# RIGIDITY IN FINITE-ELEMENT MATRICES: SUFFICIENT CONDITIONS FOR THE RIGIDITY OF STRUCTURES AND SUBSTRUCTURES

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Abstract. We present an algebraic theory of rigidity for finite-element matrices. The theory provides a formal algebraic definition of finite-element matrices; notions of rigidity of finite-element matrices and of mutual rigidity between two such matrices; and sufficient conditions for rigidity and mutual rigidity.

We also present a novel sparsification technique, called *fretsaw extension*, for finite-element matrices. We show that this sparsification technique generates matrices that are mutually-rigid with the original matrix. We also show that one particular construction algorithm for fretsaw extensions generates matrices that can be factored with essentially no fill. This algorithm can be used to construct preconditioners for finite-element matrices.

Both our theory and our algorithms are applicable to a wide-range of finite-element matrices, including matrices arising from finite-element discretizations of both scalar and vector partial differential equations (e.g., electrostatics and linear elasticity).

Both the theory and the algorithms are purely algebraic-combinatorial. They manipulate only the element matrices and are oblivious to the geometry, the material properties, and the discretization details of the underlying continuous problem.

## 1. Introduction

This paper presents an algebraic-combinatorial theory of rigidity for finite-element matrices and applies this theory to two important problems: determining whether a finite-element matrix represents a rigid structure, and determining whether a matrix representing a structure and a matrix representing a substructure have the same range and null space. The paper addresses these problems by providing simple sufficient conditions for rigidity and null-spaces equality, and by providing linear-time algorithms (assuming bounded element degrees) to test these conditions.

Our results employ three new technical tools, one combinatorial and two algebraic. One algebraic tool is a purely-algebraic definition of the rigidity relationships between two rank-deficient matrices. The other algebraic tool is a definition of a finite-element matrix $A$ as a sum of of symmetric semi-definite matrices $\{A_e\}_{e=1}^{k}$ that all satisfy a certain condition. The combinatorial tool is a graph, called the *rigidity graph*, that represents the rigidity relationships between the terms $A_e$ of a finite-element matrix $A = \sum_e A_e$. These tools may be applicable to the solution of other problems involving finite-element matrices.

The concept of rigidity is usually associated with elastic structures and with finite-element models of such structures. An elastic structure is rigid if any deformation of it that is not a translation and/or rotation requires energy. A coin is rigid; a door hinge is not. Our theory of rigidity is consistent with the traditional concept of rigidity, but it is purely algebraic and more general. By purely algebraic, we mean that our theory uses only the element matrices $A_e$ and a basis for the rigid body motions (e.g. translations and rotations) of the structure. Our theory and algorithms do not depend on the geometry of the structure and on the details of the finite-element discretization. Our theory generalizes the concept of rigidity in a natural way from finite-element models of elastic structures to models of other physical systems, such as electrostatics.

On the other hand, our theory only provides sufficient conditions for rigidity. Characterizing rigidity exactly is difficult, even if we limit our attention to specific families of elastic structures. Consider, for example a structure consisting of struts (elastic bars) connected at their endpoints by pin joints. The struts can only elongate or compress, and the struts connected to a pin are free to rotate around the pin. The rigidity of such structures in two-dimensions has been extensively studied and is now well understood. However, the conditions that characterize the rigidity of two-dimensional structure are expensive to check [13] and they do not generalize easily to three-dimensional trusses and to other structures. Our theory of rigidity avoids these difficulties by focusing on characterizations that are simple and general but only sufficient. In fact, structures consisting of struts always fail our sufficient conditions.

Our new theory is essentially an algebraic-combinatorial characterization of finite-element models of structures that are, informally speaking, "evidently rigid". Models of structures that are rigid due to complex non-local interactions between parts of the structure will usually fail our conditions. The main contributions of this paper are formal and easily-computed characterizations of "evidently-rigid" structures. We, therefore, call structures that pass our test *evidently-rigid*. We apply these characterizations to the construction of algorithms that find certain minimally-rigid substructures of a rigid structure.

The results in this paper are a step toward the generalization of results in spectral graph theory from Laplacians to finite-element matrices. We are particularly interested in an area of spectral graph theory called *support theory* or *support preconditioning*. This area is mostly concerned with constructing an approximation $B$ to a matrix $A$ in three steps: (1) building a graph $G_A$ that represents $A$, (2) approximating $G_A$ by a simpler graph $G_B$, and (3) building the matrix $B$ that corresponds to $G_B$. The graph $G_B$ should be simpler in some way than $G_A$ (e.g., smaller balanced vertex separators) and the generalized eigenvalues $\lambda$ of $Ax = \lambda Bx$ should not be very large or very small. Much progress has been made in this area, but only when $A$ is a Laplacian [2, 10, 17, 18, 4], a diagonally-dominant symmetric matrix (i.e., $G_A$ is a signed graph) [3, 10], or can be well approximated by a Laplacian [5].

This paper makes three contributions to support preconditioning of finite-element matrices. First, the paper provides a reasonable definition of what a finite-element matrix is: a sum of element matrices whose null spaces are derived from a single global null space. Second, the paper provides a graph model of finite-element matrices, and proposes graph algorithms for sparsifying the coefficient matrix $A$. Three, the paper provides simple combinatorial conditions that allow us to show that the range and null space of the sparsified matrix (the preconditioner) $B$ are the same as those of $A$. The qualitative range and null-space equalities are weaker statements than quantitative bounds on the generalized eigenvalues, but they are a step toward eigenvalue bounds. A weighted rigidity graph may allow us to bound eigenvalues and generalized eigenvalues. The same technical tools may also be applicable to the generalization of other results in spectral graph theory, such as Cheeger-type bounds [6, 8, 1].

The paper is quite technical and fairly complex. It may seem strange that all of this complexity is needed to prove results that are physically intuitive. If a structure is evidently rigid, why is all the algebraic and notational complexity needed? The answer appears to be that the complexity is a result of our insistence on a purely algebraic and combinatorial analysis. We do not rely directly on any physical or continuous properties of the structures that we analyze. Our analysis reaches the physically-intuitive conclusions, but the algebraic path toward these conclusions is complex. We believe that the generality and software-engineering advantages of a purely-algebraic approach are worth the complexity of the paper. Furthermore, the analysis is complex, but the algorithms that we propose are both general and simple.

The paper is organized as follows. Finite-element matrices are sums of very sparse terms called element matrices. Most of the rows and columns in each element matrix contain only zeros. Such matrices have a trivial null space that the zero columns generate, and sometimes another null subspace that is more interesting. Our study of rigidity is essentially a study of these nontrivial subspaces. Section 2 defines these subspaces and analyzes their properties. The combinatorial structure that we use, the rigidity graph, is defined by rigidity relationships between pairs of element matrices. These relationships are defined and explored in Sections 3 and 4. One of our ultimate goals in this paper is to show that a connected rigidity graph implies that the underlying structure is rigid. Unfortunately, this is not true for collections of arbitrary element matrices; they must have something in common for their rigidity graph to be useful. This common property is called null-space compatibility. Its definition and significance are explained in Section 5. The rigidity graph itself is defined in Section 6, along with a proof that a connected rigidity graph implies the rigidity of the structure. Section 7 studies three families of finite-element matrices and their rigidity graphs, to further illustrate the concepts presented earlier. In Section 8 we present two methods for sparsifying a finite-element matrix while preserving its null space. The more sophisticated method, called spanning-tree fretsaw extension, always leads to simplified finite-element matrices that can be factored with essentially no fill. We present two numerical examples of the use of spanning-tree fretsaw extension as preconditioners in Section 9. We conclude the paper with a few open problems in Section 10.

## 2. The Essential Null Space of a Matrix

Rigidity is closely related to relationships between null spaces. We therefore start our analysis with definitions and lemmas concerning the null space of matrices with zero columns.

**Definition 2.1.** Let $A$ be an $m$-by-$n$ matrix, let $\mathcal{Z}_A \subseteq \{1, \ldots n\}$ be the set of its zero columns, and let $\mathcal{N}_A$ be the set of its nonzero columns. The *essential null space* of $A$ is the space of vectors $x$ satisfying

- $Ax = 0$ and
- $x_i = 0$ for $i \in \mathcal{Z}_A$.

The *trivial null space* of $A_e$ is the space of vectors $x$ satisfying $x_i = 0$ for $i \in \mathcal{N}_A$. We denote the two spaces by $\operatorname{enull}(A)$ and $\operatorname{tnull}(A)$.

Clearly, the essential and trivial null spaces of a matrix are orthogonal and their union is simply the null space of the matrix.

**Definition 2.2.** A *restriction* of a vector $y$ to the indices $\mathcal{N}_A$ is the vector

$$x_i = \begin{cases} y_i & i \in \mathcal{N}_A \\ 0 & \text{otherwise.} \end{cases}$$

(The restriction is a projection.) An *extension with respect to* $\mathcal{N}_A$ of a vector $x$ satisfying $x_i = 0$ for $i \in \mathcal{Z}_A$ is any vector $y$ such that $y_i = x_i$ for all $i \in \mathcal{N}_A$.

**Lemma 2.3.** *Let $y$ be the extension with respect to $\mathcal{N}_A$ of a vector $x \in \text{enull}(A)$. Then $y \in \text{null}(A)$.*

*Proof.* Let $z = y - x$, so $y = x + z$. Since $y_i = x_i$ for $i =\in \mathcal{N}_e$, we have $z_i = 0$ for $i \in \mathcal{N}_A$. Therefore $Az = 0$, so $Ay = Ax + Az = 0 + 0 = 0$. $\square$

**Lemma 2.4.** *The restriction $x$ of a vector $y \in \text{null}(A)$ to $\mathcal{N}_A$ is in enull$(A)$.*

*Proof.* Follows directly from $\text{null}(A) = \text{enull}(A) \cup \text{tnull}(A)$. $\square$

**Lemma 2.5.** *Let $A$ be an $n$-by-$n$ symmetric positive semidefinite matrix, let $B$ be an $n$-by-$n$ positive semidefinite matrix, and let $x \in$ null$(A + B)$. Then $x \in$ null$(A)$.*

*Proof.* Suppose for contradiction that $Ax \neq 0$. $A$ has a decomposition $A = LL^T$. Since $Ax \neq 0$, we also have $L^T x \neq 0$, so $x^T LL^T x = x^T Ax > 0$. Therefore, $x^T Bx = x^T (A + B)x - x^T Ax = 0 - x^T Ax < 0$, a contradiction. $\square$

A column that is nonzero in both $A$ and $B$ can be a zero in $A + B$ due to cancellation. The next lemma shows that this cannot happen when the terms are symmetric positive semidefinite matrices (SPSD).

**Lemma 2.6.** *Let $A$ and $B$ be $n$-by-$n$ symmetric positive semidefinite matrices. Then $\mathcal{N}_{A+B} = \mathcal{N}_A \cup \mathcal{N}_B$.*

*Proof.* Clearly $\mathcal{N}_{A+B} \subseteq \mathcal{N}_A \cup \mathcal{N}_B$. Suppose for contradiction that the lemma does not hold. Then there is a column index $j$ in $\mathcal{N}_A$ or in $\mathcal{N}_B$ that is not in $\mathcal{N}_{A+B}$. Without loss of generality assume that $j \in \mathcal{N}_A$. Let $x$ be the $j$th unit vector. Since $j \in \mathcal{N}_A$, $Ax$, which is simply the $j$th column of $A$, is nonzero. But since $j \notin \mathcal{N}_{A+B}$, we also have $(A + B)x = 0$, a contradiction to Lemma 2.5. $\square$

The last lemma in this section shows the relationship between null-space containment and the sets $\mathcal{N}$ and $\mathcal{Z}$.

