Jenga *

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Abstract

Jenga is a popular block game played by two players. Each player in her turn has to remove a block from a stack, without toppling the stack, and then add it the top of the stack. We analyze the game mathematically and describe the optimal strategies of both players. We show that 'physics', that seems to play a dominant role in this game, does not really add much to the complexity of the (idealized) game, and that Jenga is, in fact, a Nim-like game. In particular, we show that a game that starts with n full layers of blocks is a win for the first player if and only if $n = 2$, or $n \equiv 1, 2 \pmod 3$ and $n \geq 4$. We also suggest some several natural extensions of the game.

1 Introduction

Jenga, see Figure 1, is a popular block game played by players of all ages around the world. We analyze an idealized version of this game and show that it is equivalent to a very simple Nim-like game. (See Berlekamp *et al.* [BCG82] for information on Nim and similar games.) We then give a complete solution of this simple combinatorial game, giving the optimal strategies of both players. We also consider some natural physical, and non-physical, generalizations of Jenga.

2 Rules of the game

The game is played using $3n$ wooden blocks each of dimensions $1 \times \frac{1}{3} \times \frac{1}{3}$. In the commercially available game, $n = 18$. The initial position of the game in an n-level tower of dimensions $1 \times 1 \times n$. Each level is composed of three adjacent blocks that are at right angles to the blocks of the level below them. The two players then take turns alternately. The official instructions attached to the game say that each player, in her turn, has to remove a block from anywhere *below* the highest completed level, and then stack it on top of the tower, at right angles to the blocks just below it. The stacked block should not start a new level, unless the highest level is full. The player that topples the tower loses the game.

Figure 1: A typical position in the game of Jenga.

In a real game of Jenga, when a player tries to remove a block from the stack, she inevitably changes, by small amounts, the position of some of the other blocks. We consider an idealized version of the game in which each player simply chooses the block to be removed. This block is then carefully removed by a robot, without changing the positions of the other blocks. The resulting tower then either continues to stand, or it collapses. The same holds for the placement of the removed block on the top of the tower.

3 Stability of towers

A stack of blocks of height k is represented by a k-tuple $(a_1, a_2, \ldots, a_k)$, where $a_i = a_{i1}\, a_{i2}\, a_{i3} \in \{0,1\}^3 - \{000\}$, for $1 \leq i \leq k$. We assume that the dimensions of each block are $1 \times \frac{1}{3} \times \frac{1}{3}$. If $a_{ij} = 1$, then there is a block in the j-th slot of the i-th level, i.e., at $[\frac{j-1}{3}, \frac{j}{3}] \times [0,1] \times [\frac{i-1}{3}, \frac{i}{3}]$, if i is odd, and at $[0,1] \times [\frac{j-1}{3}, \frac{j}{3}] \times [\frac{i-1}{3}, \frac{i}{3}]$, if i is even. We refer to such a stack as an *alternating tower*, or just a *tower*, for short. For example, the tower shown in Figure 1 is $(111, 011, 111, 011, 101, 111, \ldots)$.

* Jenga (R) is a Milton Bradley Game.

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DEFINITION 3.1. (STABILITY) *A tower* $(a_1, a_2, \ldots, a_k)$ *is said to be* stable *if and only if a physical realization of it does not collapse, even if blocks are very slightly displaced from their intended positions. A tower is said to be* semi-stable *if and only if an exact physical realization of it might stand, at least temporarily, but an infinitesimal displacement of one of its blocks would collapse it.*

DEFINITION 3.2. (VALIDITY) *A tower* $(a_1, a_2, \ldots, a_k)$ *is said to be* valid, *if and only if* $a_i \neq 100$ *and* $a_i \neq 001$, *for every* $1 \leq i < k$, *and if* $k \geq 3$ *then* (a_{k-2}, a_{k-1}, a_k) *is not one of* $(010, 010, 100)$, $(010, 010, 001)$, $(011, 010, 100)$ *and* $(110, 010, 001)$.

LEMMA 3.1. (STABILITY LEMMA) *A tower is stable if and only if it is valid.*

Proof: By elementary physics, a tower of height k is stable if and only if, for every $1 \leq i < k$, the *center of gravity* of levels $i+1$ up to k lies above the *interior* of the *convex hull* of the *area of contact* between levels i and $i+1$. A tower is semi-stable if for some i, the center of gravity lies above a *boundary* point of the convex hull of the contact area.

