Structured Coalitions in Resource Selection Games

MICHAL FELDMAN
Hebrew University of Jerusalem and Microsoft Israel R&D Center
and
MOSHE TENNENHOLTZ
Microsoft Israel R&D Center and Technion

We study stability against coalitional deviations in resource selection games where the coalitions have a certain *structure*. In particular, the agents are partitioned into coalitions, and only deviations by the prescribed coalitions are considered. This is in contrast to the classical concept of strong equilibrium according to which any subset of the agents may deviate. In resource selection games, each agent selects a resource from a set of resources, and its payoff is an increasing (or nondecreasing) function of the number of agents selecting its resource. While it has been shown that a strong equilibrium always exists in resource selection games, a closer look reveals severe limitations to the applicability of the existence result even in the simplest case of two identical resources with increasing cost functions. First, these games do not possess a super strong equilibrium in which a fruitful deviation benefits at least one deviator without hurting any other deviator. Second, a strong equilibrium may not exist when the game is played repeatedly. We prove that for any given partition, there exists a super strong equilibrium for resource selection games of identical resources with increasing cost functions. In addition, we show similar existence results for a variety of other classes of resource selection games. For the case of repeated games, we characterize partitions that guarantee the existence of a strong equilibrium. Together, our work introduces a natural concept, which turns out to lead to positive and applicable results in one of the basic domains studied in the literature.

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Authors' addresses: M. Feldman, School of Business Administration, The Hebrew University of Jerusalem, Mount Scopus, Jerusalem 91905, Israel; email: mfeldman@huji.ac.il.

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1. INTRODUCTION

When considering a prescribed behavior in a multi-agent system, it makes little sense to assume that an agent will stick to its part of that behavior if deviating from it can increase its payoff. This leads to much interest in the study of Nash equilibrium in games. A Nash equilibrium is an action profile of the agents for which unilateral deviations are not beneficial. When agents are allowed to use mixed actions, a Nash equilibrium always exists. Moreover, in the context of congestion games [Rosenthal 1973; Monderer and Shapley 1996], there always exists a pure action equilibrium. However, Nash equilibrium does not take into account deviations by non-singleton sets of agents. While stability against deviations by subsets of the agents, captured by the notion of strong equilibrium [Aumann 1959], is a most natural requirement, it is well-known that obtaining such stability is possible only in rare situations. In the context of congestion games, Holzman and Law-Yone [1997, 2003] characterized the networks where a strong equilibrium always exists. From a pragmatic perspective, the most important part of their results is the existence of strong equilibrium in *resource selection games*.

A resource selection game (RSG) consists of a set of n agents and a set of m resources. Each agent chooses a resource from among the set of resources, and its cost is a nondecreasing function of the number of agents who have chosen its selected resource. Resource selection games are fundamental and central to work in various disciplines, such as operations research, computer science, game theory, and economics. However, a closer look at the its fundamental result shows severe limitations to its applicability. In particular, the following issues arise.

(1) In the original definition of strong equilibrium, a deviation is considered profitable only if it is strictly beneficial to all agents. However, it makes much sense to consider super strong equilibrium in which a beneficial deviation improves the payoff of at least one of the deviator without hurting any other deviator.
(2) The results on existence of strong equilibrium are obtained for one-shot games, while it is desired to have stability also in the case of a repeated game.

As it turns out, the important basic results about resource selection games fail to generalize to either super strong equilibrium in one-shot RSG's or to strong equilibrium in repeated RSG's. Consider the basic setting of two identical resources with strictly increasing cost functions. This setting is fundamental to many studies in electronic commerce, operations research, communication networks, and economics. Apparently, there are simple instances of that setting in which there is no super strong equilibrium and simple instances of that setting in which there is no strong equilibrium when the game is played repeatedly.

In order to deal with these issues, we introduce the study of *structured coalitions* and apply it in the context of resource selection games. The concept of structured coalitions introduces a social context into the study of group deviations by explicitly stating a structure over the coalitions, allowing only for deviations in which the set of deviators constitutes a coalition.

In particular, we consider a coalitional structure given by a partition $T = \{T_1, \ldots, T_k\}$ of the agents into disjoint coalitions, allowing only for deviations in which the deviating coalition constitutes an element of the partition. More precisely, given a partition T, a *T-strong equilibrium* (T-SE) is an action profile from which no coalition T_i can deviate and improve the utility of each agent $J \in T_i$. Similarly, a *T-super strong equilibrium* (T-SSE) is defined analogously with respect to the super strong equilibrium notion. The reader should be careful not to confuse this notion with the notion of a k-strong equilibrium [Andelman et al. 2007], which restricts the size of the deviating coalition.

Obviously, these notions make much sense in the context of games that take into account the social structure of the set of participants. In fact, one way to view structured coalitions is as an extension of work on social context games [Ashlagi et al. 2008]. In a social context game, an agent's utility is affected by the payoffs of its friends where friends are defined using some topological or graph-theoretic structure. However, unlike previous work on social context games, which dealt with single agent deviations, here we consider the situation where members of a coalition coordinate their activity and potential deviations as in strong equilibrium. Our study lies in the framework of *noncooperative games*; in particular, no side payments are considered or allowed.

Adding social structure to games has been observed to be an important next step in applying game-theoretic concepts to existing social and organizational structures. This has been already observed by Ashlagi et al. [2008], Herings et al. [2003], and references therein. More specifically, if we are to apply game-theoretic solution concepts to a realistic social context, a natural means is to refer explicitly to the social structure in defining the solution concepts. Our work takes a step in that important direction by looking at the society as a partition into subsocieties. Extension into overlapping subsocieties may be an interesting direction for future work.

Previous work on coalitional congestion games [Hayrapetyan et al. 2006; Kuniavsky and Smorodinsky 2007] has considered side payments in the context of congestion games. In this context, each player is represented by a set of agents, each of which is a participant in the resource selection game, and the utility of the player is equal to the sum of its agents' utilities. Side payments, however, deviate from the nontransferable utility assumption which is the basic assumption in work on strong equilibrium (see, e.g., Epstein et al. [2007]; Andelman et al. [2007]; Fiat et al. [2007]; Leonardi and Sankowski [2007]; and Fotakis et al. [2006]). Our work reconsiders deviations by coalitions in the classical nontransferable utility setting.

In the context of one-shot games, a positive result showing the existence of equilibrium when monetary transfers are allowed implies the existence of a T-SSE. Indeed, one of our results can be deduced from these relationships. In most cases, however, the existence of monetary transfers yields negative results. In fact, even if we have two identical resources with strictly increasing cost functions, it has been shown that if coalitions are not restricted to have a size of at most two, then no equilibrium exists when monetary transfers are allowed. Our work shows positive results about the existence of T-SSE in this setting and in wider sets of resource selection games.

The article is structured as follows. Section 2 introduces our model and basic definitions and preliminaries. In Section 3, we consider the existence of a T-SSE for one shot RSG's, and in Section 4, we consider the existence of a T-SE for repeated RSG's. Together, our analysis addresses the aforementioned two basic issues.

In Section 3, we first concentrate on resource selection games with strictly increasing cost functions; this is a most classical type of game. It is well known that, even in the case of two identical resources, there is no super-strong equilibrium. In Theorem 1, we show that every RSG with an arbitrary number of identical, strictly increasing resources admits a T-SSE for every patrition T. We then extend our results to deal with the case of two nonidentical resources with strictly increasing cost functions. Theorem 2 shows that every RSG with two strictly increasing resources admits a T-SSE for every partition T. In all related cases, these are the first positive results on equilibrium existence when group deviations are considered, and deviations are not required to strictly benefit all agents. In Theorem 3, we consider the case of general resource selection games with nondecreasing resources. We show that a T-SSE exists in this case if all the coalitions are bounded by size 2. Since this restricted case is the only one for which a positive result is known in games with monetary transfers (see Kuniavsky and Smorodinsky [2007]) we know that a superstrong equilibrium exists in that setting; however, we provide an alternative proof for that case. Unfortunately, as we will illustrate, our results do not scale to arbitrary congestion games. Our results suggest a host of open questions regarding the existence of T-SSE in one-shot resource selection games. The most interesting challenge is to characterize the coalitional structures that guarantee the existence of a T-SSE in general RSG's.