**Lemma 2.7.** *Let $A$ be an $m$-by-$n$ matrix and let $B$ be an $l$-by-$n$ matrix with* $\text{null}(B) \subseteq \text{null}(A)$. *Then $\mathcal{Z}_B \subseteq \mathcal{Z}_A$ and $\mathcal{N}_A \subseteq \mathcal{N}_B$.*

*Proof.* Let $j \in \mathcal{Z}_B$ and let $e_j$ be the $j$th unit vector. By definition, $Be_j = 0$. By the assumption on the null spaces, $Ae_j = 0$. This implies that $j \in \mathcal{Z}_A$. Therefore, $\mathcal{Z}_B \subseteq \mathcal{Z}_A$, so the complements of these sets satisfy $\mathcal{N}_A \subseteq \mathcal{N}_B$. $\square$

## 3. Rigidity Relationships

This section introduces the main notion of this paper: rigidity relationships.

**Definition 3.1.** An $m$-by-$n$ matrix $A$ is *rigid with respect to* another $\ell$-by-$n$ matrix $B$ if for every vector $x \in \text{enull}(A)$ there is a unique vector $y \in \text{enull}(B)$, called the *rigid mapping* of $x$, such that $y_i = x_i$ for all $i \in \mathcal{N}_A \cap \mathcal{N}_B$. The two matrices are called *mutually rigid* if they are rigid with respect to each other.

**Example 3.2.** Mutual rigidity does not follow automatically from one-sided rigidity. Consider, for example,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

$A$ is rigid with respect to $B$, because vectors in enull$(A)$ have the form $\begin{bmatrix} 0 & \alpha & \alpha & 0 \end{bmatrix}^T$, and they have a unique extension to vectors in enull$(B)$, namely $\begin{bmatrix} \alpha & \alpha & \alpha & \alpha \end{bmatrix}^T$. But vectors in enull$(B)$, which have the form $\begin{bmatrix} \alpha & \alpha & \beta & \beta \end{bmatrix}^T$, are not in null$(A)$ unless $\alpha = \beta$.

**Example 3.3.** Let $A_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$. The two matrices are mutually rigid. We have

$$\begin{aligned} \text{enull}\,(A_1) &= \text{span}\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T \\ \text{enull}\,(A_2) &= \text{span}\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T \end{aligned}$$

Therefore, for every $x = \begin{bmatrix} \alpha & \alpha & 0 & 0 \end{bmatrix}^T \in$ enull$(A_1)$, there is a unique $y = \begin{bmatrix} 0 & \alpha & \alpha & 0 \end{bmatrix}^T \in$ enull$(A_1)$, and symmetrically for $A_2$.

Now let $A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$ and $A_4 = \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}$. $A_1$ is not rigid with respect to these two. It is not rigid with respect to $A_3$ because enull$(A_3) = \{0\}$, so for an $x \in$ enull$(A_1)$ there is no rigid $y$ in enull$(A_3)$. $A_1$ is not rigid with respect to $A_4$ because for $x = \begin{bmatrix} \alpha & \alpha & 0 & 0 \end{bmatrix}^T \in$ enull$(A_1)$, any $y = \begin{bmatrix} 0 & 0 & \beta & \beta \end{bmatrix}^T$ is in enull$(A_4)$, so the mapping is not unique.

We now show how to test whether a matrix $A$ is rigid with respect to another matrix $B$. For an $m$-by-$n$ matrix $A$, we define $\Xi_A$ and $\Xi_{\bar{A}}$ to be the $n$-by-$n$ diagonal matrices

$$\left[\Xi_A\right]_{jj} = \begin{cases} 1 & j \in \mathcal{N}_A \\ 0 & j \in \mathcal{Z}_A \end{cases} \quad \text{and} \quad \left[\Xi_{\bar{A}}\right]_{jj} = \begin{cases} 0 & j \in \mathcal{N}_A \\ 1 & j \in \mathcal{Z}_A \end{cases}.$$

For two matrices $A$ and $B$ with $n$ columns each, we define $\Xi_{A,B}$ to be the $n$-by-$n$ diagonal matrix

$$\left[\Xi_{A,B}\right]_{jj} = \begin{cases} 1 & j \in \mathcal{N}_A \cap \mathcal{N}_B \\ 0 & \text{otherwise}. \end{cases}$$

Let $x$ be a vector in enull$(A)$. If $x$ has a rigid mapping to $y \in$ enull$(B)$, then $y$ must satisfy the equations

$$\begin{aligned} \Xi_{\bar{B}} y &= 0 \\ B y &= 0 \\ \Xi_{A,B} y &= \Xi_{A,B} x. \end{aligned}$$

The first two conditions constrain $y$ to be in enull$(B)$ and the third condition constrains $y$ to be a mapping of $x$. If this linear system is inconsistent, then $x$ has no rigid mapping to $y \in$ enull$(B)$, so $A$ is not rigid with respect to $B$. Even if the system is consistent for all $x \in$ enull$(A)$, $A$ is not necessarily rigid with respect to $B$. If the coefficient matrix $R_{A,B} = \begin{bmatrix} \Xi_{\bar{B}}^T & B^T & \Xi_{A,B}^T \end{bmatrix}^T$ is rank deficient, the mappings are not unique.

Therefore, to test rigidity we must check that for all $x \in \text{enull}(A)$, the vector $\begin{bmatrix} 0 & 0 & x^T \Xi_{A,B}^T \end{bmatrix}^T$ is spanned by the columns of $R_{A,B}$, and that the columns of $R_{A,B}$ are linearly independent. We now derive equivalent conditions, but on a much smaller system. First, we drop rows and columns $\mathcal{Z}_B$ from the coefficient matrix and rows $\mathcal{Z}_B$ from $y$. These rows correspond to equations that constrain $y_i = 0$ for $i \in \mathcal{Z}_B$. Since these elements of $y$ are not used in any of the other equations, we can drop them without making an inconsistent system into a consistent one. Also, these columns are linearly independent, and all the other columns are independent of them. Therefore, dropping these $|\mathcal{Z}_B|$ rows and columns reduces the rank of $R_{A,B}$ by exactly $|\mathcal{Z}_B|$; therefore, $R_{A,B}$ is full rank if and only if the remaining rows and columns form a full-rank matrix. Now we drop all the zero rows from the system: Rows $\mathcal{N}_B$ in the $\Xi_{\bar{B}}$ block of $R_{A,B}$, the zero rows from the $B$ block, and the zero rows from the $\Xi_{A,B}$ block. These rows correspond to equations that are consistent for any $x$ and any $y$; being zero, they do not affect the rank of $R_{A,B}$.

We assume without loss of generality that columns $\mathcal{N}_A \cap \mathcal{N}_B$ are the last among the nonzero columns of $B$. We denote by $\breve{B}$ the matrix formed by dropping all the zero rows and columns of $B$ and by $y_{\mathcal{N}_B}$ the vector formed by dropping elements $\mathcal{Z}_B$ from $y$. (For any $n$-vector $v$ and a set $S \subseteq \{1, \ldots n\}$, the notation $v_S$ means the $|S|$-vector formed by dropping the elements of $v$ whose indices are not in $S$, and similarly for matrices.) Our reduced system is

$$\breve{R}_{A,B} y_{\mathcal{N}_B} = \begin{bmatrix} \multicolumn{3}{c}{\breve{B}} \\ 0 & | & I \end{bmatrix} y_{\mathcal{N}_B} = \begin{bmatrix} 0 \\ x_{\mathcal{N}_A \cap \mathcal{N}_B} \end{bmatrix} ,$$

where the order of the identity matrix is $|\mathcal{N}_A \cap \mathcal{N}_B|$. To test whether $A$ is rigid with respect to $B$, we construct a matrix $N_A$ whose columns span $\text{enull}(A)$, and check

(1) whether $\breve{R}_{A,B}$ has full rank, and
(2) whether for every column $x$ in $N_A$,

$$\breve{R}_{A,B} \breve{R}_{A,B}^{+} \begin{bmatrix} 0 \\ x_{\mathcal{N}_A \cap \mathcal{N}_B} \end{bmatrix} = \begin{bmatrix} 0 \\ x_{\mathcal{N}_A \cap \mathcal{N}_B} \end{bmatrix} .$$

If $B$ has only few nonzero rows and columns and if the number of columns in $N_A$ is small, then this is is an inexpensive computation. The construction is illustrated in Figure 3.1.

The next three lemmas show the relationship between null-space containment and rigidity.

**Lemma 3.4.** *Let $A$ be an $m$-by-$n$ matrix and let $B$ be an $l$-by-$n$ matrix. If* $\text{null}(A) \subseteq \text{null}(B)$*, then $A$ is rigid with respect to $B$.*

*Proof.* Let $x \in \text{enull}(A)$. Therefore, $x \in \text{null}(A)$ and $x \in \text{null}(B)$. Define $y = \Xi_B x$. We have that $x_i = y_i$ for all $i \in \mathcal{N}_A \cap \mathcal{N}_B$. By Lemma 2.4, $y \in \text{enull}(B)$. Therefore $y$ is a rigid mapping of $x$ in $\text{enull}(B)$.

We now show that $y$ is the unique mapping of $x$. Let $\hat{y}$ be a rigid mapping of $x$ in $\text{enull}(B)$. By Lemma 2.7, $\mathcal{N}_B \subseteq \mathcal{N}_A$. The equalities $y_i = x_i = \hat{y}_i$ hold for every $i \in \mathcal{N}_A \cap \mathcal{N}_B = \mathcal{N}_B$. Therefore, $y = \hat{y}$, so $y$ is the unique rigid mapping of $x$ in $\text{enull}(B)$. This implies that $A$ is rigid with respect to $B$. □

**Lemma 3.5.** *Let $A$ be an $m$-by-$n$ matrix and let $B$ be an $l$-by-$n$ matrix. If $\mathcal{N}_B \subseteq \mathcal{N}_A$ and $A$ is rigid with respect to $B$, then* $\text{null}(A) \subseteq \text{null}(B)$.

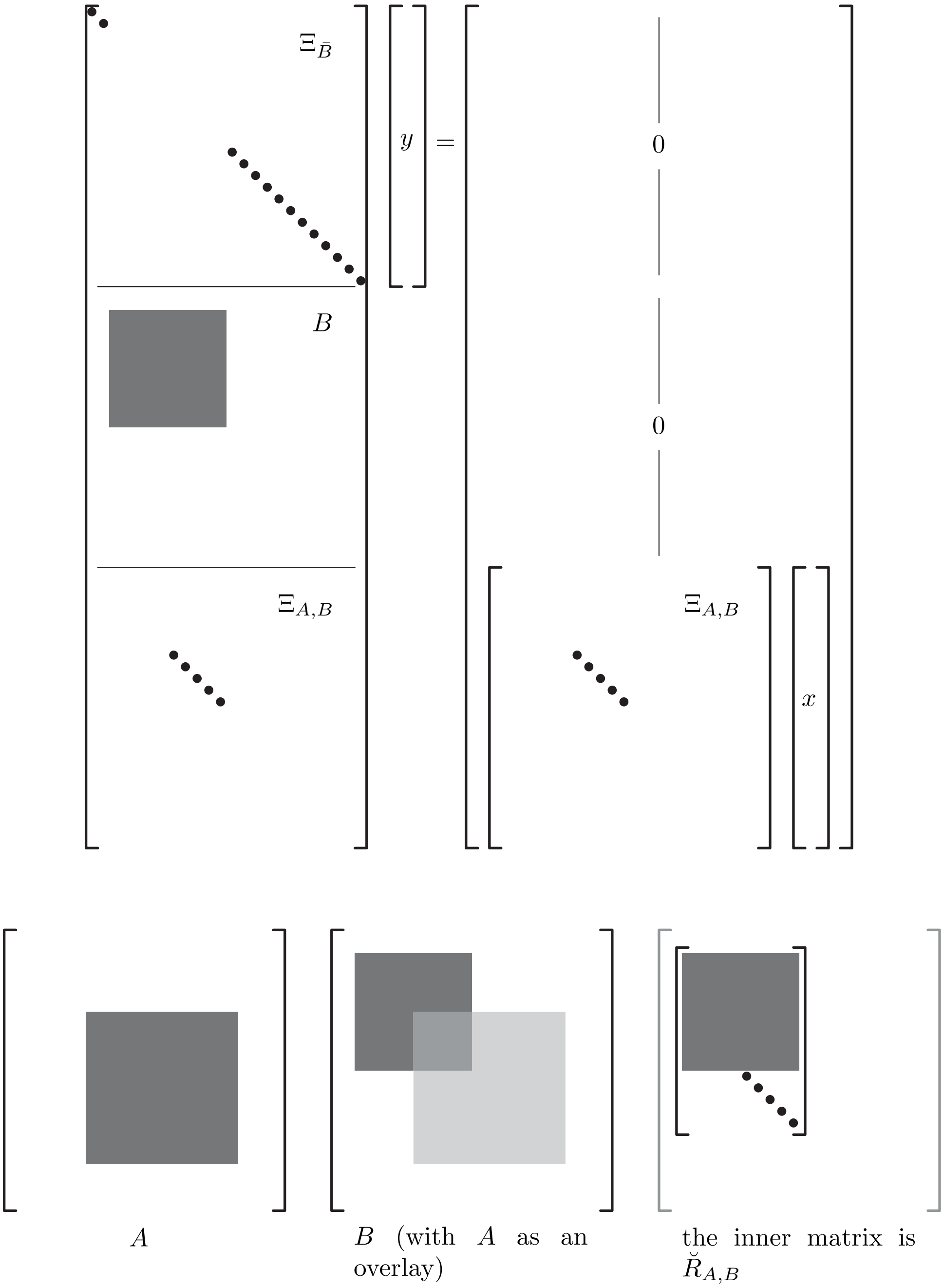


FIGURE 3.1. Testing rigidity. The top part of the figure shows the entire linear system, and the bottom part shows the construction of $\breve{R}_{A,B}$.

