translated into economic terms, there are stable situations, where some niche clientele will not be serviced by any agent. Or, in political terms, there may be a group of voters that does not participate in the political process. Contrariwise, we also find that in low-intensity competitions (*i.e.*, where the number of agents is less than four) the level of client participation is at least 50% of the maximum.

The structure of this paper is as follows:

- In Section 2 we introduce a general framework, based on the Hotelling-Downs model, that captures the limited at-

*The work was done while the author was affiliated with Tel-Aviv University traction of agents (be they vendors, political parties, or other competitors) and sharing of support (where the same client has several agents she is attracted to);

- We analyze the resulting model and show that pure Nash equilibria always exist (see Section 3);
- With respect to the measure of client participation, we consider bounds on the "Price of Anarchy" (PoA). We compare the least possible client participation under equilibrium with the best possible participation level. We provide detailed bounds as a function of the size of the attraction interval of an agent, and the agent utility function. Sections 4 and 5 study the "winner takes all" utility, while Sections 6 and 7 address the "support maximization" utility of an agent.
- Finally, our analysis allows us to compare the effects of the two utility functions. In particular, we show that, as the number of agents (be they traders or political parties) increases, the "support maximizers" utility is better suited to guarantee higher participation levels (assuming agents in equilibria).

We conclude with a set of possible future developments of our framework (Section 8). Due to space limitations, we only keep those proofs that are illustrative of our techniques, and omit all others.

2 Model and preliminaries

Consider a setting where a continuum of clients are distributed along the interval $[0, 1]$ according to a known density function $f(x)$. A client is represented by a point $x \in [0, 1]$, denoting her preference along the interval $[0, 1]$. Clients are non-strategic, and their position is drawn from a publicly known distribution.

The strategic interaction occurs among n agents, with indices $1, \ldots, n$. The set of actions for an agent is to choose a center in the interval $(0, 1]$. Let $c_i \in (0, 1]$ be the action of agent i, $1 \leq i \leq n$.

Given c_i, this determines the *attraction interval* for agent i: $[c_i - \frac{w}{2}, c_i + \frac{w}{2}]$ — an interval of width w centered around c_i (in this paper we assume that all agents have the same width w). A joint action profile is given by a vector of such "centers" $\mathbf{c} = (c_1, \ldots, c_n)$.

Every client is *attracted* to all agents whose attraction intervals contain x. *I.e.*, client x is attracted to agent i if $x \in (c_i - \frac{w}{2}, c_i + \frac{w}{2}]$. Let I_x be the set of agents that attract client x, *i.e.*,

$$I_x = \{1 \leq i \leq n \,|x \in (c_i - \frac{w}{2}, c_i + \frac{w}{2}]\}.$$

In our model, a client that is attracted to several agents divides her support equally amongst them. *I.e.*, for every agent i and client x, the attraction of x to i is given by

$$a_{x,i} = \begin{cases} 1/|I_x| & i \in I_x \\ 0 & otherwise \end{cases} .$$

Given a vector of agents with joint action profile $\mathbf{c}$ and a density function $f(x)$, the total *support* of agent $1 \leq i \leq n$ is defined to be

$$n_i(\mathbf{c}) = \int_{c_i - \frac{w}{2}}^{c_i + \frac{w}{2}} a_{x,i} f(x) dx.$$

Agents derive *utility* from the support they receive from clients, and locate their centers strategically to maximize their utility. We consider two different agent utility functions:

1. The *winner takes all* utility corresponds to a setting where only agents with maximal support derive non-zero utility. The utility of agent i, denoted u_i^W, splits the payoff equally amongst all agents with maximal support. *I.e.*, let the set of *winners* be
$$W(\mathbf{c}) = \arg\max_{i \in N} n_i(\mathbf{c}).$$
Then, the "winner takes all" utility of agent i is given by
$$u_i^W(\mathbf{c}) = \begin{cases} \frac{1}{|W(\mathbf{c})|}, & i \in W(\mathbf{c}); \\ 0, & otherwise. \end{cases} .$$
2. The *support maximizers* utility corresponds to settings where the utility of every agent i (denoted by u_i^S) is the total support it receives:
$$u_i^S(\mathbf{c}) = n_i(\mathbf{c}).$$
Note that some clients may not be attracted to any agent; therefore, $\sum_{i \in N} n_i(\mathbf{c})$ might be strictly smaller than 1 (and is at most 1).