In Section 4, we consider repeated resource selection games. In that setting, strong equilibrium (in the classical sense of Aumann [1959]) does not exist even if we have two identical resources with increasing cost functions, and we allow deviations of size two. We consider general repeated resource selection games with nondecreasing cost functions and show that there exists a T-SE when all elements in the partition are of size of at most 2, as well as when all elements in the partition are of size of at least 2. The preceding conditions are in a sense complete: we show the existence of a repeated resource-selection game where the society consists of a singleton and a triplet under which there is no T-SE. In addition, we characterize partitions that admit a T-SE for a restricted case where the resources are identical and there is a majority of singletons in the partition. In this case, we show that if the number of agents is odd, then there is a T-SE if all coalitions are of size of at most 3, and that if there exists a coalition of size greater than 3, then there exists a resource selection game that admits no T-SE. If the number of agents is even, then there is a T-SE if all coalitions are of size of at most 2, and if there exists a coalition of size greater than 2, then there exists a resource selection game that admits no T-SE.

2. MODEL AND PRELIMINARIES

A game is denoted by a tuple $G = \langle N, \{S_i\}_{i=1}^n, \{c_i\}_{i=1}^n \rangle$, where N is a set of n agents, S_i is a finite action space for agent $i \in N$, and $c_i(\cdot)$ is a cost function of agent i. We denote by $S = S_1 \times \cdots \times S_n$ the action profile space of the n agents. For an action profile $s \in S$, we denote by s_{-i} the actions of agents $j \neq i$, that is, $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. Similarly, for a set of agents $\Gamma \subseteq N$ (also called a *coalition*), we denote by s_Γ and $s_{-\Gamma}$ the action profiles of agents $j \in \Gamma$ and $j \notin \Gamma$, respectively. A cost function of agent i is a function $c_i : S \to \mathbb{R}$ that maps each action profile $s \in S$ to a real number. Throughout this article we restrict attention to pure actions.

Nash Equilibrium (NE). An action profile $s \in S$ is a Nash Equilibrium if no agent $i \in N$ can benefit from unilaterally deviating from its action to another action, that is, for every $i \in N$, $s'_i \in S_i$, it holds that $c_i(s_{-i}, s'_i) \geq c_i(s)$.

Resilience to Coalitions. An action profile of a set of agents $\Gamma \subseteq N$ specifies an action for each agent in the coalition, that is, $\gamma \in \times_{i\in\Gamma} S_i$. An action profile $s \in S$ is not resilient to a *strongly profitable* deviation of a coalition Γ if there is an action profile γ of Γ such that $c_i(s_{-\Gamma}, \gamma) < c_i(s)$ for every $i \in \Gamma$ (i.e., the agents in the coalition can deviate in such a way that each agent reduces its cost).

Definition 1. A *strong equilibrium* (SE) is an action profile that is resilient to a strongly profitable deviation of any coalition $\Gamma \subseteq N$.

An action profile $s \in S$ is not resilient to a *weakly profitable* deviation of a coalition Γ if there is an action profile γ of Γ such that $c_i(s_{-\Gamma}, \gamma) \leq c_i(s)$ for every $i \in \Gamma$, and there exists an agent $i \in \Gamma$ such that $c_i(s_{-\Gamma}, \gamma) < c_i(s)$ (i.e., the agents in the coalition can deviate in such a way that none of the agents increases its cost, and at least one agent strictly reduces its cost).

Definition 2. A *super strong equilibrium* (SSE) is an action profile that is resilient to a weakly profitable deviation of any coalition $\Gamma \subseteq N$.

We consider coalitional structures that are given by a partition of the agents $T = (T_1, \ldots, T_k)$, where $T_i \cap T_j = \emptyset$ for every i, j and $\bigcup_{i\in[k]} T_i = N$. Given a partition T, we define the following.

Definition 3. A *T-strong equilibrium* (T-SE) is an action profile that is resilient to a strongly profitable deviation of any coalition $T_i \in T$.

A *T-super strong equilibrium* ($T - SSE$) is a profile that is resilient to a weakly profitable deviation of any coalition $T_i \in T$.

Clearly, for every T, if an action profile is either an SE or a T-SSE, it is also a T-SE. It is important to note that, while the set of SE is contained in the set of NE, the set of T-SE or T-SSE is not necessarily contained in the set of Nash equilibria (nor does the set of Nash equilibria contained in the set of T-SE (or T-SSE)). It might be the case that a single agent can deviate unilaterally and strictly improve its own payoff, but if such a deviation reduces the payoff of a member of its coalition (or does not improve it), it will not be considered a beneficial deviation.

As a warm-up, let us consider two extreme cases, namely the *singleton* case and the *fully distributed* case. In the singleton case, there is a single coalition that contains all of the agents; that is, $T = \{N\}$. In this case, the set of T-SSE outcomes coincides with the set of Pareto-optimal outcomes; thus there always exists a T-SSE. In the fully distributed case, each individual constitutes a coalition; that is, $T = \{\{1\}, \ldots, \{n\}\}$. In this case, the set of T-SE coincides with the set of T-SSE and with the set of NE. Thus, any game that admits a Nash equilibrium admits a T-SSE as well. A direct corollary of the these observations is that every 2-player game that admits a Nash equilibrium admits a T-SSE for any T (as it must be one of the former or the latter cases). This is interesting, for example, in the context of potential (or congestion) games where many of the counterexamples refuting the existence of an SE are 2-player games (see, e.g., SE in cost-sharing connection games [Epstein et al. 2007]).

2.1 Resource Selection Games

A resource selection setting is characterized by a tuple $\langle M, N, \{b_i(\cdot)\}_{i=1}^{m}\rangle$, where $M = \{M_1, \ldots, M_m\}$ is a set of resources, $N = \{1, \ldots, n\}$ is a set of n agents (jobs) and a cost function of resource M_i is a function $b_i : N \rightarrow \mathbb{R}$, where $b_i(l)$ denotes the cost of an agent on resource M_i when resource M_i has a load of l agents, $1 \leq l \leq n$. A resource selection setting has *identical resources* if for every $i, i' \in [m]$ and for every $l \in [n]$ $b_i(l) = b_{i'}(l)$. In settings of identical resources, we denote the cost vector of all the resources by $b = (b(1), \ldots, b(n))$.

A *one-shot resource selection game* (RSG) has N as the set of agents and the set of resources as the set of actions for each player; that is, the action space S_J of agent $J \in N$ is the set M. In an action profile $s = (s_1, \ldots, s_n) \in S = S_1 \times \cdots \times S_n$, s_J denotes the resource selected by agent J, and the load of resource M_i, denoted $l_i(s)$, is the number of agents that chose resource M_i in the profile s. The cost of an agent J who chose resource M_i in a profile $s \in S$ is $c_J(s) = b_i(l_i(s))$.

We assume that the cost function $b_i(\cdot)$ is nondecreasing for every i. that is, it holds that $b_i(l) \leq b_i(l+1)$ for every $i \in [m]$ and every $1 \leq l \leq n-1$. In some cases, we will consider settings with strictly increasing cost functions; that is, $b_i(l) < b_i(l+1)$ for every $i \in [m]$ and every $1 \leq l \leq n-1$. In this case, if $b_i(l) \leq b_i(l')$, then $l \leq l'$. When clear in the context, we slightly abuse notation and refer to resources with increasing (strictly increasing, respectively) cost functions as increasing (strictly increasing, respectively) resources.