*Proof.* Let $x \in \text{null}(A)$. We can write $x$ as $y + z$ where $y \in \text{enull}(A)$ and $z \in \text{tnull}(A)$. We have that $z \in \text{tnull}(B)$, since $\mathcal{N}_B \subseteq \mathcal{N}_A$. Therefore, $z \in \text{null}(B)$.

We now show that $y$ is also in null$(B)$. Let $u$ be $y$'s rigid mapping to enull$(B)$. We have that $u_i = y_i$ for every $i \in \mathcal{N}_A \cap \mathcal{N}_B = \mathcal{N}_B$. Therefore, we can write $y$ as $y = u + u'$ where $u'_i \neq 0$ only for $i \in \mathcal{N}_A \backslash \mathcal{N}_B$. It clear that $u' \in \text{tnull}(B) \subseteq \text{null}(B)$. Therefore, $y = u + u' \in \text{null}(B)$ and $x = y + z \in \text{null}(B)$. Since any $x \in \text{null}(A)$ is also in null$(B)$, the lemma holds. $\square$

**Corollary 3.6.** *Let $A$ be an $m$-by-$n$ matrix and let $B$ be an $l$-by-$n$ matrix, such that $\mathcal{N}_A = \mathcal{N}_B$. Then $A$ and $B$ are mutually rigid if and only if* $\mathrm{null}(A) = \mathrm{null}(B)$.

*Proof.* Directly follows from Lemma 3.4 and Lemma 3.5. □

The last lemma in this section shows that rigidity relationships are maintained in certain Schur complements.

**Lemma 3.7.** *Let $A$ and $B$ be $n$-by-$n$ matrices of the form*

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

*where $A_{11}$ and $B_{11}$ are $k$-by-$k$ matrices for some $0 < k < n$. Assume that $A$ is rigid with respect to $B$, that $\mathcal{N}_A = \{1, \ldots, k\}$, that $\mathcal{N}_B = \{1, \ldots, n\}$, and that $B_{22}$ is nonsingular. Then* $\mathrm{null}(A_{11}) \subseteq \mathrm{null}(B_{11} - B_{12}B_{22}^{-1}B_{21})$ *and $A_{11}$ is rigid with respect to $B_{11} - B_{12}B_{22}^{-1}B_{21}$.*

*Moreover, if $A$ and $B$ are mutually rigid, then* $\mathrm{null}(A_{11}) = \mathrm{null}(B_{11} - B_{12}B_{22}^{-1}B_{21})$.

*Proof.* Let $x \in \mathrm{null}(A_{11})$. Let $\hat{x}$ be the vector of size $n$ that equals $x$ in its first $k$ coordinates and that contains zeros in its last $(n-k)$ coordinates. Clearly, $\hat{x} \in \mathrm{null}(A)$. Since there are no zero columns in $A_{11}$ we also have that $\hat{x} \in \mathrm{enull}(A)$. Let $\hat{y}$ be the rigid mapping of $\hat{x}$ in $\mathrm{enull}(B)$. The equalities $\hat{y}_i = \hat{x}_i = x_i$ hold for all $i \in \{1, \ldots, k\}$. Let $y$ be a vector of size $(n-k)$ consisting of the last $(n-k)$ elements of $\hat{y}$. Writing the equation $B\hat{y} = 0$ in terms of $x$ and $y$, we obtain

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} B_{11}x + B_{12}y \\ B_{21}x + B_{22}y \end{bmatrix} = 0$$

Multiplying the second block row by $B_{22}^{-1}$ gives $y = -B_{22}^{-1}B_{21}x$. Substituting $y$ with $-B_{22}^{-1}B_{21}x$ in the first block row, we get $B_{11}x - B_{12}B_{22}^{-1}B_{21}x = 0$. Therefore, $x \in \mathrm{null}(B_{11} - B_{12}B_{22}^{-1}B_{21})$, so $\mathrm{null}(A_{11}) \subseteq \mathrm{null}(B_{11} - B_{12}B_{22}^{-1}B_{21})$. The containment of the null spaces, along with Lemma 3.4, shows that $A_{11}$ is rigid with respect to $B_{11} - B_{12}B_{22}^{-1}B_{21}$.

Now assume $A$ and $B$ are mutually rigid (we add the assumption that $B$ is rigid with respect to $A$). Let $x \in \mathrm{null}(B_{11} - B_{12}B_{22}^{-1}B_{21})$. Let $\hat{x}$ be the vector of size $n$ that equals $x$ in its first $k$ coordinates and equals $-B_{22}^{-1}B_{21}x$ in its last $(n-k)$ coordinates. The vector $\hat{x}$ is in $\mathrm{enull}(B)$, since

$$B\hat{x} = \begin{bmatrix} B_{11}x + B_{12}(-B_{22}^{-1}B_{21}x) \\ B_{21}x + B_{22}(-B_{22}^{-1}B_{21}x) \end{bmatrix} = 0 \,.$$

Because $B$ is rigid with respect to $A$, the vector $\hat{x}$ has a unique mapping to $\mathrm{enull}(A)$. Since $\mathcal{N}_A \subseteq \mathcal{N}_B$, this mapping is $\Xi_A\hat{x}$. Therefore, $A\Xi_A\hat{x} = 0$, so $x \in \mathrm{null}(A_{11})$. This implies that $\mathrm{null}(B_{11} - B_{12}B_{22}^{-1}B_{21}) \subseteq \mathrm{null}(A_{11})$. Therefore, $\mathrm{null}(B_{11} - B_{12}B_{22}^{-1}B_{21}) = \mathrm{null}(A_{11})$. This concludes the proof of the lemma. □

## 4. Rigidity of Sums

Finite-element matrices are sums of mostly-zero matrices. This section extends our study of rigidity to sums of matrices.

**Lemma 4.1.** *Let $A$ and $B$ be symmetric positive-semidefinite $n$-by-$n$ matrices. The matrix $(A+B)$ is always rigid with respect to $A$ and with respect to $B$.*

*Proof.* Let $x \in \text{enull}(A+B)$. By Lemma 2.5, $x \in \text{null}(A)$, and by Lemma 2.4 the restriction $x'$ of $x$ to $\mathcal{N}_A$ is in enull$(A)$. We now only have to prove that $x'$ is unique. A rigid mapping must coincide with $x$ at $\mathcal{N}_{A+B} \cap \mathcal{N}_A$. Because $A$, $B$, and $A+B$ are SPSD, Lemma 2.6 implies that $\mathcal{N}_{A+B} = \mathcal{N}_A \cup \mathcal{N}_B$, so $\mathcal{N}_{A+B} \cap \mathcal{N}_A = \mathcal{N}_A$. Therefore, the mapping must coincide with $x$ in all the indices in $\mathcal{N}_A$, so it must be $x'$. $\square$

The previous lemma showed that a sum of SPSD matrices is rigid with respect to the terms of the sum, but the terms are not always rigid with respect to the sum, even when the terms are SPSD. For example, $A = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$ is not rigid with respect to $A + \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, because $A$ is rank deficient but the sum is not. Hence, vectors in enull$(A)$ have no mapping at all to the essential null space of the sum.

Also, the lemma holds for SPSD matrices but not for general matrices. Let $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and let $B = \left[\begin{smallmatrix} 0 & -1 \\ 0 & -1 \end{smallmatrix}\right]$. Their sum is $A + B = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right]$. The vector $\begin{bmatrix} 1 & 1 \end{bmatrix}^T \in \text{enull}(A+B)$, but this vector has no mapping into $\text{enull}(A) = \{0\}$.

The next lemma strengthens both the hypothesis and the consequence of Lemma 4.1. It shows that if the terms are mutually rigid, then rigidity between the terms and the sum is mutual.

**Lemma 4.2.** *Let $A$ and $B$ be mutually rigid symmetric positive-semidefinite $n$-by-$n$ matrices. Then $A$ and $A+B$ are mutually rigid, and $B$ and $A+B$ are mutually rigid.*

*Proof.* By Lemma 4.1, the sum is rigid with respect to the terms. So all we need to prove is the opposite direction.

Let $x \neq 0$ be a vector in enull$(A)$. Let $y$ be the rigid mapping of $x$ into enull$(B)$. We now show that $x$ has a rigid mapping into enull$(A+B)$; we shall show the uniqueness of the mapping later. We define

$$w_i = \begin{cases} x_i & i \in \mathcal{N}_A \\ x_i = y_i & i \in \mathcal{N}_A \cap \mathcal{N}_B \\ y_i & i \in \mathcal{N}_B \\ 0 & \text{otherwise.} \end{cases}$$

Because $\mathcal{N}_{A+B} \subseteq \mathcal{N}_A \cup \mathcal{N}_B$, we have $w_i = 0$ for $i \in \mathcal{Z}_{A+B}$. Therefore, to show that $w \in \text{enull}(A+B)$, we only need to show that $(A+B)w = 0$. This is indeed the case because $w$ is an extension of both $x$ and $y$, so $Aw = Bw = 0$.

We now show that $w$ is the only rigid mapping of $x$ into enull$(A+B)$. Suppose that there is another rigid mapping $w' \neq w$. Under this supposition, there must be $w'_i \neq w_i$ for some $i \in \mathcal{N}_B \setminus \mathcal{N}_A$, so the restriction $y'$ of $w'$ to $\mathcal{N}_B$ must be different from $y$. By Lemmas 2.5 and 2.4, $y' \in \text{enull}(B)$. The vectors $y$ and $y'$ are both in enull$(B)$ and both coincide with $x$ on $\mathcal{N}_A \cap \mathcal{N}_B$, so they are two different rigid mappings of $x$, contradicting the hypothesis that $A$ and $B$ are mutually rigid. $\square$

**Lemma 4.3.** *Let $A$ and $B$ be symmetric positive-semidefinite $n$-by-$n$ matrices, and let $C$ be an $m$-by-$n$ matrix . $C$ and $A+B$ are mutually rigid if and only if $C$ and $A+\alpha B$ are mutually rigid for any real $\alpha > 0$.*

*Proof.* Let $\alpha > 0$. We first show that $\text{enull}(A+\alpha B) = \text{enull}(A+B)$. It is clear that $\mathcal{N}_{\alpha B} = \mathcal{N}_B$ and that $\alpha B$ is symmetric positive-semidefinite. From Lemma 2.6 we have $\mathcal{N}_{A+\alpha B} = \mathcal{N}_A \cup \mathcal{N}_{\alpha B} = \mathcal{N}_A \cup \mathcal{N}_B = \mathcal{N}_{A+B}$. Let $x \in \text{enull}(A+B) \subseteq \text{null}(A+B)$. By Lemma 2.5, we have $x \in \text{null}(A)$ and $x \in \text{null}(B)$.