Consider now the *participation rate*, *i.e.*, the fraction of clients that are attracted to at least one agent.

$$P(\mathbf{c}) = \int_0^1 \sum_{i \in N} a_{x,i} f(x) dx.$$

One may argue that belonging to at least one attraction interval implies that a client has access to a desired service, or seeks to participate in the political process. This may be viewed as one measure of public welfare. Moreover, the participation rate measures how the set of agents, as a whole, is relevant to the market or the political system that they inhabit.

With this is mind, the *participation rate* is our objective function. (Unless stated otherwise, we assume that the distribution of clients on the $[0, 1]$ interval is uniform.)

For illustration consider the setting in Figure 1. There are 3 agents, whose action profile is given by $(c_1, c_2, c_3) = (0.3, 0.4, 0.55)$ (appear above the interval for clarity). In this example, clients located in the interval $[0.1, 0.2)$ support agent 1 exclusively, those located in the interval $[0.2, 0.35]$ support both agents 1 and 2, those located in the interval $[0.35, 0.5]$ support all three agents, etc. It holds that $I_x = \{1, 2, 3\}$; therefore, $a_{x,1} = a_{x,2} = a_{x,3} = 1/3$. Similarly, $I_y = \{3\}$; thus $a_{y,1} = a_{y,2} = 0$ and $a_{y,3} = 1$. Finally, $I_z = \emptyset$, so $a_{z,1} = a_{z,2} = a_{z,3} = 0$.

In this example, the total support of agents $1 \leq i \leq 3$ are

$$\begin{aligned} n_1(\mathbf{c}) &= 0.1 + 0.15 \cdot \frac{1}{2} + 0.15 \cdot \frac{1}{3} = 0.225 \\ n_2(\mathbf{c}) &= 0.15 \cdot \frac{1}{2} + 0.15 * \frac{1}{3} + 0.1 \cdot \frac{1}{2} = 0.175 \\ n_3(\mathbf{c}) &= 0.15 \cdot \frac{1}{3} + 0.1 \cdot \frac{1}{2} + 0.15 = 0.25. \end{aligned}$$

Consequently, agent 3 is the unique winner. Note that agents located in $[0, 0.1) \cup (0.75, 1]$ are in the attraction interval of no agent; consequently, the participation rate is $P(\mathbf{c}) = 0.65$.

To summarize, a game is fully defined by the number of agents (n), the width of the attraction intervals (w), and the appropriate utility function (winner takes all vs. total support). Let $G^W = G^W(n, w)$ be the game under winner takes all utilities, and let $G^S = G^S(n, w)$ be the game under total support utilities. (We omit the superscript W or S if the relevant utility function is clear from the context, or if the statement is true for both utility functions.)

A *Nash equilibrium* (NE) of a game is an action profile $\mathbf{c}$ such that no agent can increase her utility by a unilateral deviation. We denote the sets of Nash equilibria in the winner takes all and the total support games by NE^W and NE^S, respectively. In cases where the utility function is clear from the context we omit the superscript and denote the utility function and the set of NE by u and NE, respectively.

As stated above, the objective function of interest is the participation rate $P(\mathbf{c})$. There is no reason to assume that a Nash equilibria of the agents (of whatever utility function) will optimize the participation. It is common to quantify the efficiency loss by the *price of anarchy* (Koutsoupias and Papadimitriou 1999; Nisan et al. 2007). In our context, we define this to be the ratio between the participation ratio of the worst case Nash equilibrium and the maximal participation ratio attainable by any positioning of the agents. We define the price of anarchy,

$$PoA(G) = \frac{\min_{\mathbf{c} \in NE(G)} P(\mathbf{c})}{\max_{\mathbf{c}} P(\mathbf{c})}.$$

The price of anarchy with respect to the winner takes all and the support maximizers utilities will be denoted as PoA^W and POA^S, respectively. We slightly abuse the notation as follows: when discussing a range of games with a fixed number of agents, n, and variable attraction interval width w, we will write $PoA(w) = PoA(G(n, w))$.