Every RSG is a congestion game, and thus admits a Nash equilibrium in pure actions. In addition, it has been shown in Holzman and Law-Yone [1997] that every RSG with nondecreasing cost functions admits a SE and thus also admits a T-SE for any T. Yet, as we will see, an RSG with nondecreasing cost functions might not admit a T-SSE, nor will a repeated RSG necessarily admit a T-SE. These two matters will be our focus in the following two sections, respectively.

3. *T*-SUPER STRONG EQUILIBRIUM (*T*-SSE) EXISTENCE

Every RSG admits an SE [Holzman and Law-Yone 1997] and, therefore, admits a T-SE as well. However, an RSG might not admit any SSE. This nonexistence may occur even for an RSG with two identical strictly increasing resources as demonstrated in the following example. Consider an RSG with two identical resources with cost function $b = (1, 2, 3)$ and 3 agents. A profile in which all three agents share the same resource can obviously not be an SSE (and not even an NE). Suppose without loss of generality that agents 1, 2 are assigned to M_1, and agent 3 is assigned to M_2. Then, agents 1 and 2 can deviate to a profile in which agent 1 migrates to M_2, incurring the same cost as before, while agent 2 reduces its cost from 2 to 1. The assertion follows.

3.1 The Case of Identical, Strictly Increasing Resources

While an SSE might not exist even under the restricted setting of two identical strictly increasing resources, as we shall soon show, every RSG with identical strictly increasing resources admits a T-SSE for every T. Before formulating the theorem, we present the following lemma and definition that will prove useful in the proof.

LEMMA 1. *Let G be an RSG with m identical strictly increasing resources, and let s be a Nash equilibrium of G. Let $s' = (s'_\Gamma, s_{-\Gamma})$ be a weakly profitable deviation of a coalition Γ. It holds that $l_i(s') \leq l_i(s) + 1$ for every $i \in [m]$.*

PROOF. Since s is a NE, for every $i, j \in [m]$, $b_i(s) \leq b_j(l_j(s) + 1)$ (otherwise an agent assigned to M_i can improve by deviating to M_j). Suppose by way of contradiction that there exists a resource M_k such that $l_k(s') \geq l_k(s) + 2$, then there must exist an agent J for which $s'_J = M_k$ and $s_J = M_j$ for some $j \neq k$. We show that this player's cost increases after the deviation. It holds that $c_J(s) = b_j(s) \leq b_k(l_k(s) + 1) < b_k(l_k(s) + 2) \leq b_k(l_k(s')) = c_J(s')$, where the first inequality holds since s is an NE, and the second inequality follows from strict monotonicity. □

We proceed with the following definition of a *(T,s)-spread-out* assignment. This definition will be used extensively in the proofs throughout the article. Let $l(s) = (l_1(s), \ldots, l_m(s))$ be the congestion vector of a profile s, sorted in nonincreasing order. A *(T,s)*-spread-out assignment is an assignment that is obtained by spreading out the members of each coalition on the resources in a nonincreasing order of $|T_i|$ according to the sorted vector $l(s)$.

An example is in order. Suppose there exist three resources and a coalition structure $T = \{T_1, T_2, T_3\}$ such that $|T_1| = 4$ and $|T_2| = 3$ and $|T_3| = 2$. Consider a profile s where $l_1(s) = 4, l_2(s) = 3, l_3(s) = 2$. A*(T,s)*-spread-out assignment

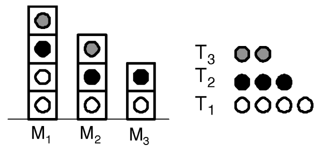

Fig. 1. An example of a (T, s)-spread-out assignment with 3 resources, a coalitional structure with $|T_1| = 4$, $|T_2| = 3$, $|T_3| = 2$, and s such that $l_1(s) = 4, l_2(s) = 3, l_3(s) = 2$.

would assign two members of T_1 on M_1 and one member of T_1 on each one of M_2, M_3. It will then assign one member of T_2 on each one of M_2, M_3, M_1. Finally, it will assign one member of T_3 on each one of M_2, M_1. A demonstration of the example is given in Figure 1.

With this we are ready to state the theorem.

THEOREM 1. *Every RSG with identical strictly increasing resources admits a T-SSE for every T.*

PROOF. Let s be a Nash equilibrium of G. We claim that a (T, s)-spread-out assignment is a T-SSE of G. Suppose by way of contradiction there is a weakly profitable deviation of some coalition to a profile s'. Since the resources are identical and strictly increasing, there must exist L such that $l_i(s) \in \{L, L+1\}$ for every i. Let k denote the number of resources with load L and refer to agents assigned to resources with load L and $L+1$ as low and high agents, respectively.

We first claim that $l_i(s') \geq L$ for every i. Suppose by way of contradiction that there exists an i such that $l_i(s') \leq L-1$. In order to assign all the low jobs, there must exist k additional resources with load at most L. It follows that there must exist an i for which $l_i(s') \geq L+2$, contradicting Lemma 1. We conclude that $l_i(s') \in \{L, L+1\}$ for every i. Since the total number of agents remains the same, there must exist k resources with load L and $m-k$ resources with load $L+1$ in s'. For every low job J, it must hold that $l_i(s') \leq L$. Therefore, for every high job, it must hold that $l_i(s') = L+1$. It follows that no job in the coalition strictly improves its load, which establishes the proof of the assertion. □

3.2 The Case of Two Strictly Increasing Resources

In this section, we consider the more general setting of nonidentical resources, but provide results only for the case of two strictly increasing resources. We begin with several characteristics of RSG's with two resources.

We distinguish between two types of coalitional deviations in settings with two resources, namely *unidirectional* and *bidirectional* deviations. In a unidirectional deviation, there exists a coalition Γ such that a subset $\Gamma' \subseteq \Gamma$ deviates from one resource to the second one. In a bidirectional deviation, there exists a coalition Γ and a partition (Γ_1, Γ_2) (i.e., $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$), where $|\Gamma_i| > 0$ for $i = 1, 2$, such that the members of Γ_1 deviate from M_1 to M_2 and the members of Γ_2 deviate from M_2 to M_1.

We establish several Lemmata that will be used in the proof of Theorem 2.

LEMMA 2. *Let G be an RSG with two strictly increasing resources with cost functions $b_1(\cdot)$ and $b_2(\cdot)$, respectively. Let s be a Nash equilibrium of G. If $b_1(l_1(s)) \le b_2(l_2(s))$, then the only possible weakly profitable bidirectional deviation is one in which the number of jobs that migrate from M_2 to M_1 is greater by* 1 *than the number of jobs that migrate from M_1 to M_2.*

PROOF. Observe that in every weakly profitable deviation s' the following must hold: (i) each resource can increase by at most a single job (by Lemma 1), (ii) either $l_1(s) \neq l_1(s')$ or $l_2(s) \neq l_2(s')$ (otherwise at least one job incurs a higher cost), and (iii) the load on M_2 cannot increase, since, if it does, then there must exist a coalition member that incurs a cost of $b_2(l_2(s)+1) > b_2(l_2(s)) \ge b_1(l_1(s))$, where $b_1(l_1(s))$ is its original cost. Therefore, it must be the case that M_1 increases by a single job, and by strict monotonicity $b_1(l_1(s')) > b_1(l_1(s))$. Therefore, no coalition member stays on M_1 in s'. □

OBSERVATION 1. *Let G be an RSG with two strictly increasing resources, and let s be a Nash equilibrium of G. If there exists a weakly profitable unidirectional deviation of a coalition T_i, then $s_J = s_{J'}$ for every $J, J' \in T_i$.*

PROOF. Assume without loss of generality that $s_J = M_2$ for every $J \in \Gamma$. If there exists some $J \in T_i$ such that $s_J = M_1$, then $c_J(s) < c_J(s')$, meaning that job J incurs a higher cost in s', contradicting weak profitability of the deviation. □

Definition 4. A vector $(\sigma_1, \sigma_2, \ldots \sigma_m)$ is lexicographically smaller than $(\hat{\sigma}_1, \hat{\sigma}_2, \ldots \hat{\sigma}_m)$ if for some i, $\sigma_i < \hat{\sigma}_i$ and $\sigma_k = \hat{\sigma}_k$ for all $k < i$.