Therefore, $x \in \text{null}(A + \alpha B)$. Clearly, $x_i = 0$ for every $i \notin \mathcal{N}_{A+\alpha B}$. Therefore, $x \in \text{enull}(A + \alpha B)$. This shows that $\text{enull}(A + B) \subseteq \text{enull}(A + \alpha B)$. The proof of the other inclusion direction is the same, so $\text{enull}(A + \alpha B) = \text{enull}(A + B)$.

The lemma follows directly from the definition of mutual rigidity and the fact that $\text{enull}(A + \alpha B) = \text{enull}(A + B)$. □

In some special cases, mutual rigidity between sums allows us to infer that the terms of the sums are mutually rigid and vice versa.

**Lemma 4.4.** *Let $A$, $B$ and $C$ be $n$-by-$n$ symmetric positive-semidefinite matrices such that $\mathcal{N}_C \cap \mathcal{N}_A = \mathcal{N}_C \cap \mathcal{N}_B = \emptyset$. Then $A$ and $B$ are mutually rigid if and only if $A + C$ and $B + C$ are mutually rigid.*

*Proof.* Assume that $A$ and $B$ are mutually rigid. We show that $A + C$ is rigid with respect to $B + C$. By symmetry, $B + C$ is rigid with respect to $A + C$, so the two sums are mutually rigid.

Let $x \in \text{enull}(A + C)$. By Lemma 2.5, $x \in \text{null}(A)$ and $x \in \text{null}(C)$. Let $\hat{x}$ be $x$'s restriction to $\mathcal{N}_A$ and let $\bar{x}$ be its restriction to $\mathcal{N}_C$. By Lemma 2.4, $\hat{x} \in \text{enull}(A)$ and $\bar{x} \in \text{enull}(C)$. Let $\hat{y}$ be $\hat{x}$'s unique rigid mapping to $\text{enull}(B)$. We define the vector

$$y_i = \begin{cases} \hat{y}_i & i \in \mathcal{N}_B \\ \bar{x}_i & i \in \mathcal{N}_C \\ 0 & \text{otherwise} . \end{cases}$$

The definition is valid because $\mathcal{N}_C \cap \mathcal{N}_B = \emptyset$. We show that $y$ is the unique rigid mapping of $x$ in $\text{enull}(B + C)$. Multiplying $B + C$ by $y$ we obtain $(B + C)y = By + Cy = B\hat{y} + C\bar{x} = 0 + 0 = 0$. Since $y_i = 0$ for all $i \notin \mathcal{N}_B \cup \mathcal{N}_C = \mathcal{N}_{B+C}$, $y \in \text{enull}(B + C)$. By definition, $y_i = x_i$ for all $i \in (\mathcal{N}_A \cap \mathcal{N}_B) \cup \mathcal{N}_C = (\mathcal{N}_A \cup \mathcal{N}_C) \cap (\mathcal{N}_B \cup \mathcal{N}_C) = \mathcal{N}_{A+C} \cap \mathcal{N}_{B+C}$. Therefore, $y$ is a rigid mapping of $x$ in $\text{enull}(B + C)$.

We now show that this mapping is indeed unique. Assume that there exists $u \in \text{enull}(B + C)$ that satisfies $u_i = x_i$ for all $i \in \mathcal{N}_{A+C} \cap \mathcal{N}_{B+C}$. We have that $u_i = x_i = y_i$ for all $i \in \mathcal{N}_C \subseteq \mathcal{N}_{A+C} \cap \mathcal{N}_{B+C}$. Let $\hat{u}$ be $u$'s restriction to $\mathcal{N}_B$. We have $\hat{u} \in \text{enull}(B)$ and $\hat{u}_i = x_i = \hat{x}_i$ for all $i \in \mathcal{N}_A \cap \mathcal{N}_B$. Therefore, $\hat{u}$ is a rigid mapping of $\hat{x}$ in $\text{enull}(B)$. Since $A$ and $B$ are mutually rigid, $\hat{u}$ must equal $\hat{y}$. Therefore, $u = y$ and $y$ is the unique rigid mapping of $x$ in $\text{enull}(B + C)$. This shows that $A + C$ is rigid with respect to $B + C$. Figure 4.1 (a) presents this notation graphically.

We now show the other direction. Assume $A + C$ and $B + C$ are mutually rigid. We show that $A$ is rigid with respect to $B$; mutual rigidity follows by symmetry. The notation for this part of the proof is presented graphically in part (b) of Figure 4.1. Let $\hat{x} \in \text{enull}(A)$. Since $\mathcal{N}_C \cap \mathcal{N}_A = \emptyset$, $\hat{x} \in \text{tnull}(C)$. We also have $\hat{x}_i = 0$ for all $i \notin \mathcal{N}_A \cup \mathcal{N}_C = \mathcal{N}_{A+C}$, so $\hat{x} \in \text{enull}(A + C)$. Let $\hat{y}$ be $\hat{x}$'s rigid mapping to $\text{enull}(B + C)$. We show that $\hat{y}$ is $\hat{x}$'s rigid mapping to $\text{enull}(B)$. By Lemma 2.5, $\hat{y} \in \text{null}(B)$. Also, $\hat{y}_i = \hat{x}_i = 0$ for all $i$ in $i \in \mathcal{N}_C \subseteq \mathcal{N}_{B+C}$. Therefore, $\hat{y}_i = 0$ for all $i \notin \mathcal{N}_B$, so $\hat{y} \in \text{enull}(B)$. By definition, $\hat{x}_i = \hat{y}_i$ for all $(\mathcal{N}_A \cap \mathcal{N}_B) \subseteq \mathcal{N}_{A+C} \cap \mathcal{N}_{B+C}$. This implies that $\hat{y}$ is a rigid mapping of $\hat{x}$ in $\text{enull}(B)$.

Finally, we claim that $\hat{y}$ is a unique rigid mapping of $\hat{x}$. Assume that there exist $\hat{u}$ in $\text{enull}(B)$ that satisfies $\hat{x}_i = \hat{u}_i$ for all $i \in \mathcal{N}_A \cap \mathcal{N}_B$. We have that $\hat{x}_i = \hat{u}_i = 0$ for all $i \in \mathcal{N}_C$. Since $\hat{u}$ is also in $\text{enull}(B + C)$ then it is a rigid mapping of $\hat{x} \in \text{enull}(A+C)$ in $\text{enull}(B+C)$ . Because $A+B$ is rigid with respect

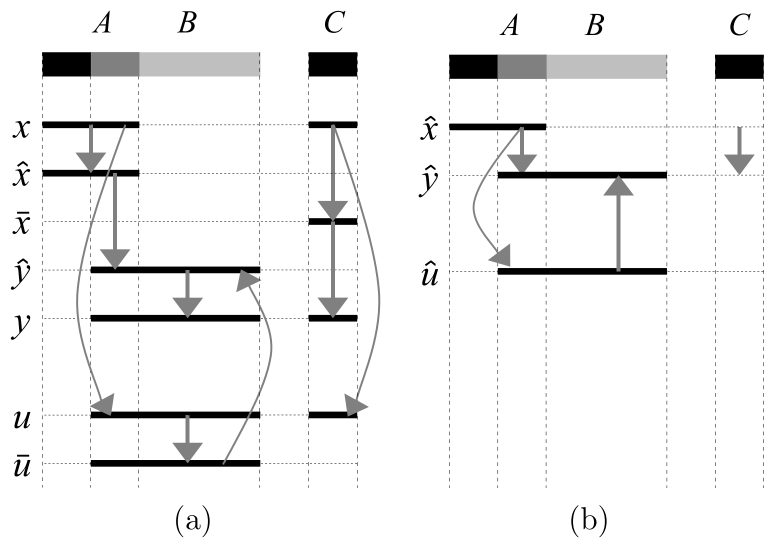


FIGURE 4.1. An illustration of the notation of Lemma 4.4: (a) the vectors defined in the proof of the mutual rigidity of $A+B$ and $A+C$, and (b) the vectors defined in the proof of the mutual rigidity of $A$ and $B$.

to $A+C$, we have that $\hat{u} = \hat{y}$. Therefore, $\hat{y}$ is indeed unique. This implies that $A$ is rigid with respect to $B$, which concludes the proof of the lemma. $\square$

We would like to build larger assemblies of mutually-rigid matrices from chains of mutual rigidity, but this is not always possible, as the next example shows.

**Example 4.5.** Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

These matrices are all SPSD, and their essential null spaces are spanned by $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, respectively. The matrices $A$ and $B$ are mutually rigid, and so are $B$ and $C$. The essential null space of $A+B$ is spanned by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ and the essential null space of $B+C$ is spanned by $\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T$. Therefore, $C$ is not mutually rigid with $A+B$ and $A$ is not mutually rigid with $B+C$. Moreover, neither $A$, $B$, $C$, $A+B$, nor $B+C$ is mutually rigid with $A+B+C$, because $A+B+C$ has full rank. This example is inspired by the analysis of signed graphs in [3], which shows that $A+B+C$ has full rank.

To build larger assemblies of mutually-rigid matrices, we need another tool.

## 5. Null-Space Compatibility

This section defines and explores a concept that we call null-space compatibility, which is the tool that allows us to build large assemblies of mutually-rigid matrices.

**Definition 5.1.** Let $\mathbb{N} \subseteq \mathbb{R}^n$ be a linear space. A matrix $A$ is called $\mathbb{N}$-*compatible* (or compatible with $\mathbb{N}$) if every vector in $\text{enull}(A)$ has a unique extension into a vector in $\mathbb{N}$, and if the restriction of every vector in $\mathbb{N}$ to $\mathcal{N}_A$ is always in $\text{enull}(A)$.

**Definition 5.2.** Let $A$ be an $n$-by-$n$ matrix. If $A$ is $\mathbb{N}$-compatible and $\mathcal{N}_A = \{1,\ldots,n\}$, then we say that $A$ is $\mathbb{N}$-*rigid*. If $\mathbb{N}$ is clear from the context, we simply say that $A$ is rigid.

Given a basis for $\mathbb{N}$, we can easily check the compatibility of a matrix $A$. Let the columns of $N$ be a basis for $\mathbb{N}$, and let $N_A$ be a basis for $\operatorname{enull}(A)$. $A$ is compatible with $\mathbb{N}$ if and only if $N_A = \Xi_A N_A$ and $\Xi_A N$ have the same range. This can be checked numerically using the singular value decompositions of the two matrices, for example.

**Example 5.3.** Let $\mathbb{N} = \operatorname{span}\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$. The matrices $A$ and $B$ from Example 4.5 are compatible with $\mathbb{N}$, but $C$ is not. Note that the mutual rigidity of $A$ and $C$ together with the $\mathbb{N}$-compatibility of $A$ do not imply $\mathbb{N}$-compatibility for $C$. The matrix $A+B$ from the same example is also compatible with $\mathbb{N}$, and since $\mathcal{N}_{A+B} = \{1,2,3\}$, $A+B$ is rigid.

**Lemma 5.4.** *Let $A$ be an $\mathbb{N}$-compatible matrix. Let $N$ be a matrix whose columns form a basis for $\mathbb{N}$. Then* $\operatorname{enull}(A) = \operatorname{span}(\Xi_A N)$.

*Proof.* We first show that $\operatorname{enull}(A) \subseteq \operatorname{span}(\Xi_A N)$. Let $x \in \operatorname{enull}(A)$. Since $A$ is $\mathbb{N}$-compatible, $x$ has a unique extension $w$ in $\mathbb{N}$. By definition, there exists a vector $y$ such that $w = Ny$. Substituting $w$ in the equation $x = \Xi_A w$, we get $x = \Xi_A N y$. Therefore, $x \in \operatorname{span}(\Xi_A N)$ so $\operatorname{enull}(A) \subseteq \operatorname{span}(\Xi_A N)$.