Let $\mathrm{cent}(a_1, a_2, \ldots, a_k)$ be the projection of the center of gravity of the tower $(a_1, a_2, \ldots, a_k)$ on the x-y plane. Let $\mathrm{conv}(a_1, a_2)$ be the projection, again on the x-y plane, of the area of contact between the first and second levels of the tower (a_1, a_2). If $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, we let $\mathbf{x}^R = (x_2, x_1)$. More formally then, a tower $(a_1, a_2, \ldots, a_k)$ is stable if and only if $\mathrm{cent}(a_{i+1}, a_{i+2}, \ldots, a_k)^R \in \mathrm{int}(\mathrm{conv}(a_i, a_{i+1}))$, for $1 \leq i < k$, where $\mathrm{int}(A)$ denotes the interior of a set A. The reverse operation is needed here as the orientation of the layers alternates. (A tower is semi-stable if and only if $\mathrm{cent}(a_{i+1}, a_{i+2}, \ldots, a_k)^R \in \mathrm{conv}(a_i, a_{i+1})$, for $1 \leq i < k$, and for at least one such i, $\mathrm{cent}(a_{i+1}, a_{i+2}, \ldots, a_k)^R$ is on the boundary of $\mathrm{conv}(a_i, a_{i+1})$.)

It is easy to check, by inspection, that a 2-layer tower (a_1, a_2) is stable if and only if it is valid. The only towers that are non-stable and non-valid have $a_1 \in \{100, 001\}$. These towers are not even semi-stable. (Recall that the even numbered levels are rotated by 90°.)

It is also easy to check that a 3-level tower (a_1, a_2, a_3) is stable if and only if it is valid. Here the situation is a little bit more interesting. The towers $(010, 010, 100)$, $(010, 010, 001)$, $(011, 010, 100)$ and $(110, 010, 001)$ are semi-stable and are therefore defined not to be valid.

To prove the claim for higher towers, we need the following claims:

CLAIM 3.1. *If* $(a_1, a_2, \ldots, a_k)$ *is a valid tower, and* $k \geq 3$, *then* $\mathrm{cent}(a_1, a_2, \ldots, a_k) \in (\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$.

CLAIM 3.2. *If* $a_1, a_2 \notin \{100, 001\}$, *then* $\mathrm{conv}(a_1, a_2) \supseteq [\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}]$.

The proofs of these claims are given below. We show first that they imply the validity of the lemma.

We show first, by induction on k, that if $(a_1, a_2, \ldots, a_k)$ is valid, then it is stable. We already know that the claim holds for $k = 1, 2, 3$. Suppose therefore that $k \geq 4$. It is easy to see that if $(a_1, a_2, \ldots, a_k)$ is valid, then $(a_2, \ldots, a_k)$ is also valid. By the induction hypothesis, $(a_2, \ldots, a_k)$ is stable. We only have to show, therefore, that $\mathrm{cent}(a_2, a_3, \ldots, a_k)^R \in \mathrm{int}(\mathrm{conv}(a_1, a_2))$. This follows immediately from the two claims.

We next show, again by induction on k, that if $(a_1, a_2, \ldots, a_k)$ is not valid, then it is not stable. If $(a_2, a_3, \ldots, a_k)$ is not valid, then by the induction hypothesis, it is not stable, and $(a_1, a_2, \ldots, a_k)$ is not stable either. Suppose, therefore, that $(a_2, a_3, \ldots, a_k)$ is valid, but $a_1 = 100$. (The case $a_1 = 001$ is identical.) But then clearly $\mathrm{conv}(a_1, a_2) \subseteq [0, \frac{1}{3}] \times [0, 1]$, while $\mathrm{cent}((a_2, a_3, \ldots, a_k)^R \in (\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$ and the claim follows.

All that remains, therefore, is to prove the two claims:

Proof: (of Claim 3.1) The inspection used to verify the claim of the lemma for $k = 2$ also shows that for every valid tower (a_1, a_2) we have $\mathrm{cent}(a_1, a_2) \in (\frac{1}{3}, \frac{2}{3}) \times [\frac{1}{3}, \frac{2}{3}]$. (It is easy to check, for example, that the tower that minimizes the y coordinate of $\mathrm{cent}(a_1, a_2)$ is $(010, 100)$, and that the y coordinate is then $\frac{1}{3}$. Now, it is also easy to check that if $a_1 \neq 100$ and $a_1 \neq 001$, then $\mathrm{cent}(a_1) \in (\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$. As $\mathrm{cent}(a_1, a_2, \ldots, a_k)$ is a weighted average of $\mathrm{cent}(a_1)$ and $\mathrm{cent}(a_2, a_3 \ldots, a_k)$, the claim follows easily by induction. □