An action profile s is costwise lexicographically smaller than s' if the cost vector $c(s) = (c_1(l_1(s)), \ldots, c_m(l_m(s)))$, sorted in nonincreasing order, is smaller lexicographically than $c(s')$, sorted in nonincreasing order.

LEMMA 3. *Let G be an RSG with strictly increasing resources. If an action profile s is costwise lexicographically minimal in G, then s is a Nash equilibrium of G.*

PROOF. Suppose by way of contradiction that s is not a Nash equilibrium, and let s' be a profile that is obtained from a unilateral strictly beneficial deviation of a job J. Let i be the resource J is assigned to in the profile s, and i' be the resource J is assigned to in the profile s'. Since J's cost decreases, it holds that $b_i(l_i(s)) > b_{i'}(l_{i'}(s'))$. In addition, since every resource has a strictly increasing cost and $l_i(s) > l_i(s')$, it holds that $b_i(l_i(s)) > b_i(l_i(s'))$. We get that in the obtained profile s', $b_i(l_i(s)) > b_{i'}(l_{i'}(s'))$ and $b_i(l_i(s)) > b_i(l_i(s'))$. Therefore, the profile s' is costwise lexicographically smaller than s, and the proof is established. □

Using the aforementioned lemmata, we show that every RSG with two strictly increasing resources admits a T-SSE for every T. This is in contrast to the notion of SSE where an SSE may not exist even on two identical strictly increasing resources. The proof is deferred to the appendix.

THEOREM 2. *Every RSG with two strictly increasing resources admits a T-SSE for every partition T.*

3.3 The Case of *m* Nondecreasing Resources

We next consider the more general case of nondecreasing cost functions. We show that if $|T_i| \leq 2$ for every i, then a T-SSE always exists. Before formulating our theorem, we present the following lemma.

LEMMA 4. *Let G be an RSG with nondecreasing resources, and let s be a Nash equilibrium of G. Given a coalition T_j, if it holds that $|\{J|J \in T_j$ and $s_J = M_i\}| \leq 1$ for every i, then T_j has no weakly profitable deviation.*

PROOF. Suppose by way of contradiction that there is a weakly profitable deviation of T_j to a profile $s' = (s_{-T_j}, s'_{T_j})$. It follows that there exists $J \in T_j$ such that $c_J(s') < c_J(s)$. Let i be the resource index such that $s_J = M_i$. It is easy to see that the only way for J to reduce its cost is by deviating to another resource, M_j, from which another job J' migrates (otherwise, it contradicts s being a NE). If the cost of J strictly decrease, then it must hold that $b_j(l_j(s')) < b_i(l_i(s))$, but since the load on M_j does not change, it holds that $b_j(l_j(s)) < b_i(l_i(s))$.

J' cannot migrate to M_i since $b_j(l_j(s)) < b_i(l_i(s))$, neither can J' migrate to a resource $M_{j'}$ from which no job migrates since, if it does, then it must hold that $b_{j'}(l_{j'}(s)+1) \leq b_j(l_j(s))$, which is strictly smaller than $b_i(l_i(s))$ by the preceding argument. This contradicts s being a NE. Thus, J' can only migrate to a resource from which another job migrates.

We prove by induction that each job that leaves a resource must migrate to a resource from which another job migrates. The base of the induction is the first job, as described. Suppose that the claim holds up to the t^{th} resource $M_{j_1}, M_{j_2}, \ldots M_{j_t}$. For every $k \in \{1, \ldots, t\}$, it holds that $b_{j_k}(l_{j_k}(s)) \leq \cdots \leq b_{j_2}(l_{j_2}(s)) < b_{j_1}(l_{j_1}(s))$, thus $b_{j_k}(l_{j_k}(s) < b_{j_1}(l_{j_1}(s))$. Now consider the job that leaves resource M_t. First, it cannot migrate to M_{j_1} since otherwise $b_{j_1}(l_{j_1}(s)) \leq b_{j_t}(l_{j_t}(s))$, which is strictly smaller than $b_{j_1}(l_{j_1}(s))$ by the preceding argument. Second, it cannot migrate to any resource M_k, $k \in \{2, \ldots, t-1\}$ since, if it does, it holds that $b_{j_k}(l_{j_k}(s)+1) \leq b_{j_t}(l_{j_t}(s)) < b_{j_1}(l_{j_1}(s))$, contradicting s being a NE. Third, it cannot migrate to another resource M_w from which no job migrates since, if it does, it holds that $b_w(l_w(s)+1) \leq b_{j_t}(l_{j_t}(s)) < b_{j_1}(l_{j_1}(s))$, contradicting s being a NE. Thus, it must migrate to another resource from which another job migrates. Since the number of resources that contain jobs in T_j is finite, the last job leaving its own resource cannot migrate to any other resource. □

With this we are ready to state the theorem.

THEOREM 3. *Every RSG with nondecreasing resources in which $|T_i| \leq 2$ for every i admits a T-SSE.*

PROOF. Let s be a Nash equilibrium of the game and consider a (T, s)-spread-out assignment s_T. For the simplicity of the presentation, we abuse notation and denote the spread-out assignment by s. In s, there might be only a single resource that contains more than a single member of each coalition, denote it M_i. Since s is a NE, no singleton can deviate. By Lemma 4, no coalition (of size 2) that is assigned to different resources can deviate either. Thus, we should only consider deviations of pairs (recall that $|T_i| \leq 2$ for every i) that are assigned to the same resource. Let J, J' be two jobs that are assigned to M_i and let s' be a weakly profitable deviation of $\{J, J'\}$. We claim that if both agents migrate, it contradicts s_T being a NE. To see this, note that if they migrate to the same resource, then at least one of the jobs can strictly improve by a unilateral deviation. Similarly, if they migrate to the same resource, then obviously there exists a profitable unilateral deviation. In conclusion, we should only consider a deviation in which one of the jobs migrates from M_i while the other job stays on M_i, that is, a profile s' in which $s'_{J'} = M_i$ and $s'_J = M_k$, for some $k \neq i$.

It must hold that $c_J(s') \leq c_J(s)$, thus $b_k(l_k(s')) = b_k(l_k(s) + 1) \leq b_i(l_i(s))$, and, by s being a Nash equilibrium, it must hold that $b_k(l_k(s) + 1) \geq b_i(l_i(s))$; thus $b_k(l_k(s) + 1) = b_i(l_i(s))$. Therefore, the cost of agent J' must strictly decrease, that is, it must hold that $b_i(l_i(s) - 1) < b_i(l_i(s))$.

We next claim that s' is also a NE. Since $b_k(l_k(s')) = b_k(l_k(s) + 1) = b_i(l_i(s))$, a unilateral deviation from M_i to M_k or in the other direction is not profitable. A unilateral deviation from $M_l, l \neq i, k$ to M_i is not profitable either since by s being a NE, a job from M_l cannot improve by deviating to M_k, thus by $b_k(l_k(s) + 1) = b_i(l_i(s))$, it cannot improve by deviating to M_i either. By s being a NE, it follows that all other unilateral deviations are not profitable either. A similar argument shows that after each deviation of a pair that is assigned to M_i, we should again consider only such deviations. But this process is limited by the number of pairs that are assigned to M_i, which is finite. We conclude that this process must converge to a T-SSE. □

Remark. The last theorem in fact follows from a result introduced by Kuniavsky and Smorodinsky [2007]. They prove this result in a setting with *transferrable utilities*, while in our case utilities are nontransferrable. Yet our alternative proof is presented here due to its simplicity.