We now show that $\operatorname{span}(\Xi_A N) \subseteq \operatorname{enull}(A)$. Let $x = \Xi_A N y \in \operatorname{span}(\Xi_A N)$. Define $w = Ny \in \mathbb{N}$. Since $A$ is $\mathbb{N}$-compatible, $x = \Xi_A w \in \operatorname{enull}(A)$. This shows that $\operatorname{span}(\Xi_A N) \subseteq \operatorname{enull}(A)$. □

The definition of null-space compatibility is related to the definition of mutual rigidity, but it defines compatibility with respect to a space, not with respect to a particular matrix having that space as a null space. Here is the relationship of $\mathbb{N}$-compatibility with mutual rigidity.

**Lemma 5.5.** *Let $\mathbb{N} \subseteq \mathbb{R}^m$ be a linear space, and let $B$ be some matrix with no zero columns whose null space is $\mathbb{N}$. Another matrix $A$ is $\mathbb{N}$-compatible if and only if $A$ and $B$ are mutually rigid.*

*Proof.* The equivalence follows from the fact that $\operatorname{enull}(B) = \operatorname{null}(B) = \mathbb{N}$ (because $\mathcal{Z}_B = \emptyset$) and from the fact that $\mathcal{N}_A \cap \mathcal{N}_B = \mathcal{N}_A$. □

If the dimension of $\mathbb{N}$ is small, the $\mathbb{N}$-compatibility test given after Definition 5.2 can be much more efficient than the test for mutual rigidity given earlier.

**Example 5.6.** Two matrices that are both compatible with some null space $\mathbb{N}$ are not necessarily mutually rigid. For example

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

are both compatible with $\mathbb{N} = \operatorname{span}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, but they are not mutually rigid. Also, their sum is not $\mathbb{N}$-compatible. Since $\mathcal{N}_{A+B} = \{1,2,3,4\}$, $\operatorname{enull}(A) = \operatorname{null}(A)$, so $A+B$ is $\mathbb{N}$-compatible if and only if $\operatorname{null}(A) = \mathbb{N}$. However,

$$\text{enull}(A+B) = \text{enull}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \text{span}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \neq \mathbb{N} = \text{span}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} .$$

The next two lemmas are key results that will allow us to build large assemblies of mutually-rigid matrices.

**Lemma 5.7.** *Let $A$ and $B$ be mutually rigid, both $\mathbb{N}$-compatible for some null space $\mathbb{N}$. Then $A+B$ is also $\mathbb{N}$-compatible.*

*Proof.* Let $u$ be a vector in $\text{enull}(A+B)$, let $x$ be the restriction of $u$ to $\mathcal{N}_A$, and let $y$ be the restriction to $\mathcal{N}_B$. Let $w$ be the extension of $x$ to a vector in $\mathbb{N}$. We claim that $w$ is a unique extension of $u$ to $\mathbb{N}$. If $w$ is not an extension of $u$, then they must differ on an index in $\mathcal{N}_B \setminus \mathcal{N}_A$, so the restriction of $w$ to $\mathcal{N}_B$ is some $y' \neq y$. But both $y$ and $y'$ are rigid mappings of $x$, a contradiction to the mutual rigidity of $A$ and $B$.

We now show that $w$ is the unique extension of $u$ to $\mathbb{N}$. If there is another extension, its restriction to $\mathcal{N}_A$ must differ from $x$, so it cannot be an extension of $u$.

We now show that the restriction $u$ of a vector $w \in \mathbb{N}$ is in $\text{enull}(A+B)$. The restriction $x$ of $u$ to $\mathcal{N}_A$ is also the restriction of $w$ to $\mathcal{N}_A$, so $Ax = Au = Aw = 0$. Similarly for the restrictions to $\mathcal{N}_B$. Therefore $(A+B)u = 0$, so $u \in \text{enull}(A+B)$. $\square$

We now introduce a technical lemma that shows how to transform a null-space extension of a vector into a rigid mapping of the same vector.

**Lemma 5.8.** *Let $A$ and $B$ be mutually rigid matrices, both compatible with some null space $\mathbb{N}$. Let $x \in \text{enull}(A)$ and let $w \in \mathbb{N}$ be its unique extension to $\mathbb{N}$. The vector $\Xi_B w$ is the unique rigid mapping of $x$ to $\text{enull}(B)$. In particular, if some vector $u$ is a rigid mapping of $x$ to $\text{enull}(B)$, then the unique extension of $u$ to $\mathbb{N}$ is $w$.*

*Proof.* Let $y = \Xi_B w$. We first show that $y$ is the unique rigid mapping of $x$ to $\text{enull}(B)$. The vector $y$ is in $\text{enull}(B)$, since $B$ is $\mathbb{N}$-compatible and $w \in \mathbb{N}$. From the definition of $w$ and $y$ we have that $x_i = w_i = y_i$ for all $i \in \mathcal{N}_A \cap \mathcal{N}_B$. Therefore, $y$ is a mapping of $x$ in $\text{enull}(B)$ and it is unique because $A$ and $B$ are mutually rigid.

Let $u$ be a rigid mapping of $x$ to $\text{enull}(B)$. Since this mapping is unique, $u = y = \Xi_B w$. The vector $w$ is an extension of $u$ to $\mathbb{N}$. The matrix $B$ is $\mathbb{N}$-compatible so this extension is unique. $\square$

The following lemma is the main result of this section. Compare this lemma to Example 4.5: in the example, the three matrices were not all compatible with some null space $\mathbb{N}$; the conclusion of this lemma does not hold in that example.

**Lemma 5.9.** *Let $A$, $B$, and $C$ be symmetric positive-semidefinite $n$-by-$n$ matrices, all compatible with some null space $\mathbb{N}$. Let $A$ and $B$ be mutually rigid and let $B$ and $C$ be mutually rigid. Then $A+B$ and $C$ are mutually rigid (and similarly for $B+C$ and $A$).*

*Proof.* We first show that $A+B$ is rigid with respect to $C$. Let $u$ be in $\text{enull}(A+B)$. By Lemma 5.7, the matrix $A+B$ is $\mathbb{N}$-compatible. Let $w$ be $u$'s extension to $\mathbb{N}$. Define $x = \Xi_A^T w$, $y = \Xi_B^T w$ and $z = \Xi_C^T w$. By definition, $u = \Xi_{A+B}^T w$ and therefore $z_i = u_i$ for all $i \in \mathcal{N}_{A+B} \cap \mathcal{N}_C$. Since $C$ is $\mathbb{N}$ compatible, $z \in \operatorname{enull}(C)$. Therefore, $z$ is a rigid mapping of $u$ in $\operatorname{enull}(C)$.

We show that $z$ is unique. The matrices $A + B$ and $A$ are mutually rigid according to Lemma 4.2. According to Lemma 5.8, $x$ is the unique rigid mapping of $u$ in $\operatorname{enull}(A)$, $y$ is the unique rigid mapping of $x$ in $\operatorname{enull}(B)$ and $z$ is the unique rigid mapping of $y$ in $\operatorname{enull}(C)$. Therefore, $z$ is the unique rigid mapping of $u$ in $\operatorname{enull}(C)$ and $A + B$ is rigid with respect to $C$.

We show now that $C$ is rigid with respect to $A + B$. Let $z$ be in $\operatorname{enull}(C)$. The matrix $C$ is $\mathbb{N}$-compatible. Let $w$ be $z$'s extension to $\mathbb{N}$. Define $x = \Xi_A^T w$, $y = \Xi_B^T w$ and $u = \Xi_{A+B}^T w$. By definition, $z = \Xi_C^T w$ and therefore $z_i = u_i$ for all $i \in \mathcal{N}_{A+B} \cap \mathcal{N}_C$. Therefore, $u$ is a rigid mapping of $z$ in $\operatorname{enull}(A + B)$.

We show that $u$ is unique. According to Lemma 5.8, $y$ is the unique rigid mapping of $z$ in $\operatorname{enull}(B)$, $x$ is the unique rigid mapping of $y$ in $\operatorname{enull}(A)$ and $u$ is the unique rigid mapping of $x$ in $\operatorname{enull}(A + B)$. Therefore, $u$ is the unique rigid mapping of $z$ in $\operatorname{enull}(A+B)$. This implies that $C$ is rigid with respect to $A+B$ and concludes the proof of the lemma. □

The last lemma of this section characterizes the rigidity and $\mathbb{N}$-compatibility of certain larger sums.

**Lemma 5.10.** *Let $A$ and $B_1, B_2, \ldots, B_k$ be symmetric positive-semidefinite matrices, all compatible with some null space $\mathbb{N}$. Let $A$ and $B_i$ be mutually rigid for $i = 1, \ldots, k$. Then $A$ and $A + \sum_{i=1}^k B_i$ are mutually rigid and $A + \sum_{i=1}^k B_i$ is $\mathbb{N}$-compatible.*

*Proof.* We prove the lemma by induction on $k$. The case $k = 1$ is trivial by Lemma 4.2 and Lemma 5.7. We assume that the claim is correct for $k$ smaller than $n$ and show that it is correct for $k = n$. By the inductive assumption, $A$ and $A + \sum_{i=1}^{n-1} B_i$ are mutually rigid and $A + \sum_{i=1}^{n-1} B_i$ is $\mathbb{N}$-compatible. $A$ and $B_n$ are mutually rigid. Therefore, by Lemma 5.9 we have that $B_n$ and $A + A + \sum_{i=1}^{n-1} B_i = 2A + \sum_{i=1}^{n-1} B_i$ are mutually rigid. By lemma 4.3, $B_n$ and $A + \sum_{i=1}^{n-1} B_i$ are mutually rigid. Therefore, by Lemma 5.9, we have that $A$ and $B_n + A + \sum_{i=1}^{n-1} B_i = A + \sum_{i=1}^{n} B_i$ are mutually rigid. By Lemma 5.7, we also have that $B_n + A + \sum_{i=1}^{n-1} B_i = A + \sum_{i=1}^{n} B_i$ is $\mathbb{N}$-compatible. This concludes the proof of the lemma. □

## 6. The Rigidity Graph

Mutual rigidity relationships in a collection of $\mathbb{N}$-compatible SPSD matrices define a graph structure that we can use to demonstrate the rigidity of finite-element matrices.

**Definition 6.1.** Let $A_1, A_2, \ldots, A_k$ be $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices for some null space $\mathbb{N}$. The *rigidity graph* $G = (V, E)$ of $\{A_1, \ldots, A_k\}$ is the undirected graph with $V = \{A_1, \ldots, A_k\}$ and

$$E = \{(A_e, A_f) : A_e \text{ and } A_f \text{ are mutually rigid}\} \ .$$

We could also define the rigidity graph of a collection of matrices that are not necessarily $\mathbb{N}$-compatible, but Example 4.5 suggests that such a definition might not have interesting applications. On the other hand, the $\mathbb{N}$-compatibility requirement in the definition enables an important result, which we state and prove next.

**Lemma 6.2.** *Let $G$ be the rigidity graph of a collection $A_1, A_2, \ldots, A_k$ of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices. Let $H$ be a connected subgraph of $G$, and let $A_e$ be a vertex in $H$. Then $A_e$ and $\sum_{A_f \in V(H)} A_f$ are mutually rigid, and $\sum_{A_f \in V(H)} A_f$ is $\mathbb{N}$-compatible.*

*Proof.* Let $T$ be a depth-first-search tree of $H$ whose root is $A_e$. Denote by $\{T_1, T_2, \ldots, T_c\}$ the trees in forest formed from $T$ by removing $A_e$. We show by induction on the height $h$ of $T$ that the following claims holds: $A_e$ and $A_e + \sum_{i=1}^{c} \sum_{A_f \in T_i} A_f$ are mutually rigid, and $A_e + \sum_{i=1}^{c} \sum_{A_f \in T_i} A_f$ is $\mathbb{N}$-compatible.