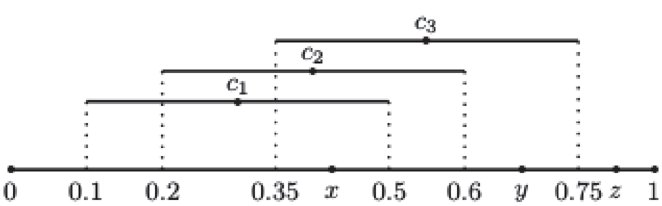

Figure 1: Example: Attraction intervals

3 Existence of Nash Equilibrium

As mentioned above, many extensions of the Hotelling-Downs model have instances without pure Nash equlibria. We first show that both variants of our model have no such shortcoming. The "winner-takes-all" utility function is neither continuous in agent strategies nor in the density function, and one may possibly suspect that no pure Nash equilibria exist. Contrary to such intuition, our next two theorems show that pure Nash equilibria always exist for both utilities.

Theorem 3.1. *For any pair (n, w), the game $G^W(n, w)$ has a pure Nash equilibrium.*

Theorem 3.2. *For any pair (n, w) game $G^S(n, w)$ has a pure Nash equilibrium.*

Having demonstrated that pure NEs always exist, the price of anarchy with respect to pure strategies is well defined. In the following sections we study the price of anarchy as a function of the attraction interval size w and the number of agents n.

4 Price of anarchy, 2 and 3 agents, "winner takes all" utilities

Arguably, the simplest and most natural setting is two agents, winner takes all. Indeed, the original Hotelling-Downs model was presented for two agents.

Proposition 4.1. *In the case of $n = 2$, the price of anarchy with respect to winner takes all utilities is given by*

$$PoA^W(w) = \begin{cases} \frac{1}{2}, & w \leq \frac{1}{2}; \\ w, & otherwise. \end{cases}.$$

In the case of three agents, the price of anarchy changes dramatically and gives rise to complex patterns as a function of the attraction interval size. These patterns are given in the following theorem, and depicted in Figure 2.

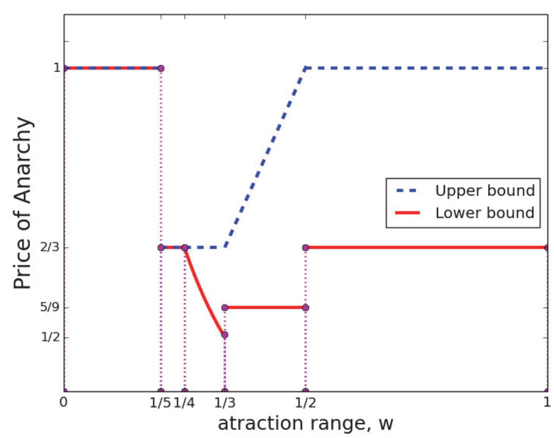

Figure 2: Theorem 4.2: PoA bounds as a function of w

Theorem 4.2. *In the case of $n = 3$, the price of anarchy with respect to winner takes all utilities, as a function of the attraction interval W, is given in the following table.*

w range	*PoA: Lower bound*	*PoA: Upper bound*
$w \leq 1/5$	1	1
$1/5 < w \leq 1/4$	$2/3$	$2/3$
$1/4 < w \leq 1/3$	$1/(6w)$	$2/3$
$1/3 < w \leq 1/2$	$5/9$	$2w$
$w > 1/2$	$2/3$	1

Proof. The proof is via case analysis, one for every row in the table. In what follows we only consider the 4^{th} row of the table. *I.e.*, we show that for $1/3 < w \leq 1/2$, the price of anarchy with respect to the winner takes all utility is at least $5/9$ and at most $2w$.