4. *T*-STRONG EQUILIBRIUM (*T*-SE) EXISTENCE IN REPEATED RSG'S

Let G be a one-shot game and let R be an integer. A *repeated game* $\hat{G} = < G, R >$ is a setting in which the game G is played repeatedly R periods. Recall that $s = (s_1, \ldots, s_n)$ denotes an action profile for the n agents in the one-shot game. Similarly, we denote by $s^t = (s^t_1, \ldots, s^t_n) \in S_1 \times \cdots \times S_n$ an action profile that is played in period t of the repeated game. A strategy of agent J is a set of functions $g^t_J : (S_1 \times \cdots \times S_n)^{t-1} \to S_J$, which maps a history $(s^1, \ldots, s^{t-1})$ up to period $t-1$, into an action $s_J \in S_J$ for every $t \in \{1, \ldots, R\}$, where a null action is defined for time $t = 1$. Let $g_J = \{g^t_J\}^R_{t=1}$ denote the strategy of agent $J \in [n]$. The cost of agent J in $\hat{G} = < G, R >$ under a strategy profile $g = (g_1, \ldots, g_n)$ is given by $c_J(g) = \sum_{t=1}^{R} c_J(s^t_1, \ldots, s^t_n)$, where $s^t_J = g^t_J(s^1, \ldots, s^{t-1})$. We also denote by $s^t(g)$ the action profile that is induced by the strategy profile g in period t.

The notion of T-SE can be extended to settings of repeated games. A strategy profile is said to be a T-SE of a repeated game if there is no strongly-profitable deviation of a coalition $T_i \in T$. There is a crucial difference between SE and Nash equilibrium in repeated games. Suppose s is a Nash equilibrium of the game G. Then, playing s in every round of the repeated game must be a Nash equilibrium of the repeated game. In contrast, if s is an SE of the game G, it is not necessarily the case that playing s in every round of the repeated game is an SE of the repeated game.

For example, while every RSG admits an SE, even on the very simple RSG that is composed of two identical resources with nondecreasing cost functions and 3 agents, its repeated version might not admit an SE.

OBSERVATION 2 [TENNENHOLTZ AND ZOHAR 2009]. *There exists a repeated RSG with two identical nondecreasing resources and three agents that does not admit an SE.*

Similarly, if s is a T-SE of the game G, it is not necessarily the case that playing s in every round of the repeated game is a T-SE of the repeated game. This is in fact exemplified later on in Proposition 1. Thus, characterizing the set of repeated games that admit a T-SE is a challenging goal.

4.1 The General Case

The following theorem shows that every repeated RSG with m nondecreasing resources admits a T-SE if T contains no singletons. We first define a Γ-minimal agent and present several lemmata that will be used in the proof of the theorem.

Let G be a one-shot game and let s be an action profile in G. An agent $i \in \Gamma$ is said to be Γ^s-minimal if for any action profile $s' = (s'_\Gamma, s_{-\Gamma})$, it holds that $c_i(s) \leq c_i(s')$.

LEMMA 5. *Let $\hat{G} = \langle G, R\rangle$ be a repeated RSG game, and let s be an action profile of G such that, for every i, there exists an agent $J \in T_i$ that is T_i^s-minimal. Then, playing s in every round of $\hat{G}$ is a T-SE of $\hat{G}$.*

PROOF. For every coalition T_i, there exists an agent $J \in T_i$ whose cost cannot be reduced in any round of the game by a deviation of its coalition. Since this is true for every coalition T_i, no strongly profitable deviation exists, thus playing s in every period is a T-SE of $\hat{G}$. □

The following theorem identifies a family of T-structures for which every repeated RSG admits a T-SE.

THEOREM 4. *Let $\hat{G} = \langle G, R\rangle$ be a repeated RSG with an arbitrary number m of nondecreasing resources. If $|T_i| \geq 2$ for every i, then $\hat{G}$ admits a T-SE.*

PROOF. Let $C \subseteq N$ be a set containing an arbitrary single agent from every coalition T_i, and let M_j be a resource satisfying $b_j(|C|) \geq b_i(|C|)$ for every $i \neq j$. Since $|T_i| \geq 2$ for every i, it holds that $|N \setminus C| \geq |C|$. Let $G' =< M, C, b'(\cdot), c(\cdot) >$ be the game induced by G, where all agents $J \in N \setminus C$ are assigned to M_j; i.e., G' is a one-shot game with $|C|$ agents and with cost functions $b'_j(l) = b_j(l + |N \setminus C|)$ and $b'_i(l) = b_i(l)$ for every $i \neq j$.

We shall prove that there exists a Nash equilibrium s' in G' in which $s'_J \neq M_j$ for every $J \in C$. To see this, consider best-response dynamics starting from an arbitrary assignment of the agents $J \in C$ on resources $M \setminus M_j$. Since this is a potential game, best-response dynamics must converge to a NE. For every profile s of the agents $J \in C$, it holds that $c_k(|C|) \leq c_j(|C|)$ by the choice of resource M_j, and $c_j(|C|) \leq c_j(|N \setminus C|)$ from monotonicity. Thus, $c_k(|C|) \leq c_j(|N \setminus C|)$. In addition, for every $k \in [m]$, it holds that $c_k(l_k(s)) \leq c_k(|C|)$ (from monotonicity). Together, we obtain that, for every $k \in [m]$, $c_k(l_k(s)) \leq c_j(|N \setminus C|)$. Therefore, we can assume without loss of generality that in the best-response dynamics, agents do not migrate to resource M_j.

Let s be an action profile in G in which $s_J = M_j$ for every $J \in N \setminus C$, and $s_J = s'_J$ for every $J \in C$ for some Nash equilibrium s' of G' such that $s'_J \neq M_j$ for every $J \in C$ (such a Nash equilibrium exists by the last argument). We claim that playing s in every round of $\hat{G}$ is a T-SE of the repeated game. For every $i = 1, \ldots, |T|$, let J_i be the agent J such that $J \in T_i \cap C$. We claim that s is T_i-minimal for J_i for every i. First, since s is a NE, J_i cannot reduce its cost by a unilateral deviation. In addition, $s_J = M_j$ for every $J \in T_i \setminus \{J_i\}$. Thus, to show that s is T_i-minimal for J_i, it is sufficient to show that J_i's cost will not decrease in a profile s'' where all the agents in $T_i \setminus \{J_i\}$ migrate from M_j and J_i migrates to M_j. Let k be the index such that $s_{J_i} = M_k$. It holds that $l_k(s) \leq |C|$. In addition, since there is still one agent from every coalition except for the coalition T_i, it holds that $l_j(s'') \geq |C|$. It follows that

$$c_{J_i}(s) = b_k(l_k(s)) \leq b_k(|C|) \leq b_j(|C|) \leq b_j(l_j(s'')) = c_{J_i}(s''),$$

where the last inequality follows from the aforementioned argument and the inequality preceding it follows by the choice of resource M_j. Thus, s is T_i-minimal for J_i in the one-shot game, and by Lemma 5, playing s in every round of the game constitutes a T-SE. □

We shall soon show that every repeated RSG with nondecreasing resources admits a T-SE also in the case where $|T_i| \leq 2$ for every i. The following observation will be used in proving the next theorem.

OBSERVATION 3. *Let G be an RSG with nondecreasing resources, and let s be a Nash equilibrium of G such that no resource contains more than a single representative of every coalition. Then, playing s in every round constitutes a T-SE of the repeated game.*

PROOF. It follows from Lemma 4, s is a T-SE of the one-shot game in every round of the repeated game. Thus, for every coalition T_i, there exists an agent $J \in T_i$ for which s is T_i-minimal. Thus, by Lemma 5, no coalition has a strongly profitable deviation in the repeated game, and the assertion follows. □

We are now ready to establish the following theorem.

THEOREM 5. *Let $\hat{G} = \langle G, R \rangle$ be a repeated RSG with an arbitrary number m of nondecreasing resources. If $|T_i| \leq 2$ for every i, then $\hat{G}$ admits a T-SE.*

PROOF. Let G be a one-shot RSG with m nondecreasing resources. Let $C \subseteq N$ be a set containing one representative of every T_i such that $|T_i| = 2$, and let C' be the set of their "partners". Let M_j be a resource satisfying $b_j(|C|) \geq b_i(|C|)$ for every $i \neq j$. Let $G' =< M, N \setminus C, b'(\cdot), c(\cdot) >$ be the game induced by G where all agents $J \in C$ are assigned to M_j; that is, $b'_j(l) = b_j(l + |C|)$ and $b'_i(l) = b_i(l)$ for every $i \neq j$.

