the $u_1, u_2 ... u_n ...$ sequence. If we remove the first element of u_i and the first element of u_j we will have a new sequence $u_1, u_2, , , u_{i-1}, \hat{v_i}, , , \hat{v_j}, , , \hat{v_k}$. All the words denoted by $\hat{v_n}$ are words that are actually the u_ns located in the same position of the original sequence without their first element. Using the same argument that we used on the first elements of the u_is, we must have a subsequence of the $\hat{v_n}$ with a non-increasing first elements. Taking that sequence and appending it to the $u_1, u_2, , , u_{i-1}$, we will have a new sequence $u_1, u_2, , , u_{i-1}, v_i, , , v_j, , , v_k$. Since our sequence is minimal in it's words length and the sequence we have just built has elements (v_i) with smaller length, it is not a counter example. We can't embed any of the u_ns where n=1,2,,i-1 in the v_js j=i,i+1,i+2,,,. if such an embedding was possible, and a u_n was embedded in a v_j then in the original sequence, u_n would have been embedded in the u_j which contradicts our assumption. Since our newly built sequence is not a counter example, we must have k ,l where $k, l > i - 1$ such that $v_k \lhd v_l$ and $a_k \lesssim a_l$. This fact means that $u_k \lhd u_l$ in contradiction to our assumption on the u sequence.

Back to the "Tree Theorem's proof"...

Let's denote a homeomorphic embedding between trees by $\leqslant$.

The tree theorem states that if we have a well quasi order on the tree nodes then the homeomorphic embedding order is a well quasi order.

Suppose it is not so, then we have an infinite number of trees $T_1, T_2 ... T_n ...$ such that $T_i \nleqslant T_j$ for every $i < j$. In a similar way to what we did with the embedding sequence proof, we will build a minimal counter example where the number of nodes replaces the sequence length, u_is where i=1,2,3...

In a similar way to the embedding sequence proof, no one of the u_is is the empty tree (it would have been embedded in it's follower). Let's look on the trees roots since the elements are well quasi order, there is an infinite subsequence of trees $u_i, , , u_j, , , u_k, , ,$ which roots r_ns have the following relation $r_i \lesssim r_j$. Let S be the set of all subtrees created from the removal of the roots from the sub-sequence mentioned in the preceding sentence.

Lemma

S is well quasi ordered.

Proof: Let's look on an infinite sequence of S. Suppose S is not well quasi ordered, then we have a counter example S_n which is an infinite sequence that shares no nodes between it's elements. Let S_k be the subtree of $\{u_i\}$ with the smallest index. Suppose that this subtree belongs to the u_j tree. Let's look at the following sequence: $u_1, u_2, , , u_{j-1}, S_k, S_{k+1}, , ,$. In a similar way to what we did in the embedding sequence proof, we can't embed any of the u_ns where n=1,2,,,j-1 in any of S_k since it would have contradicted the fact that the $\{u_i\}$ sequence is a counter example. The S_ks themselves cannot be embedded in each other and we have received a counter example with trees smaller in their node number than the u_is sequence and this contradicts the minimality of the u_i sequence. Hence S is well quasi ordered. $\Box lemma$

Definition:
An order relation $\leqq$, that contains a homeomorphic embedding is a an order relation which has the following characteristic:
If T $\leqslant$ (homeomorphic embedded) $T_1 \rightarrow$ T $\leqq T_1$.
Deduction: Any order relation on trees that contains a homeomorphic embedding is well founded order.
Proof: Let's suppose it is not so then there is an infinite decreasing sequence of trees such that $T_1 > T_2 > T_3 > .. > T_i > ... > T_j....$ On the other hand there exist i,j such as $j > i$ where $T_i \leqslant T_j$ (since homeomorphic embedding is well quasi order). But following the fact that the order relation is a relation that contains homeomorphic embedding, this means that $T_i \leqq T_j$ which contradict our first assumption. $\Box$