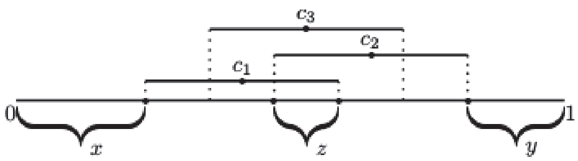

Figure 3: Illustration for Theorem 4.2, case $\frac{1}{3} < w \leq \frac{1}{2}$.

We first show that $PoA^W(w) \geq \frac{5}{9}$. For $w > 1/3$ and 3 agents, one can easily achieve full participation by an attraction vector $(\frac{1}{6}, \frac{1}{2}, \frac{5}{6})$. Therefore, the price of anarchy is simply the minimal participation rate over all Nash equilibria.

Notice also that it always holds that $PoA(w) \geq w$ since the lowest participation is obtained in the case where all attraction intervals coincide, in which case the participation is exactly w.

Let us now assume that there is an equilibrium attraction vector $\mathbf{c}'$ so that $PoA^W(w) \leq P(\mathbf{c}') < \frac{5}{9}$, and achieve a contradiction. Without loss of generality, we may assume that agents are numbered so that c_1 is the left-most agent and c_2 is the right-most agent. Notice that the attraction intervals of these two agents must intersect (see Figure 3 for an illustration), otherwise they cover more than $\frac{5}{9}$ of the client population. Furthermore, all the clients that are attracted to agent 3 must also be attracted to either agent 1, agent 2, or both of them. Evidently, agent 3 cannot be a winner.

Denote the length of the left uncovered interval by x, and the length of the right uncovered interval by y (see Figure 3 for an illustration). One can easily verify that $x = c_1 - \frac{w}{2}$, and $y = 1 - c_2 - \frac{w}{2}$. Finally, denote the length of the intersection of the attraction intervals of agents 1 and 2 by z. We proceed by a case analysis, depending on the relative size of x, y and z.

Suppose $x > z, y > z$. We have assumed that $\mathbf{c}'$ is a NE, thus, agent 3 has no strategy that would make her a winner. In particular, agent 3 cannot become a winner by shifting its interval to begin at 0 or by shifting its interval to end at 1. This fact is formalized by the following inequalities:

$$x + \frac{w-x}{2} \leq \frac{z}{2} + w - z \qquad (1)$$

$$y + \frac{w-y}{2} \leq \frac{z}{2} + w - z \qquad (2)$$

Combining Inequalities 1 and 2 we obtain

$$2w - 2z \geq x + y.$$

We also know that $x + y + 2w - z = 1$, that is, $x + y = 1 - 2w + z$. Combining these equations we obtain:

$$\begin{aligned} 2w - 2z \geq x + y \quad &\Leftrightarrow \quad 2w - 2z \geq 1 - 2w + z \Leftrightarrow \qquad (3) \\ &\Leftrightarrow \quad 4w - 1 \geq 3z \\ &\Leftrightarrow \quad \frac{4w-1}{3} \geq z \end{aligned}$$

Thus,

$$P(c') = 2w - z \geq 2w - \frac{4w-1}{3} = \frac{2w+1}{3} \qquad (4)$$

By assumption, $w > \frac{1}{3}$. Hence, $P(c') \geq \frac{2w+1}{3} > \frac{5}{9}$, contradicting the assumption that $P(c') < \frac{5}{9}$.

Suppose $x \leq z, y \geq z$. Consider again an attempt by agent 3 to become a winner by shifting her interval to begin at 0. In this situation her interval will contain points which already belong to both intervals of agents 1 and 2. The intersection of all three intervals will be of size at least $z - x$. Combined with the fact that $\mathbf{c}'$ is a NE, this implies inequality 5 below. In a similar manner to the derivation of inequality 2, the fact that it is not beneficial for agent 3 to move her interval to the right end implies Inequality 6.