We shall prove that there exists a Nash equilibrium of the agents $J \in N \setminus C$ in the game G' in which $s_J \neq M_j$ for every $J \in C'$. Since a Nash equilibrium is not sensitive to the identities of the agents, it is sufficient to show that there exists a Nash equilibrium in which $|\{J | s_J \in M \setminus M_j\}| \geq |C'|$. To see this, consider best-response dynamics starting from an arbitrary assignment of the agents $J \in N \setminus C$. Since this is a potential game, best-response dynamics must converge to a NE. We will show that for any profile s in which $l_j(s) = |N \setminus C|$, no agent would migrate to resource M_j as a best response. This means that in any Nash equilibrium that is obtained by best-response dynamics, the number of agents choosing resource M_j would never exceed $N \setminus C$, thus the number of agents on resources $M - M_j$ would be at least $|C|$. We next prove that, if $l_j(s) = |N \setminus C|$, no agent would migrate to resource M_j as a best response. If $l_j(s) = |N \setminus C|$, then for every $k \neq j$, $b_k(l_k(s)) \leq b_k(|C|)$. By the choice of resource M_j, $b_k(|C|) \leq b_j(|C|)$. By monotonicity, $b_j(|C|) \leq b_j(|N \setminus C|)$, and since $l_j(s) = |N \setminus C|$, we obtain $b_j(|C|) \leq b_j(l_j(s))$. Together, we get that for every profile s in which $l_j(s) = N \setminus C$ and every $k \neq j$, it holds that $b_k(l_k(s)) \leq b_j(l_j(s))$. Therefore, no agent will migrate to resource M_j, as promised. We conclude that there exists a Nash equilibrium s of G in which the members of each coalition of size 2 are spread out.

We claim that playing s in every round is a T-SE of the repeated game. Obviously, none of the singletons can deviate since playing a Nash equilibrium in every round of the repeated game is also a Nash equilibrium of the repeated game. In addition, since in s each coalition is spread out, Observation 3 guarantees that no coalition of size 2 has a profitable migration. This concludes the proof. □

The preceding theorems assert that if the coalition structure is such that $|T_i| \leq 2$ for every i or $|T_i| \geq 2$ for every i, then every repeated game with non decreasing resources admits a T-SE. We next show that this is tight. In particular, the following proposition demonstrates that there exists a repeated RSG with a coalition structure that does not adhere to the structure just described, which does not admit a T-SE.

PROPOSITION 1. *There exists a repeated RSG with two identical nondecreasing resources such that $|T_1| = 1$ and $|T_2| = 3$ that does not admit a T-SE.*

PROOF. Let G be an RSG with two identical resources and four agents. The coalition structure is $T = \{T_1, T_2\}$, where $|T_1| = 3$ and $|T_2| = 1$. Suppose that $b(1)+2b(3) < 3b(2)$, and $b(2) < b(3)$, and consider the game $\hat{G} = \langle G, 3 \rangle$. Suppose by way of contradiction that the repeated game just discussed admits a T-SE, and let g be a T-SE of the game. In the third (and last) round of the game, the singleton will never share a resource with more than one additional plyer, since if it does, it will incur a cost of at least $b(3)$, while it can incur a cost of at most $b(2)$ by deviting. In addition, the three agents in T_1 will not all share a resource since if one of them deviates, the cost of each one of them will decrease from $b(3)$ to $b(2)$. Therefore, in the third round, every resource should be assigned exactly two agents. Using a backward induction argument, under the strategy profile g, in every round of the game two agents should be assigned to every resource, that is, $l_i(s^t(g)) = 2$ for every $i \in [m]$ and every $t \in \{1, \ldots, R\}$. Suppose the agents in T_1 deviate in a way that each agent in T_1 is left alone in one of the rounds and has a load of 3 in the other two rounds. Let g' denote the obtained profile. For every agent $J \in T_1$, it holds that $c_J(g') = 2c(3)+c(1) < 3c(2) = c_J(g)$. Therefore, g' is a strongly profitable deviation of T_1 and the game admits no T-SE. □

Remark. The construction given in the proof of Proposition 1 implies that the existence of T-SE in one-shot RSGs does not apply to general congestion games. This can easily be verified by constructing a congestion game that consists of three networks that are composed serially where each network is composed of two parallel edges with the cost functions and T-structure given in the aforementioned example.

4.2 The Case of Majority of Singletons

For special cases, we have a more refined characterization. In particular, if the majority of the agents are singletons and the resources are identical, we fully characterize the T-structures that admit a T-SE in repeated games. The characterization is slightly different for the cases of odd and even numbers of agents.

4.2.1 *Odd Number of Agents.* When the number of agents is odd, we find that every repeated RSG with identical resources admits a T-SE if and only if $|T_i| \leq 3$ for every i.

THEOREM 6. *If the number of agents is odd and there is a majority of singletons, every RSG with identical strictly increasing resources admits a T-SE if and only if $|T_i| \leq 3$ for every i.*

PROOF. For the first direction, we show that for every coalitional structure such that there exists i for which $|T_i| \geq 4$, there exists a repeated RSG that does not admit a T-SE.

Let x denote the size of the largest coalition in T, and consider an RSG with two identical strictly increasing resources that is repeated x rounds. Suppose also that the resource cost function satisfy the following condition.

$$(x-1) \cdot b\left(\left\lceil \frac{n}{2} \right\rceil + x - 1\right) + b\left(\left\lceil \frac{n}{2} \right\rceil - x + 1\right) < x \cdot b\left(\left\lfloor \frac{n}{2} \right\rfloor\right). \tag{1}$$

This is a valid cost function since $x \geq 4$. Suppose towards contradiction that the aforementioned game admits a T-SE, and let g be a T-SE of the game. In the last round of the game, it must hold that $|l_1(s^R(g)) - l_2(s^R(g))| \leq 1$. We show the impossibility of $l_2(s^R(g)) + 1 < l_1(s^R(g))$. An analogous argument shows the impossibility of $l_1(s^R(g)) + 1 < l_2(s^R(g))$. If $l_2(s^R(g)) + 1 < l_1(s^R(g))$, then by monotonicity $b(l_2(s^R(g)) + 1) < b(l_1(s^R(g)))$. Since there is a majority of singletons, there must exist a singleton on M_1. That singleton can benefit by migrating to M_2, contradicting g is a T-SE. Following a backward induction argument, it must hold that, for every round t of the game $|l_1(s^t(g)) - l_2(s^t(g))| \leq 1$, the load on one resource is $\lfloor \frac{n}{2} \rfloor$ and therefore the load on the other resource is $\lceil \frac{n}{2} \rceil$. Therefore, every agent is assigned in every round of the game to a resource

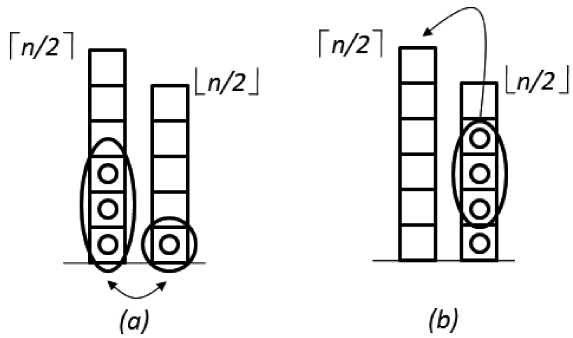

Fig. 2. (a) A bidirectional deviation leaving a single member alone; (b) A unidirectional deviation leaving a single member in the less loaded machine.

with load of at least $\lfloor \frac{n}{2} \rfloor$, and its total cost in the repeated game is at least $x \cdot b(\lfloor \frac{n}{2} \rfloor)$.