The claim holds trivially for $k = 1$ (a single-vertex tree), because $A_e$ is $\mathbb{N}$-compatible and is mutually rigid with itself.

Now, we assume that the inductive claim is correct for trees with height $h$ or less and we show it is correct for trees with height $h + 1$. Let $T$ be a tree of height $h + 1$ whose root is $A_e$, vertex, and let $T_1, T_2, \ldots, T_c$ be the subtrees defined above. The height of every $T_i$ is $h$ or less. Let $A_i$ be the root vertex of $T_i$, and let $F_i$ be the forest of $A_i'$s descendants. By definition, $A_e$ and $A_i$ are mutually rigid. By the inductive claim on $T_i$, we have that $A_i + \sum_{A_f \in F_i} A_f$ is $\mathbb{N}$-compatible and mutually rigid with $A_i$. We note that all the sums of the form $\sum A_f$ are symmetric and positive-semidefinite. Therefore, by Lemma 5.9 $A_e$ and $A_i + (A_i + \sum_{A_f \in F_i} A_f) = 2A_i + \sum_{A_f \in F_i} A_f$ are mutually rigid. By Lemma 4.3, we have that $A_e$ and $A_i + \sum_{A_f \in F_i} A_f = \sum_{A_f \in T_i} A_f$ are mutually rigid for every $i$. By Corollary 5.10, we have that $A_e$ and $A_e + \sum_{i=1}^{c} \sum_{A_f \in T_i} A_f$ are mutually rigid and $A_e + \sum_{i=1}^{c} \sum_{A_f \in T_i} A_f$ is $\mathbb{N}$-compatible. This concludes the proof of the lemma. $\square$

The next result generalizes the previous lemma.

**Theorem 6.3.** *Let $G$ be the rigidity graph of a collection $A_1, A_2, \ldots, A_k$ of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices. Let $H_1$ and $H_2$ be two connected subgraphs of $G$ that share a vertex $A_e$. Then $B = \sum_{A_f \in V(H_1)} A_f$ and $C = \sum_{A_f \in V(H_2)} A_f$ are mutually rigid.*

*Proof.* According to Lemma 6.2, $B$ and $A_e$ are mutually rigid and so are $C$ and $A_e$. By Lemma 5.9, we have that $B$ and $A_e + C$ are mutually rigid. The sum $A_e + C$ equals $\sum_{A_f \in V(H_2) \setminus A_e} A_f + 2A_e$. By Lemma 4.3, we have that $B$ and $\sum_{A_f \in V(H_2)} A_f = C$ are mutually rigid. $\square$

The next theorem shows that the rigidity graph can sometimes tell us that a finite-element matrix is rigid in the sense that its null space is exactly $\mathbb{N}$. This is only a sufficient condition; it is not necessary.

**Theorem 6.4.** *Let $G$ be the rigidity graph of a collection $A_1, A_2, \ldots, A_k$ of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices. Let $A = \sum_{e=1}^{k} A_e$. Let $N$ be a matrix whose columns form a basis for $\mathbb{N}$. If $G$ is connected, then* $\mathrm{enull}(A) = \mathrm{span}(\Xi_A N)$. *In particular, if $\bigcup_{e=1}^{k} \mathcal{N}_{A_e} = \{1, \ldots n\}$, then* $\mathrm{null}(A) = \mathbb{N}$ *and $A$ is rigid.*

*Proof.* According to Lemma 6.2, $A$ is $\mathbb{N}$-compatible. Therefore, by Lemma 5.4, $\mathrm{enull}(A) = \mathrm{span}(\Xi_A N)$.

If $\bigcup_{e=1}^{k} \mathcal{N}_{A_e} = \{1, \ldots n\}$, then $\Xi_A N = N$ and $\mathrm{enull}(A) = \mathrm{null}(A)$. Therefore, $\mathrm{null}(A) = \mathrm{enull}(A) = \mathrm{span}(\Xi_A N) = \mathrm{span}(N) = \mathbb{N}$. By definition, $A$ is rigid. $\square$

When the rigidity graph is not connected, the null space may or may not be $\mathbb{N}$. To show that a disconnected rigidity graph sometimes corresponds to a null space larger than $\mathbb{N}$, consider

$$A_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} .$$

Both are compatible with $\mathbb{N} = \operatorname{span} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, but they are not mutually rigid. Therefore, their rigidity graph consists of two disconnected vertices. The null space of $A_1 + A_2$ is spanned by both $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$, so it is indeed larger than $\mathbb{N}$, even though $\mathcal{N}_{A_1} \cup \mathcal{N}_{A_2} = \{1, 2, 3, 4\}$. Examples in which the rigidity graph is not connected but the null space of the sum is $\mathbb{N}$ are more complex; we show an example later in the paper, in section 7.3.

## 7. Three Case Studies

This section presents three families of $\mathbb{N}$-compatible SPSD matrices for two different $\mathbb{N}$s. One is well known and we present it without proofs. The second and third are more complex and we present them in full.

### 7.1. Laplacians.

The first family of matrices that we present consists of Laplacians, matrices that are often used in spectral graph theory and in other areas. The results in this sections are all adaptations of known results, so we omit the proofs. All the matrices and vectors are of order $n$.

**Definition 7.1.** For a pair $(k, j)$ of indices, $1 \le k < j \le n$, we define the vector $u^{(k,j)}$,

$$u_i^{(k,j)} = \begin{cases} +1 & i = k \\ -1 & i = j \\ 0 & \text{otherwise} . \end{cases}$$

We define the $(k, j)$ *edge matrix* using $A^{(k,j)} = u^{(k,j)}\, u^{(k,j)\,T}$.

**Lemma 7.2.** *Let , $1 \le k < j \le n$ and let $A^{(k,j)}$ be an edge matrix. Then*

(1) *$A^{(k,j)}$ is symmetric positive-semidefinite.*
(2) *$\mathcal{N}_{A^{(k,j)}} = \{k, j\}$.*
(3) *$A^{(k,j)}$ is compatible with $\mathbb{N}_1 = \operatorname{span} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$.*

The next lemma gives a sufficient and necessary condition for two edge matrices to be mutually rigid.

**Lemma 7.3.** *The edge matrices $A^{(i,j)}$ and $A^{(k,\ell)}$ are mutually rigid if and only if $|\{i, j\} \cap \{k, \ell\}| \ge 1$.*

Laplacians are sums of edge matrices (sometimes of positively-scaled edge matrices). They are often defined using an undirected graph $G = (\{1, 2, \dots, n\}, E)$,

$$A^{(G)} = \sum_{(i,j) \in E} A^{(i,j)} .$$

Each edge matrix $A^{(i,j)}$ is then associated with an edge $(i, j) \in E$ in the graph $G$. Lemma 7.3 states that two edge matrices are mutually rigid if and only if the corresponding edges are incident on a common vertex.

The rigidity graph of $\{A^{(i,j)} | (i, j) \in E\}$ is a dual of $G$,

$$G_{\text{dual}} = (E, \{(e, f) : e \text{ and } f \text{ share a vertex in } G\}) .$$

The rigidity graph of Laplacians is special in that its connectivity is not only a sufficient condition for the rigidity of the Laplacian, as shown in Theorem 6.4, but also a necessary condition.

**Lemma 7.4.** *Let $G = (\{1, 2, \ldots, n\}, E)$ be an undirected graph. If $A^{(G)} = \sum_E A^{(i,j)}$ is rigid then the rigidity graph $G_{dual}$ of $\{A^{(i,j)} | (i, j) \in E\}$ is connected.*

*Proof.* We first show that if $A^{(G)}$ is rigid, then $G$ is a connected graph. Assume for contradiction that $G$ is not connected. Therefore, there are two nonempty sets of vertices $S$ and $\bar{S} = \{1, \ldots, n\}\backslash S$ that are disconnected. Define the vector $x$,

$$x_i = \begin{cases} +1 & i \in S \\ -1 & i \in \bar{S} \ . \end{cases}$$

By definition, $A^{(i,j)}x = 0$ for every $(i, j) \in E$. Therefore, $A^{(G)}x = \sum_{(i,j)\in E} A^{(i,j)}x = 0$. The vector $x$ is in $\text{enull}(A^{(G)})$ and has no extension to $\mathbb{N}_1$. This contradicts the assumption that $A^{(G)}$ is rigid, since this assumption implies that it is $\mathbb{N}_1$-compatible. Therefore, $G$ is a connected graph.

It is clear that if $G$ is connected, then $G_{\text{dual}}$ is connected. This concludes the proof of the lemma. □

All the results on the rigidity of Laplacians hold for weighted Laplacians, which are sums of positively-scaled edge matrices.

7.2. **Elastic Struts in Two Dimensions.** The second family of matrices model a collection of pin-jointed struts. Such a collection may form a rigid structure called a truss (e.g., a triangle in two dimensions) or a non-rigid structure called a mechanism (e.g., two struts connected by a pin). The rigidity graph of such a structure, however, is never connected: it has no edges at all. Therefore, the rigidity graph can never show that the underlying structure is rigid.

We note that there is a combinatorial technique that can determine whether such a structure is rigid, under a technical assumption on the geometrical location of the pins. The structure is modeled by a graph in which vertices correspond to pins (assuming there is a pin at the end of each strut) and in which edges correspond to struts. If the pins are an an appropriately-defined general position, then several equivalent conditions on the graph characterize exactly the rigidity of the structure [12, 14, 16, 19, 13]. These conditions can be tested in $O(n^2)$ operations [11].

Our technique is more general but less precise than these techniques. It applies to any finite-element matrix, but it only provides sufficient conditions for rigidity. In the cases of two-dimensional struts, our sufficient conditions are never satisfied. We show later in this section that our technique does work for other families of elastic structures.

**Definition 7.5.** Let $P = \{p_l\}_{l=1}^n$ be a set of different points in the plain, $p_l = (x_l, y_l)$. For every $p_i \neq p_j$, let $v^{(i,j)}$ be the $2P$-by-1 vector defined by

$$v_i^{(k,j)} = \begin{cases} (x_k - x_j)/r_{k,j} & i = 2k - 1 \\ (y_k - y)/r_{k,j} & i = 2k \\ -(x_k - x_j)/r_{k,j} & i = 2j - 1 \\ -(y_k - y_j)/r_{k,j} & i = 2j \\ 0 & \text{otherwise} \end{cases} ,$$

where $r_{k,j} = \sqrt{(x_k - x_j)^2 + (y_k - y_j)^2}$. We define the $(i,j)$ *strut matrix* to be $A^{(i,j)} = v^{(i,j)}\, {v^{(i,j)}}^T$.

**Definition 7.6.** Given a collection $P = \{p_l\}_{l=1}^n$ of points in the plane, we define the translation and rotation vectors

$$N_P^{(x)} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad N_P^{(y)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad N_P^{(r)} = \begin{bmatrix} -y_1 \\ x_1 \\ -y_2 \\ x_2 \\ \vdots \\ -y_n \\ x_n \end{bmatrix}.$$

The *planar null-space* of the collection is $\mathbb{N}_P = \operatorname{span} N_P$, where $N_P = \begin{bmatrix} N_P^{(x)} & N_P^{(y)} & N_P^{(r)} \end{bmatrix}$.

The next lemma shows that the strut matrices are $\mathbb{N}_P$ compatible.

**Lemma 7.7.** *Let $P = \{p_l\}_{l=1}^n$ be a set of different points in the plain. Then*

(1) *$A^{(i,j)}$ is symmetric and positive semidefinite.*
(2) *$\mathcal{N}_{A^{(i,j)}} = \{2i-1, 2i, 2j-1, 2j\}$.*
(3) *$A^{(i,j)}$ is $\mathbb{N}_P$-compatible.*

*Proof.* The first two claims in the lemma follow directly from the definition of $A^{(i,j)}$. We show that $A^{(i,j)}$ is $\mathbb{N}_P$-compatible by showing that the columns of $\Xi_{A^{(i,j)}} N_P$ form a basis for $\operatorname{enull}(A^{(i,j)})$.