The proof of Claim 3.2 is immediate. This completes the proof of the stability lemma. □

4 Combinatorial formulation

Each position in the game is, therefore, characterized by a valid tower $(a_1, a_2, \ldots, a_k)$, for some $k \geq 1$. In the initial position, $a_1 = a_2 = \ldots = a_k = 111$. The rules of the game instruct each player to "remove a block from anywhere *below* the highest completed story. Then stack it on top of the tower, at right angles to the blocks just below it." If follows easily, by induction, that in any tower obtained during a valid game of Jenga, we have $a_{k-1} = 111$ or $a_k = 111$. (Note, in particular, that this means that no semi-stable towers can be encountered during the game. Is this the rational behind the requirements that blocks should only be removed from stories below the highest completed story, and not, for example, from anywhere below the top story?)

The configuration of blocks at each of the levels of a valid tower, except possibly the highest level, is, up to symmetry, of one of these types: III, II-, I-I, -I-, where I denotes a block, and - denotes an empty position. Levels of type I-I and -I- are *inactive*, in the sense that any block removed from them would topple the tower. There are, therefore, only three types of moves that a player can make without toppling a tower:

I-I Remove the middle block from a story of type III.

II- Remove a side block from a story of type III.

-I- Remove the side block from a story of type II-.

The name given to each type of move is the configuration in which the corresponding level is left after this move. In each case, the removed block is placed at the top level, at right angles to the blocks below it. (Actually, the rules do not specify *where* at the top level this block should be placed. But, as it is placed on top of a full story, it follows from the stability lemma that if the removal of the block did not topple the tower, placing that block at *any* position on the top level would not topple it either.

As a further consequence of the stability lemma, note that if there are several levels at which a given type of move may be applied, then it *does not matter* which one of these levels is chosen. In other words, a position in the game is characterized, combinatorially, by a triplet (x, y, z), where:

x – number of III levels, below highest full level.

y – number of II- levels, below highest full level.

z – number of blocks on top of the highest full level

Here x, y are non-negative integers, while $z \in \{0, 1, 2\}$. Note that the number and type of the inactive levels is not recorded, as it has no influence on the continuation of the game. The three possible moves, in this notation, are therefore:

I-I $(x, y, z) \to (x - 1, y, z + 1)$ (valid if $x > 0$),

II- $(x, y, z) \to (x - 1, y + 1, z + 1)$ (valid if $x > 0$),

-I- $(x, y, z) \to (x, y - 1, z + 1)$ (valid if $y > 0$),

where position $(x, y, 3)$, when it occurs, is immediately converted to $(x + 1, y, 0)$.

A tower of n full levels corresponds to position $(n - 1, 0, 0)$, as only full levels below the highest full level are counted. The positions $(0, 0, z)$, where $z\{0, 1, 2\}$, are losing positions, as the player whose turn is to move has no legal moves.

5 Analysis

Let $V(x, y, z)$ be the *value* of position (x, y, z), where the value of a position is 1 if the first player can force a win from this position, and 0 otherwise. Clearly, for $z \in \{0, 1, 2\}$ we have:

$$V(x, y, z) =$$

$$\begin{cases} 1 - \min \left\{ \begin{array}{c} V(x-1, y, z+1) \\ V(x-1, y+1, z+1) \\ V(x, y-1, z+1) \end{array} \right\} & \text{if } x > 0, y > 0, \\ 1 - \min \left\{ \begin{array}{c} V(x-1, 0, z+1) \\ V(x-1, 1, z+1) \end{array} \right\} & \text{if } x > 0, y = 0, \\ 1 - V(0, y-1, z+1) & \text{if } x = 0, y > 0, \\ 0 & \text{if } x = y = 0, \end{cases}$$

while

$$V(x, y, 3) = V(x + 1, y, 0) .$$

This gives a simple recursive procedure for computing $V(x, y, z)$ for any (x, y, z). Would this recursive procedure always halt? Yes, as, for example, each recursive call decreases $x + \frac{1}{2}y + \frac{1}{3}z$ by at least $\frac{1}{6}$, while the 'normalization' rule $(x, y, 3) \to (x + 1, y, 0)$ does not increase it. (The existence of some weight function that could be used to show that the recursion is well defined is not surprising. We know that each Jenga game must end as each move *raises* the center of gravity of the tower.)