We next contradict the stability of g by constructing a strictly profitable deviation. Consider the coalition of size x. The deviation of the coalition will be such that one of the coalition members will be left alone in one of the rounds (a different member in every round). A coalition member who is left alone will incur a cost of at most $b(\lceil \frac{n}{2} \rceil - x + 1)$. To see this, note that the most extreme case is when one of the coalition members is assigned to the less loaded resource in every round of the game (see Figure 2(a) for an illustration of a coalition of size 4). In addition, every other coalition member will incur a cost of at most $b(\lceil \frac{n}{2} \rceil + x - 1)$. To see this, note that the most extreme case is when all the coalition members are assigned to the less-loaded resource and $x - 1$ of them migrate to the more loaded one (see Figure 2(b) for an illustration of a coalition of size 4).

In the assignment that is obtained from the deviation, the total cost of every coalition member is at most $(x - 1) \cdot b(\lceil \frac{n}{2} \rceil + x - 1) + b(\lceil \frac{n}{2} \rceil - x + 1)$, which is smaller than $x \cdot b(\lfloor \frac{n}{2} \rfloor)$ (its cost in the original profile s) by Equation (1). This concludes the proof of the first direction of the assertion of the theorem.

We now turn to prove the second direction of the assertion. In particular, we show that, for every repeated RSG with identical nondecreasing resources with a majority of singletons and an odd number of agents, if $|T_i| \leq 3$ for every i, then the game admits a T-SE.

We distinguish between two cases:

Case (a): *The number of resources is greater than* 2. Let s be a Nash equilibrium of the one-shot game, and consider a (T, s)-spread-out assignment of the one-shot game. Since the number of resources is greater than 2, no two members of any one coalition will be assigned to the same resource (recall that the resources are identical). By Observation 3, playing s in every round of the game is a T-SE of the repeated game.

Case (b): The number of resources is 2. We claim that in the case of two resources, there always exists a Nash equilibrium s of the one-shot game such that the load on the two resources differ by at most 1, that is, $|l_2(s) - l_1(s)| \leq 1$. Suppose towards contradiction that, for every profile s which is a Nash equilibrium of the one-shot game, it holds that $|l_2(s) - l_1(s)| > 1$. Then either $b(l_2(s)) \geq b(l_1(s)+1)$ or $b(l_1(s)) \geq b(l_2(s)+1)$. The profile in which a single agent migrates from M_1 to M_2 (respectively, from M_2 to M_1) is a Nash equilibrium as well, which arrives at a contradiction. Therefore, the one-shot game admits a Nash equilibrium in which one of the resources has load $\lceil \frac{n}{2} \rceil$ and the second resource has load $\lfloor \frac{n}{2} \rfloor$).

Among all the Nash equilibria of the one-shot game in which the loads on the resources differ by 1, consider the assignment s^*, constructed as follows. For every coalition of size 3, assign one its members to M_1, and add one singleton to M_1. Assign the other two members of this coalition to M_2. At the end of this process, the two resources have equal load. We repeat this process for every coalition of size 3. Note that since there is a majority of singletons, there is a sufficient number of singletons to support this process. For coalitions of size 2, assign one of the agents to M_1 and the other to M_2. Let x denote the number of singletons that have not been assigned yet. It is easy to verify that x is odd. Assign $\lfloor \frac{x}{2} \rfloor$ singletons to M_1 and $\lceil \frac{x}{2} \rceil$ singletons to M_2. It holds that $l_2(s^*) = l_1(s^*) + 1$.

We claim that playing s^* in every round constitutes a T-SE of the repeated game. By Lemma 5, it is sufficient to show that for every coalition T_i, there exists an agent $J \in T_i$ which is $T_i^{s^*}$-minimal in the one-shot game. The claim holds trivially for coalitions of size 1, and, for coalitions of size 2, the claim follows from Lemma 4. It remains to show that the claim holds for coalitions of size 3. Let T_i be a coalition of size 3, and consider the member $J \in T_1$ that is assigned to M_1. If J migrates to M_2, then its cost will be at least $b(l_2(s^*) - 1)$ (since the lowest cost will occur if the other two agents of the coalition migrate from M_2 to M_1). But since $l_2(s^*) - 1 = l_1(s^*)$, the cost of J after the deviation will be at least its original cost. Therefore, the assignment s^* is $T_i^{s^*}$-minimal for the member that is assigned to M_1. This concludes the proof. □

4.2.2 *Even Number of Agents.* When the number of agents is even, we find that every repeated RSG with identical resources admits a T-SE if and only if $|T_i| \leq 2$ for every i.

THEOREM 7. *If the number of agents is even and there is a majority of singletons, every RSG with identical strictly increasing resources admits a T-SE if and only if $|T_i| \leq 2$ for every i.*

PROOF. For the first direction, we show that, for every coalitional structure such that there exists i for which $|T_i| \geq 3$, there exists a repeated RSG that does not admit a T-SE.

Let x denote the size of the largest coalition in T, and consider an RSG with two identical strictly increasing resources that is repeated x rounds. Suppose also that the resource cost function satisfies the following condition:

$$(x-1) \cdot b\left(\frac{n}{2}+1\right) + b\left(\frac{n}{2}-1\right) < x \cdot b\left(\frac{n}{2}\right). \tag{2}$$

Suppose towards contradiction that the aforementioned game admits a T-SE, and let g be a T-SE. In the last round of the game, it must hold that

$l_1(s^R(G)) = l_2(s^R(G))$ since otherwise the gap between the resources' loads is at least 2, and so a singleton that is assigned to the more loaded resource (there must be such an agent since there is a majority of singletons) can improve its cost by a migration. Following a backward induction argument, in every T-SE of the repeated game, the cost of every agent in every round of the repeated game is $b(\frac{n}{2})$, and its total cost is $x \cdot b(\frac{n}{2})$.

We next contradict the stability of g by constructing a strictly profitable deviation. Consider the coalition of size x. There must exist a resource that contains at least $\lceil \frac{x}{2} \rceil \geq 2$ agents of this coalition. The deviation of the coalition will be such that one of the coalition members will be left alone in one of the rounds (a different member in every round) on that resource. Every coalition member will incur a cost of at most $b(\frac{n}{2} - 1)$ at the round in which it is left alone, while incurring a cost of at most $b(\frac{n}{2} + 1)$ in the other rounds. Thus, each member of the coalition incurs a cost of at most $(x - 1) \cdot b(\frac{n}{2} + 1) + b(\frac{n}{2} - 1)$, which is strictly smaller than $x \cdot b(\frac{n}{2})$ by Equation (2). This concludes the proof.

We now turn to prove the second direction of the assertion. In particular, we show that, for every repeated RSG with identical nondecreasing resources with a majority of singletons and an even number of agents, if $|T_i| \leq 2$ for every i, then the game admits a T-SE.

Let s be a Nash equilibrium of the one-shot game, and consider a (T, s)-spread-out assignment of the one-shot game. Since $|M| \geq 2$ and $|T_i| \leq 2$ for every i, no two members of the same coalition are assigned to the same resource. By Observation 3, repeating this assignment in every round of the game constitutes a T-SE of the repeated game. □

5. CONCLUSIONS

Stability against coalitional deviations is a main interest in game theory and in many social environments. In contrast to the notion of Nash equilibrium, it accounts for situations where agents can coordinate their actions and their deviations in particular. The focus of this article is those situations where the atomic unit of deviation is a coalition, and the prespecified coalitions constitute a partition over the set of agents. We study this notion in resource selection games where general stability to coalitional deviations is not guaranteed when using the notion of super strong equilibrium or when requiring stability in repeated settings. Our results suggest that when the agents are partitioned into prespecified coalitions both of the these limitations are mitigated to some extent. In particular, we present several existence results in a variety of classes of resource selection games that do not admit stable outcomes in the more general sense. We believe that the introduction of the notion of stability is very natural and that this notion occurs in many social situations in real life where agents are indeed partitioned into groups such as families, friends, and clubs. Our work raises a host of interesting directions for future research. The main three are: (i) to close the gaps that are left in resource selection settings, (ii) to study our notion of stability in additional settings such as more general congestion games, and (iii) to study related notions of stability where the coalitions are prespecified but do not necessarily constitute a partition of the agents, for example, where there could be an overlap between the coalitions.