A direct calculation, which we omit, shows that $A \Xi_{A^{(i,j)}} N_P = 0$. The points $p_i$ and $p_j$ are different, so the rank of $\Xi_{A^{(i,j)}} N_P$ is 3. The rank of $A^{(i,j)}$ is 1, so its essential null space has dimension 3. Therefore, $\Xi_{A^{(i,j)}} N_P$ spans $\operatorname{enull}(A^{(i,j)})$, so $A^{(i,j)}$ is $\mathbb{N}_P$ compatible. $\square$

The following lemma indicates that the rigidity graph of a collection of strut matrices contains only trivial edges (self loops, which are always present).

**Lemma 7.8.** *Let $P = \{p_l\}_{l=1}^n$ be a set of different points in the plain. Let $A^{(i,j)}$ and $A^{(k,\ell)}$ be two strut matrices. $A^{(i,j)}$ and $A^{(k,\ell)}$ are mutually rigid if and only if $\{i,j\} = \{k,\ell\}$.*

*Proof.* Assume $A^{(i,j)}$ and $A^{(k,\ell)}$ are mutually rigid. Let $A = A^{(i,j)} + A^{(k,\ell)}$. By Lemma 5.7, $A$ is $\mathbb{N}_P$-compatible. Therefore, $\operatorname{rank}(\operatorname{enull}(A)) = \operatorname{rank}(\mathbb{N}_P) = 3$. Since $A^{(i,j)}$ and $A^{(k,\ell)}$ are mutually rigid, $|\{i,j\} \cap \{k,\ell\}| \geq 1$. Assume for contradiction that $|\{i,j\} \cap \{k,\ell\}| = 1$. This implies that $|\mathcal{N}_A| = 6$, so $\operatorname{rank}(A) = |\mathcal{N}_A| - \operatorname{rank}(\operatorname{enull}(A)) = 3$. But we also have $\operatorname{rank}(A) = \operatorname{rank}(v^{(i,j)}\, {v^{(i,j)}}^T + v^{(k,l)}\, {v^{(k,l)}}^T) \leq 2$, a contradiction. Therefore, $|\{i,j\} \cap \{k,\ell\}| = 2$ and $\{i,j\} = \{k,\ell\}$.

The other direction is immediate; a matrix is always mutually rigid with itself. $\square$

7.3. **Elastic Triangles in Two Dimensions.** We now study another family of matrices that also arise in two-dimensional linear elasticity, matrices that model triangular elements. The rigidity graph of such a collection can be connected, so the rigidity graph can sometimes tell us that the structure is rigid. There are also cases in which the structure is rigid but its rigidity graph is not connected.

**Definition 7.9.** Let $P = \{p_l\}_{l=1}^n$ be a set of different points in the plain, $p_l = (x_l, y_l)$, let $v^{(i,j)}$ and $A^{(i,j)}$ be defined as in Definition 7.5. For three different points $p_i$, $p_j$, and $p_k$, we define the $(i, j, k)$ *element matrix* in this family to be

$$A^{(i,j,k)} = A^{(i,j)} + A^{(i,k)} + A^{(j,k)} = v^{(i,j)}\, {v^{(i,j)}}^T + v^{(j,k)}\, {v^{(j,k)}}^T + v^{(k,i)}\, {v^{(k,i)}}^T .$$

The next lemma is the equivalent of Lemma 7.7. We omit the proof, which is similar to the proof of Lemma 7.7.

**Lemma 7.10.** *Let $P = \{p_l\}_{l=1}^n$ be a set of different points in the plain. Let $A$ be an $(i, j, k)$ element matrix. Then*

(1) *$A$ is symmetric and positive semidefinite.*
(2) *$\mathcal{N}_A = \{2i - 1, 2i, 2j - 1, 2j, 2k - 1, 2k\}$.*
(3) *$A$ is $\mathbb{N}_P$-compatible, where $\mathbb{N}_P$ is the planar null space defined in Definition 7.6.*

The following lemma characterizes mutual rigidity between $\mathbb{N}_P$-compatible matrices.

**Lemma 7.11.** *Let $P = \{p_l\}_{l=1}^n$ be a set of different points in the plain. Let $A$ and $B$ be $\mathbb{N}_P$-compatible matrices. Assume that there exist $i$ and $j \neq i$ such that $\{2i - 1, 2i, 2j - 1, 2j\} \subseteq \mathcal{N}_A \cap \mathcal{N}_B$. Then, $A$ and $B$ are mutually rigid.*

*Proof.* We show that $A$ is rigid with respect to $B$. Let $i$ and $j$ be two different indices such that $\{2i - 1, 2i, 2j - 1, 2j\} \subseteq \mathcal{N}_A \cap \mathcal{N}_B$. Let $u \in \text{enull}(A)$. We can write $u = \Xi_A N_P z$ for some $z$. Define the vector $v = \Xi_B N_P z$. By definition, $v \in \text{enull}(B)$ and $u_m = v_m$ for every $m \in \mathcal{N}_A \cap \mathcal{N}_B$. Therefore, $v$ is a rigid mapping of $u$ to $\text{enull}(B)$. Let $\hat{v}$ be a vector in $\text{enull}(B)$ such that $\hat{v}_m = u_m$ for every $m \in \mathcal{N}_A \cap \mathcal{N}_B$. We show that $v = \hat{v}$. We can write $\hat{v} = \Xi_B N_P w$ for some $w$. We have that $\hat{v}_m = u_m$ for every $m \in \{2i-1, 2i, 2j-1, 2j\} \subseteq \mathcal{N}_A \cap \mathcal{N}_B$. This gives us $\Xi_{A,B}(N_P z - N_P w) = 0$, which we expand into

$$\begin{aligned} z_1 - z_3 y_i &= w_1 - w_3 y_i \\ z_2 + z_3 x_i &= w_2 + w_3 x_i \\ z_1 - z_3 y_j &= w_1 - w_3 y_j \\ z_2 + z_3 x_j &= w_2 + w_3 x_j \end{aligned}$$

Subtracting the third equation from the first and the fourth equation from the second, we obtain

$$\begin{aligned} z_3(y_j - y_i) &= w_3(y_j - y_i) \\ z_3(x_i - x_j) &= w_3(x_i - x_j) \end{aligned}$$

Since $(x_i, y_i) \neq (x_j, y_j)$, we have that $z_3 = w_3$. Substituting $w_3$ with $z_3$ in the first two equations, we obtain $w_1 = z_1$ and $w_2 = z_2$. Therefore, $v = \hat{v}$. This shows that $A$ is rigid with respect to $B$. By symmetry, $B$ is rigid with respect to $A$. Therefore, $A$ and $B$ are mutually rigid. $\square$

The last lemma of this section shows how to construct the rigidity graph for this family of matrices.

**Lemma 7.12.** *Let $P = \{p_l\}_{l=1}^n$ be a set of different points in the plain. Let $A = A^{(i,j,k)}$ and let $B = A^{(p,q,r)}$ be element matrices Then, $A$ and $B$ are mutually rigid if and only if $|\{i, j, k\} \cap \{p, q, r\}| \geq 2$.*

*Proof.* If $|\{i,j,k\} \cap \{p,q,r\}| \geq 2$, there exist $m \neq l$ such that $\{2m-1, 2m, 2l-1, 2l\} \subseteq \mathcal{N}_A \cap \mathcal{N}_B$. Then, by Lemma 7.11, $A$ and $B$ are mutually rigid.

Let $c = |\{i,j,k\} \cap \{p,q,r\}|$. We show that if $A$ and $B$ are mutually rigid, then $c \geq 2$. It is clear that if $c = 0$ then $A$ and $B$ are not mutually rigid. It is sufficient to show that $c \neq 1$. Assume for contradiction that $c = 1$. Assume without loss of generality that $(i,j,k) = (1,4,5)$ and $(p,q,r) = (1,2,3)$. We show that the vector $u = 0 \in \operatorname{enull}(A)$ has two different rigid mappings in $\operatorname{enull}(B)$. The first rigid mapping is the vector $u = 0$ itself. The second rigid mapping is $v = \Xi_B N_P \begin{bmatrix} y_1 & -x_1 & 1 \end{bmatrix}^T$. This vector is different from $u = 0$, since we either $y_1 \neq y_2$ or $x_1 \neq x_2$, so

$$v = \Xi_B \begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & x_1 \\ 1 & 0 & -y_2 \\ 0 & 1 & x_2 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} y_1 \\ -x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} y_1 - y_1 \\ -x_1 + x_1 \\ y_1 - y_2 \\ -x_1 + x_2 \\ \vdots \end{bmatrix} \neq 0$$

By definition, $v \in \operatorname{enull}(B)$ and

$$\Xi_{A,B} v = \begin{bmatrix} y_1 - x_1 \cdot 0 - y_1 \\ y_1 \cdot 0 - x_1 + x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \Xi_{A,B} u \, .$$

Therefore, both $u$ and $v \neq u$ are rigid mappings of $u$ to $\operatorname{enull}(B)$, so $A$ is not rigid with respect to $B$. This contradicts our assumption and shows that $c \neq 1$. Therefore, $c = |\{i,j,k\} \cap \{p,q,r\}| \geq 2$. ☐

Informally speaking, Lemma 7.12 shows that edges in the rigidity graph correspond to pairs of triangles whose mutual rigidity is evident: they share a side. The lemma can be generalized to higher dimensions: elastic elements are mutually rigid if and only if they share a face. For elasticity, this may be a trivial statement, but it shows that our *algebraic* definition of mutual rigidity indeed captures the domain-specific notion of rigidity.

Figure 7.1 shows a few examples of triangular plane elements and their rigidity graphs. The structures in cases (a) and (b) are not rigid, and the rigidity graph is not connected. Case (c) is rigid, and the rigidity graph is connected. Case (d) is rigid, but the rigidity graph does not show it; the graph is not connected. This shows, again, that connectivity of the rigidity graph is not a necessary condition for rigidity.

## 8. Rigid Sparsifications

Our next goal is to sparsify a matrix $A$ defined as a sum of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices, but without changing $\operatorname{null}(A)$. By sparsify we mean that linear systems of the form $Bz = r$, where $B$ is the sparsification of $A$, should be easier to solve than linear systems of the form $Ax = b$. In this sense, $B$ is sparser than $A$ if it has a subset of the nonzeros of $A$, or if the pattern graph of $B$ has smaller balanced vertex separators than the pattern graph of $A$ [9, 15]. There may be other meanings.

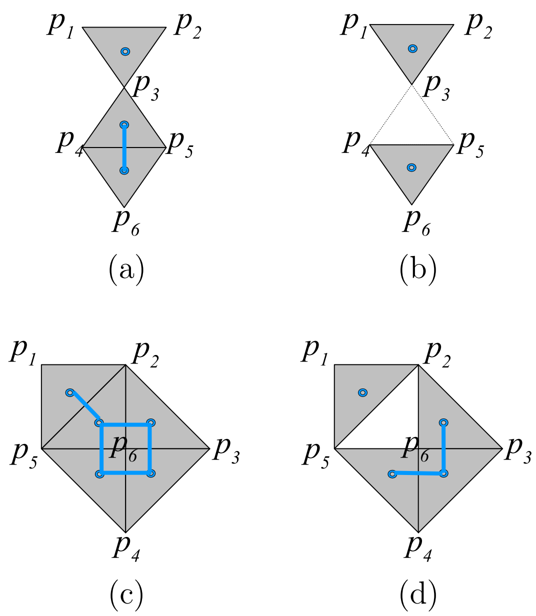


FIGURE 7.1. Triangular plane elements and their rigidity graphs (in blue).

8.1. **Dropping Vertices from the Rigidity Graph.** Perhaps the most obvious way to sparsify $A = \sum_e A_e$ is to drop some of the element matrices from the assembly. This section gives a condition that guarantees that the subset-sum has the same range and null space as $A$. The analysis is inductive: it analyzes the effect of dropping one element at a time.