$$x + \frac{w-x}{2} + \frac{z-x}{3} \leq \frac{z-x}{3} + \frac{x}{2} + w - z \qquad (5)$$

$$y + \frac{w-y}{2} \leq \frac{z}{2} + w - z \Rightarrow w - z \geq y \geq z \Rightarrow \frac{w}{2} \geq z \qquad (6)$$

By the assumption, $z \geq x$, thus, using Inequality 6, we obtain $w \geq x + y$. Combining this inequality with the fact that $x + y + 2w - z = 1$, we obtain $3w - 1 \geq z$. In turn, this entails:

$$P(c') \geq \min_{\substack{3w-1 \geq z \\ w \geq 2z \\ w \leq \frac{5}{9}}} 2w - z = \min\{\frac{2}{3}, \frac{10}{9}, \frac{7}{9}, \frac{3}{5}\} > \frac{5}{9} \qquad (7)$$

Suppose $x < z, y < z$. Inequality 5 holds as before, and similarly we obtain the following inequality:

$$y + \frac{w-y}{2} + \frac{z-y}{3} \leq \frac{z-y}{3} + \frac{y}{2} + w - z \qquad (8)$$

Inequalities 5 and 8 imply that $w - z \geq x$ and $w - z \geq y$.

Recall that we have assumed that $P(c') < \frac{5}{9}$. Hence, $x + y = 1 - P(c') > \frac{4}{9}$, so that at least one of x, y is larger than $\frac{2}{9}$. Assume w.l.o.g. that $x > \frac{2}{9}$. Since we are currently under the assumption that $z > x$, it also holds that $z > \frac{2}{9}$. It follows, then, from $w - z \geq x$, that $w > \frac{4}{9}$. Hence,

$$2w - z = P(c') < \frac{5}{9} \Rightarrow z > \frac{3}{9}.$$

Combining $w - z \geq x$, $z > \frac{3}{9}$ and $x > \frac{2}{9}$ we obtain:

$$w \geq z + x > \frac{3}{9} + \frac{2}{9} = \frac{5}{9}.$$

However, this contradicts our case assumption that $w \leq \frac{1}{2}$.

We, therefore, conclude that $PoA(w) \geq \frac{5}{9}$. Furthermore, it is possible to show that this bound is tight.

Finally, we show an upper bound of $2w$ on the price of anarchy. Consider an attraction vector $\mathbf{c}$ of the form $c_1 = \frac{w}{2}$ and $c_2 = c_3 = 1 - \frac{w}{2}$. One can easily verify that this is a NE and its participation rate is $P(\mathbf{c}) = 2w$. □

5 Price of anarchy, n agents ($n > 3$), "winner takes all"

In this section we provide an asymptotic analysis of the price of anarchy, as n grows to infinity. Roughly speaking, we find that the price of anarchy is 1 for small w, then exhibits a sharp fall, and begins to get closer to 1 again as w grows (but stays bounded away from 1). This pattern is cast in the following theorems and lemmata.

We first show that the price of anarchy is 1 for sufficiently small attraction intervals and a modest number of agents.

Theorem 5.1. *Consider any pair n, w such that $w \leq \frac{1}{3}$ and $(2n+1)w \leq 1$. Then $PoA^W(w) = 1$.*

Lemma 5.2. *Consider any pair n, w such that $w \leq \frac{1}{3}$ and $(2n+1)w > 1$. Then*

$$PoA^W(w) < \begin{cases} \frac{1-w}{2\min(nw,1)}, & \frac{1-w}{2w} \notin \mathbb{Z}; \\ \frac{1+w}{2\min(nw,1)}, & \frac{1-w}{2w} \in \mathbb{Z}. \end{cases}.$$

Lemma 5.3. *Consider any pair n, w such that $\frac{1}{3} < w < \frac{1}{2}$ and $n \geq 7$. Then $PaA^W(w) \leq \frac{1+2w}{2}$.*

Moreover, as long as the attraction interval does not cover the majority of the client population, the price of anarchy is bounded away from 1. In fact, Lemmata 5.3 and 5.2 jointly imply the following theorem.