APPENDIX

A. PROOF OF THEOREM 2

Let $\hat{s}$ be a costwise lexicographically minimal profile, and let s be a $(T, \hat{s})$-spread-out profile. Suppose without loss of generality that $l_2(s) \geq l_1(s)$.

It follows from Lemma 3 that s is a Nash equilibrium. By Lemma 1, it holds that $l_i(s') \leq l_i(s) + 1$ for every resource M_i for every weakly profitable deviation s'. If $b_1(l_1(s)) = b_2(l_2(s))$, there is no weakly profitable deviation: if the load on one of the resources grows, the cost of the jobs migrating to this resource increases. If the load on both resources remain the same, all the jobs incur the same cost in s and in s', and therefore s' is not weakly profitable. In conclusion, we need to consider only cases in which $b_1(l_1(s)) \neq b_2(l_2(s))$. We distinguish between two cases.

Case (a): $b_1(l_1(s)) < b_2(l_2(s))$. Consider a deviation s'. By Lemma 1, the load of every resource can increase by at most a single job. Since $b_1(l_1(s)) < b_2(l_2(s))$, if the load on M_2 increases, there must be a job in the coalition whose cost increases, thus it must be the case that $l_1(s') = l_1(s)+1$ and $l_2(s') = l_2(s)-1$. The deviation to s' can have one of the following two structures: (i) A unidirectional deviation from M_2 to M_1, where the jobs that stay on M_2 improve and the ones migrating to M_1 are indifferent. This case can occur only if there exists a coalition T_i such that $s_J = M_2$ for every $J \in T_i$ (by Observation 1). In this case, $b_2(l_2(s')) = b_2(l_2(s) - 1) < b_2(l_2(s))$ and $b_2(l_2(s)) = b_1(l_1(s) + 1) = b(l_1(s'))$. Therefore,

$$b_2(l_2(s')) = b_2(l_2(s) - 1) < b_1(l_1(s) + 1) = b_1(l_1(s')). \tag{3}$$

(ii) A bidirectional deviation with a structure as described in Lemma 2. In this case, it must hold that $b_2(l_2(s)) \geq b_1(l_1(s) + 1) = b(l_1(s'))$ (so that the jobs migrating to M_1 do not incur a higher cost) and $b_2(l_2(s')) = b_2(l_2(s)-1) \leq b_1(l_1(s))$ (so that the jobs migrating to M_2 do not incur a higher cost). It is easy to see that if one of the preceding two inequalities is strict, s' is costwise lexicographically smaller than s, contradicting the minimality of s. Therefore,

$$b_2(l_2(s)) = b_1(l_1(s) + 1) \tag{4}$$

and

$$b_2(l_2(s) - 1) = b_1(l_1(s)). \tag{5}$$

Integrating the strict monotonicity into the equations we get $b_2(l_2(s')) = b_2(l_2(s) - 1) < b_2(l_2(s)) = b_1(l_1(s) + 1) = b_1(l_1(s'))$. We conclude that in (i) as well as in (ii) Equation (3) must hold.

At this stage, for every i it holds that the number of agents from T_i that are on M_1 can be greater than the number of agents from T_i that are on M_1 by at most 1. This claim holds in the profile s since it is a T-spread-out assignment. In case (i) (unidirectional deviation), M_1 contains a single agent of the deviating coalition, and the other coalitions are like in s. In case (ii) (bidirectional deviation), all the members of the deviating coalition that were on M_1 deviated to M_2, and a number of agents greater than this number by 1 moved to M_1, so the claim holds for the deviating coalition, and it obviously holds for the other coalitions (just like in s).

We next consider a weakly profitable deviation s'' from s'. s'' cannot be a unidirectional deviation from M_2 to M_1 since $b_1(l_1(s')) > b_2(l_2(s'))$. s'' cannot be a unidirectional deviation from M_1 to M_2 either since M_1 does not contain whole coalitions (see Observation 1). Thus, it can only be a bidirectional deviation. In any bidirectional deviation s'', M_2 should increase by a single job, thus $l_1(s'') = l_1(s') - 1 = l_1(s)$ and $l_2(s'') = l_2(s') + 1 = l_2(s)$. Since the cost of the job migrating to M_1 cannot increase, it must hold that $b_1(l_1(s)) = b_1(l_1(s'')) \leq b_2(l_2(s')) = b_2(l_2(s) - 1)$. If the deviation to s' was bidirectional (case (ii)), then $b_2(l_2(s) - 1) = b_1(l_1(s))$ (see Equation (5)), and $b_2(l_2(s)) = b_1(l_1(s) + 1)$ (by Equation (4)). Therefore, the only agents who can strictly improve by this deviation are the jobs that stay on M_1. But since the number of agents migrating from M_1 to M_2 is greater by 1 than the number of agents moving in the opposite direction, it must hold that, in s', the number of agents from the deviating coalition assigned to M_1 is greater by at least 2 than those assigned to M_2. But this contradicts our claim previous (showing that it can be greater by at least 1).

If, however, the deviation to s' was unidirectional (case (i)), then a bidirectional from s' to s'' may still be possible, and s'' is a costwise lexicographic minimum (since the resources' loads according to s'' are identical to s). Therefore, s'' resembles s in almost all aspects except that the number of coalitions that are fully assigned to M_2 is smaller by 1 than this number in s.

Since the number of coalitions is finite, we can repeat this process only a finite number of times until there are no coalitions that are fully assigned to M_2. Then. the first stage deviation must be a bidirectional one, and the obtained assignment must be a T-SSE from the exact same reasoning as before (since the number of agents from the deviating coalition assigned to M_1 cannot be greater by more than 1 than those assigned to M_2).

Case (b): $b_1(l_1(s)) > b_2(l_2(s))$. We show that in this case s is a T-SSE. In any deviation s', M_2 should increase by a single job. By Observation 1, the deviation cannot be unidirectional since no coalition exists that is fully assigned to M_1, and a singleton cannot unilaterally deviate since s is a NE. Therefore, it can only be a bidirectional deviation of a coalition T_i, where all agents $J \in T_i$ assigned to M_2 migrate to M_1 and a number of agents that is greater than this number by 1 migrate from M_1 to M_2 (by Lemma 2). In this case, it must hold that $b_2(l_2(s)) \geq b_1(l_1(s) - 1)$ (so that the cost of the agents migrating to M_1 does not increase) and $b_2(l_2(s) + 1) \leq b_1(l_1(s))$ (so that the cost of the agents migrating to M_2 does not increase). In addition, since s is a NE, it must hold that $b_2(l_2(s) + 1) \geq b_1(l_1(s))$. Therefore, $b_2(l_2(s) + 1) = b_1(l_1(s))$.

After such a deviation, the load on M_2 increases, and since it has a strictly increasing cost function, so does its cost. Therefore, there cannot remain any coalition members on M_2. Additionally, there cannot remain any coalition members on M_1 since s is a T-spread-out assignment in which the number of coalition members on M_1 can be greater than the number of coalition members on M_2 by at most 1 for any coalition. But since there must be a coalition member whose cost strictly decreases, it can only be a member that migrated from M_2 to M_1. Thus, it must hold that $b_2(l_2(s)) > b_1(l_1(s) - 1)$. We get:

$b_2(l_2(s)) > b_1(l_1(s) - 1) = b_1(l_1(s'))$ and $b_2(l_2(s')) = b_2(l_2(s) + 1) = b_1(l_1(s))$, implying that s' is costwise lexicographically smaller than s, contradicting the minimality of s. This concludes the proof.

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