Let

$$v = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \quad \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \quad \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \right] .$$

We let $v(x, y, z)$, where $x, y, z \in \{0, 1, 2\}$ be the (x, y)-th element of the z-th matrix defining v. Thus, for example $v(0, 1, 2) = 0$ while $v(2, 1, 0) = 1$. We also let

$$v_0 = [0\ 1\ 0] .$$

We claim:

THEOREM 5.1.

$$V(x, y, z) =$$

$$\begin{cases} v(\, x \bmod 3\, ,\, y \bmod 3\, ,\, z\,) & \textit{if } x > 0 \textit{ or } z > 0, \\ v_0(\, y \bmod 3\,) & \textit{if } x = z = 0. \end{cases}$$

Proof: By simple induction. □

In particular, we get that $V(x, 0, 0) = 1$, if and only if $x = 1$, or $x \equiv 0, 1 \pmod 3$ and $x \geq 3$. (Consult the first column of the first matrix in the definition of v.) It follows, therefore, that a game that starts with a tower of height n is a win for the first player if and only if $n = 2$, or $n \equiv 1, 2 \pmod{3}$ and $n \geq 4$.

The optimal moves from any position are given by:

$$m = \begin{bmatrix} \text{I-I} & \text{-I-} & \text{II-} & \circ & \text{-I-} & \text{-I-} & \circ & \circ & \text{-I-} \\ \text{I-I} & \circ & \text{II-} & \begin{array}{c}\text{I-I}\\\text{II-}\\\text{-I-}\end{array} & \text{I-I} & \text{II-} & \begin{array}{c}\text{II-}\\\text{-I-}\end{array} & \begin{array}{c}\text{I-I}\\\text{-I-}\end{array} & \circ \\ \circ & \text{-I-} & \circ & \circ & \text{II-} & \text{I-I} & \text{I-I} & \text{II-} & \begin{array}{c}\text{I-I}\\\text{II-}\end{array} \end{bmatrix}$$

and

$$m_0 = [\, \circ \quad \text{I-I} \quad \circ \,] \,.$$

More specifically, the optimal move(s) from position (x, y, z), where $x > 0$ or $z > 0$, are given by $m(\, x \bmod 3\,,\, y \bmod 3\,,\, z\,)$, while the optimal move(s) from position $(0, y, 0)$, are given by $m_0(\, y \bmod 3)$. From some positions, there are several optimal moves. In particular, from positions of the form $(x, y, 1)$, where $x \equiv 1 (\bmod 3)$ and $y \equiv 0 (\bmod 3)$, all three moves are winning moves! Entries in m and m_0 containing $\circ$ correspond to loosing positions, so all moves from these positions are loosing moves. The claim that these are indeed the optimal moves can again be verified by induction.

There does not seem to be a more compact way of characterizing the winning positions in Jenga, and the optimal moves from each position. Note, however, that the characterization given is very compact. Given the constant-size arrays m and m_0, we can find an optimal move from any position in *constant* time.

6 Generalization

We suggest the following natural generalization of Jenga, called Jenka$_k$. This game is played by blocks of dimension $\frac{1}{k} \times \frac{1}{k} \times 1$, and each layer in the initial tower is composed of k such blocks. The rules of the game remain unchanged. As pointed out by one of the reviewers, the starting position of Jenka$_k$, for even k, is a win for the second player, as the second player can always play a move that is the mirror image of the move played by the first player. Analyzing Jenka$_k$, for $k > 3$, k odd, seems to be a more challenging task. Our analysis of Jenka$_3$ relied heavily on the stability lemma that implied, among other things, that two towers that contain the same number of layers of each type are equivalent. Unfortunately, this claim does not hold when $k \geq 4$. Thus, at least potentially, there may be an exponential number of non-equivalent towers with a given number of blocks.

Open problem 1: For which values of n can the first player force a win in a game of Jenka$_k$ that starts with n-full levels, where $k > 3$ is an odd number?

Open problem 2: What is the complexity of determining whether a given position in Jenka$_k$, for $k > 3$, is a win for the first player?

7 Concluding remarks

We presented a complete analysis of Jenka$_3$. Analyzing Jenka$_k$, for $k > 3$, k odd, remains a challenging open problem.

References

[BCG82] E.R. Berlekamp, J.H. Conway, and R.K. Guy. *Winning Ways for your mathematical plays*, volume 1. Academic Press, 1982.