Theorem 5.4. *The price of anarchy is bounded away from 1, even if the number of agents grows to infinity, given that one of the following holds*

- $w \leq \frac{1}{3}$ *and* $(2n+1)w > 1$.
- $\frac{1}{3} < w < \frac{1}{2}$ *and* $n \geq 7$.

Finally, consider the case where the attraction interval includes the majority of the clients. In this case, the upper bound on the price of anarchy (although not necessarily the price of anarchy itself) approaches 1.

Theorem 5.5. *Consider any pair n, w such that $w \geq \frac{1}{2}$ and $n \geq 5$. Then*

$$PoA^W(w) < \begin{cases} 1 - \frac{1-w}{2n^2-n-2}, & n \notin 2\mathbb{Z}; \\ 1 - \frac{(1-w)(2n+1)}{n^2(n+2)+6n+3}, & otherwise. \end{cases}.$$

6 Price of anarchy, 2 and 3 agents, support maximizers

We now consider support maximizers utilities, and re-investigate the price of anarchy. For a small number of agents we can provide exact price of anarchy values.

Lemma 6.1. *Let $\mathbf{c}$ be an NE attraction vector in a game $G^S(w,n)$. Let $i \in N$ be so that $J = [c_i - \frac{w}{2}, c_i + \frac{w}{2}]$ is next to an uncovered sub-interval. Then J does not intersect any other attraction interval.*

A straightforward conclusion from Lemma 6.1 is the following proposition.

Proposition 6.2. *Suppose that $n = 2$. Then $PoA^S(w) = 1$.*

The following theorem presents the price of anarchy for the case of 3 agents.

Theorem 6.3. *Suppose that $n = 3$. Then, as the size of the attraction interval, w increases, the bounds on the price of anarchy $PoA^S(w)$ change as follows:*

$$PoA^S(w) = \begin{cases} 1, & w \leq \frac{1}{4}; \\ \frac{w+\frac{1}{2}}{3w}, & \frac{1}{4} \leq w \leq \frac{1}{3}; \\ w + \frac{1}{2}, & \frac{1}{3} \leq w \leq \frac{1}{2}; \\ 1, & \frac{1}{2} \leq w. \end{cases}.$$

7 Price of anarchy, n agents ($n > 3$), support maximizers

In contrast to the case of *"winner-takes-all"* utilities, under *"support maximizers"* utilities, the price of anarchy is always 1. The main observation that leads to this result is given in the following lemma.

Lemma 7.1. *For any pair n and w suppose that there exists a client, who is attracted to at least 3 agents under the joint strategy $\mathbf{c}$. Then $\mathbf{c} \in NE^S(w)$ if and only if for every $x \in [0,1]$ it holds $|I_x| \geq 1$.*

Proof. Assume the contrary, *i.e.*, there is some $z \in [0,1]$ so that $|I_z| < 1$. Because all attraction intervals are closed by definition, there are an $\epsilon > 0, \eta > 0$, so that $|I_{z'}| < 1$ for all $z' \in (z-\epsilon, z+\eta)$, i.e. a non-trivial uncovered interval.

Now, let x be a client that is attracted to at least 3 agents (say, i, j and k). Without loss of generality assume that i'th interval is the leftmost and j'th interval is the rightmost of those covering x. In particular it means that, because x is covered by all three intervals, $c_i \leq c_k \leq c_j$. As a consequence, for any $y \in [c_k - \frac{w}{2}, c_k + \frac{w}{2}]$ holds that either $y \in [c_i - \frac{w}{2}, c_i + \frac{w}{2}]$ or $\in [c_j - \frac{w}{2}, c_j + \frac{w}{2}]$. Hence, $n_k(\mathbf{c}) \leq \frac{w}{2}$.

Because $\mathbf{c}$ is a Nash equilibria, any attraction interval that neighbors $(z-\epsilon, z+\eta)$ does not intersect other attraction intervals due to Lemma 6.1. Let us now assume that agent k changes it's center of attraction to c'_k so that its interval covers $(z-\epsilon, z+\eta)$. In this case, all points of k's new attraction interval are shared with at most one other agent, and there's a sub-interval of size $\epsilon + \eta > 0$ that is exclusive to k. Hence $n_k(c'_k, \mathbf{c}_{-k}) > \frac{w}{w}$. In other words, there is a beneficial deviation for agent k. This is in contradiction to $\mathbf{c}$ being a Nash equilibria. $\square$

Theorem 7.2 below follows easily from the lemma above and suggests that, however small the attraction interval of a single agent may be, the market will always be saturated (i.e. all possible clients' will have an agent to serve their interests) if the number of agents is sufficiently large.

Theorem 7.2. $PoA^S(w) = 1$, *if $n > 3$ and $nw > 2$.*

8 Conclusions

In this paper we have presented a variation of the Hotelling-Downs model, where agents have a limited effective range and clients' support can be shared. Unlike the original model, some clients may now refrain from supporting (or buying from) any agent. As a result, agents have an additional strategic dimension to consider.

In spite of adding this additional strategic consideration, and in contrast to other extensions of the Hotelling-Downs model, a pure Nash equilibria always exists in [both] our models. This allows us to study measures of social effect of pure Nash equilibria. In particular, we study the Price of Anarchy as a function of the attraction interval, w, and the number of agents n. We also consider two utility function variations: "winner-takes-all" and "support maximizers".

In particular, our analysis shows that the price of anarchy is bound away from 1 under the "winner-takes-all" utility for any number of agents. On the other hand, for "support maximizers", the price of anarchy may attain the value of 1, under appropriate conditions. Since the different utility functions correspond to different commercial and political systems, a regulator seeking to ensure high participation rates may find these results of value.

Consider, for example, a talent show, where singers compete. The broadcaster of the show is interested in the highest possible ratings, *i.e.*, that the competitors represent as many performance styles as possible and, therefore, attract as many listeners and viewers as possible. How should the music competition compensate participants? Should there be a single prize, or the prize money should be proportional to the number of fan votes? *I.e.*, in terms of our formalism, should the "winner takes all" or the "support maximizers" utility be used?[1] The former option is the cheapest for the broadcaster, but it can not guarantee the greatest variety of styles. On the other hand, while requiring a greater investment, the "support maximizers" option may guarantee the greatest style coverage. However, it would require that the number of competitors and their individual style variability (*i.e.*, their attraction intervals) are sufficiently large.

Of course our results can be applicable outside the entertainment business as well. Setting up political systems also deals with the choice of winner-take-all *vs* support maximizers utilities. *E.g.*, would state financing of political parties based on their support base, *i.e.*. adding a support maximizers utility component, increase political participation by citizens? The answer we give herein is yes, if the number of parties is sufficiently large relative to their ideological specialization. The answer is no, if the number of parties is small.

Now, of course, our conclusion regarding real world politics is a bit of a stretch. Thus far, we have only considered agents with equal attraction interval. In the real world, attraction interval may differ between agents. This, therefore, becomes the next natural step in the development of our framework.

Another direction to pursue is to investigate different ways that voters are shared among covering agents. For instance, in this manner the issues of voter apathy, and likelihood of support based on ideological distance.

In more detail, consider first the situation where some set of voter preferences is represented by all parties. Rather than participating in the election, this set of voters may fall into apathy and abstain. After all, it does not matter which exact party wins, they will be represented. It is easy to see that such behavior can be readily captured by a rule that governs how shared voters affect agent utilities – they are simply removed from the support.

Similarly, one may consider more complex functions that assign a probability of support as a function of the distance to the agent (e.g., growing smaller with distance). The probability of support may also be impacted by the positions of multiple agents, effectively responding to patterns in their strategies. E.g. favoring (even more distant) centrist parties if there are many extremist counterparts.

Finally, it would be interesting to consider alternative utility functions such as market share, as opposed to direct market support. This would capture the parliamentary elections, where parties compete to maximize their support among all those who voted, rather than their support among the entire population.

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[1] Notice that, since music styles vary along the natural single-dimensional time-line, our formalism can indeed capture the competition.