**Lemma 8.1.** *Let $G$ be the rigidity graph of a collection $A_1, A_2, \ldots, A_k$ of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices. Let $\{H_1, H_2, \ldots, H_\ell\}$ be the connected components of the $G$. Assume that $\mathcal{N}_{H_i} \cap \mathcal{N}_{H_j} = \phi$ for every $i \neq j$.*

*Let $A = \sum_{f=1}^{k} A_f$ and let*

$$B = \sum_{\substack{f=1 \\ f \neq e}}^{k} A_f = A - A_e$$

*for some vertex $A_e$. If*

1. *$G$ and the rigidity graph of $B$ have the same number of connected components, and*
2. *$\mathcal{N}_A = \mathcal{N}_B$,*

*then the matrices $A$ and $B$ have the same null space.*

*Proof.* By Corollary 3.6 and by the condition $\mathcal{N}_A = \mathcal{N}_B$, it is sufficient to show that $A$ and $B$ are mutually rigid. By Lemma 4.1, $A = B + A_e$ is rigid with respect to $B$. All that is left to show is that $B$ is rigid with respect to $A$.

Assume without loss of generality that $A_e$ is in $H_1$. Let

$$C = \sum_{\substack{A_f \in H_1 \\ f \neq e}} A_f \ .$$

The first condition of the lemma implies that $H_1 \backslash \{A_e\}$ is a nonempty connected subgraph of $G$. Therefore, there exist a vertex $A_c$ in $H_1 \backslash \{A_e\}$ such that $A_e$

and $A_c$ are mutually rigid. Applying Theorem 6.3 to $\{A_e, A_c\}$ and to $H_1 \backslash \{A_e\}$ shows that $A_e + A_c$ and $C$ are mutually rigid. Therefore, by Lemma 4.2, $C$ and $C + A_e + A_c$ are mutually rigid. Finally, by Lemma 4.3, $C$ and $C + A_e$ are mutually rigid.

Let $D = \sum_{i=2}^{\ell} \sum_{A_f \in H_i} A_f$. We have that $\mathcal{N}_D \cap \mathcal{N}_C = \phi$ and $\mathcal{N}_D \cap \mathcal{N}_{C+A_e} = \phi$, since $\mathcal{N}_{H_i}$ are pairwise disjoint. By Lemma 4.4, we have that $C+D$ and $C+A_e+D$ are mutually rigid. Therefore, $A = C + A_e + D$ and $B = C + D$ are mutually rigid, so $\text{null}(A) = \text{null}(B)$. □

In particular, if $G$ is connected, then the lemma allows us to drop element matrices only as long as the rigidity graph of the remaining elements remains connected. Clearly, there are cases where we can drop an element matrix that would disconnect the rigidity graph without changing the null space of the sum. In this case dropping the element violates the sufficient condition stated in Lemma 8.1, but without actually changing the null space. For example, dropping $A^{(2,5,6)}$ from the structure shown in Figure 7.1.(c) leads to the structure shown in Figure 7.1.(d), which is also rigid, but has a disconnected rigidity graph.

The examples shown in Figure 7.1 parts (a) and (b) show that the lemma does not hold if the $\mathcal{N}_{H_i}$ are not mutually disjoint. Dropping $A^{(3,4,5)}$ from the structure shown in part (a) of the figure gives the structure shown in part (b). The rigidity graphs of both structures have the same number of connected components, 2, and $\mathcal{N}_A = \mathcal{N}_{A-A^{(3,4,5)}}$. But the null space of the structure in (a) has dimension 4 (rigid body motions and a rotation around $p_3$), where as the null space of the structure in (b) has dimension 6 (separate rigid body motions for the two elements).

If we use Lemma 8.1 to construct a preconditioner $B$ by repeatedly dropping element matrices from the sum $A = \sum_i A_i$, the generalized eigenvalues of $(B, A)$ are clearly bounded from above by 1, since for any $\lambda$ that satisfies $Bx = \lambda Ax$ for an $x \notin \text{null}(A)$ we have

$$\begin{aligned} \lambda &\leq \max_{\substack{x \\ Ax \neq 0}} \frac{x^T B x}{x^T A x} \\ &= \max_{\substack{x \\ Ax \neq 0}} \frac{x^T \left( \sum_{i \in S \subset \{1, \ldots k\}} A_i \right) x}{x^T \left( \sum_{i=1}^k A_i \right) x} \\ &\leq 1 \, . \end{aligned}$$

8.2. **Dropping Edges from the Rigidity Graph by Fretsaw Extensions.** We now show and analyze a more sophisticated sparsification technique that drops edges from the rigidity graph.

**Definition 8.2.** An *extension mapping* for matrices with $n$ rows is a length $\ell$ vector $s = \begin{bmatrix} s_1 & s_2 & \cdots & s_\ell \end{bmatrix}$ of integers in the range 1 to $n$. The *master extension matrix* $P$ of an extension mapping $s$ is an $(n+\ell)$-by-$n$ matrix with a single 1 in each row,

$$P_{ij}^{(s)} = \begin{cases} 1 & i \leq n \text{ and } j = i \\ 1 & i > n \text{ and } j = s_{i-n} \\ 0 & \text{otherwise} \, . \end{cases}$$

An *extension matrix* $Q^{(s)}$ of an extension mapping $s$ is an $(n+\ell)$-by-$n$ with a single 1 in each column and at most a single 1 in each row such that $P_{ij}^{(s)} = 0$

implies $Q_{ij}^{(s)} = 0$. When the mapping is clear from the context, we drop the superscript from the extension matrices.

In the product $Q^{(s)}A$ of an extension matrix $Q^{(s)}$ and an arbitrary matrix $A$, each row of the product is either all zeros or a copy of some row of $A$, and each row of $A$ is mapped to a row of the product. In particular, row $i$ of $A$ is either mapped to row $i$ of the product or to row $n + j$ where $s_j = i$.

The following lemma states some properties of extension matrices and their relation to the projection matrices $\Xi_A$. We omit its proof.

**Lemma 8.3.** *Let $P = P^{(s)}$ and $Q = Q^{(s)}$ be a master extension matrix and an extension matrix for some extension mapping $s$ of length $\ell$. Let $A$ be an $n$-by-$n$ symmetric matrix and let $I_n$ be the $n$-by-$n$ identity matrix. Then $Q^TQ = P^TQ = Q^TP = I_n$ and $\Xi_{QAQ^T} = Q\Xi_AQ^T$.*

**Definition 8.4.** The extension of a subspace $\mathbb{N} \subseteq \mathbb{R}^n$ under an extension mapping $s$ is the subspace $\operatorname{span}\left(P^{(s)}N\right)$ where $N$ is a matrix whose columns form a basis for $\mathbb{N}$. To keep the notation simple, we abuse the notation and denote this space by $P^{(s)}\mathbb{N}$.

**Lemma 8.5.** *The space $P^{(s)}\mathbb{N}$ depends only on $P^{(s)}$ and on $\mathbb{N}$, not on a particular basis $N$. That is, for any two matrices $N_1$ and $N_2$ whose columns span $\mathbb{N}$, $\operatorname{span}\left(P^{(s)}N_1\right) = \operatorname{span}\left(P^{(s)}N_2\right)$.*

*Proof.* Let $x \in \operatorname{span}\left(P^{(s)}N_1\right)$. There exists a vector $y$ such that $x = P^{(s)}N_1y$. Since $N_1y \in \mathbb{N}$, there exists a vector $z$ such that $N_1y = N_2z$. Therefore, $x = P^{(s)}N_1y = P^{(s)}N_2z \in \operatorname{span}(P^{(s)}N_2)$. This implies that $\operatorname{span}\left(P^{(s)}N_1\right) \subseteq \operatorname{span}\left(P^{(s)}N_2\right)$. Equality follows by symmetry. $\square$

The extension of an $\mathbb{N}$-compatible symmetric positive semidefinite matrix $A$ retains the essential properties of $A$.

**Lemma 8.6.** *Let $A$ be an $\mathbb{N}$-compatible symmetric positive semidefinite matrix, and let $Q = Q^{(s)}$ be an extension matrix for some extension mapping $s$. Then $QAQ^T$ is symmetric positive semidefinite and compatible with $P^{(s)}\mathbb{N}$.*

*Proof.* The matrix $QAQ^T$ is symmetric since $A$ is symmetric. For an arbitrary vector $x$, let $y = Q^Tx$. We have that $x^TQAQ^Tx = y^TAy \geq 0$, since $A$ is positive semidefinite. This implies that $QAQ^T$ is positive semidefinite.

We now show that $QAQ^T$ is compatible with $P^{(s)}\mathbb{N}$. Let $N$ be a matrix whose columns form a basis for $\mathbb{N}$. It is sufficient to show that $\operatorname{span}(\Xi_{QAQ^T}P^{(s)}N) = \operatorname{enull}(QAQ^T)$. Let $x \in \operatorname{span}(\Xi_{QAQ^T}P^{(s)}N)$, so there is a vector $y$ such that $x = \Xi_{QAQ^T}P^{(s)}Ny$. By Lemma 8.3,

$$QAQ^Tx = QAQ^T\Xi_{QAQ^T}P^{(s)}Ny = QAQ^TP^{(s)}Ny = QAI_nNy = Q(AN)y = 0 .$$

The last equality is due to the fact that $A$ is $\mathbb{N}$-compatible, so $AN = 0$. Therefore, $x \in \operatorname{null}(QAQ^T)$. Moreover, by definition $x_i = 0$ for all $i \notin \mathcal{N}_{QAQ^T}$, so $x \in \operatorname{enull}(QAQ^T)$. This implies that $\operatorname{span}(\Xi_{QAQ^T}P^{(s)}N) \subseteq \operatorname{enull}(QAQ^T)$.

We now show the inclusion in the other direction. Let $x \in \operatorname{enull}(QAQ^T)$. Since $QAQ^Tx = 0$, $AQ^Tx = Q^TQAQ^Tx = 0$. Therefore $Q^Tx \in \operatorname{null}(A)$, so $\Xi_AQ^Tx \in \operatorname{enull}(A)$. Since $A$ is $\mathbb{N}$-compatible, there exists a vector $y$ such that $\Xi_AQ^Tx = \Xi_ANy$. By Lemma 8.3,

$$x = \Xi_{QAQ^T}x = Q\Xi_AQ^Tx = Q\Xi_ANy = Q\Xi_A(Q^TP^{(s)})Ny = \Xi_{QAQ^T}P^{(s)}Ny .$$

Therefore, $x \in \operatorname{span}(\Xi_{QAQ^T} P^{(s)} N)$. This shows that $\operatorname{enull}(QAQ^T) \subseteq \operatorname{span}(\Xi_{QAQ^T} P^{(s)} N)$ and concludes the proof of the lemma. $\square$

Similarly-extended matrices maintain their rigidity relationship.

**Lemma 8.7.** *Let $\{A_i\}_{i=1}^k$ be a collection of $\mathbb{N}$-compatible symmetric positive semidefinite matrices, and let $Q_e = Q_e^{(s)}$ and $Q_f = Q_f^{(s)}$ be extension matrices for some $s$, such that $[Q_e]_{:,j} = [Q_f]_{:,j}$ for any $j \in \mathcal{N}_{A_e} \cap \mathcal{N}_{A_f}$. Then $Q_e A_e Q_e^T$ is rigid with respect to $Q_f A_f Q_f^T$ if and only if $A_e$ is rigid with respect to $A_f$.*

*In particular, if $Q$ is an $(n+\ell)$-by-$n$ identity matrix (the first $n$ columns of an $(n+\ell)$-by-$(n+\ell)$ identity matrix; such a matrix is an extension matrix for any $s$), then $(A_e, A_f)$ is an edge of the rigidity graph of $\{A_i\}_{i=1}^k$ if and only if $(QA_eQ^T, QA_fQ^T)$ is an edge of the rigidity graph of $\{QA_iQ^T\}_{i=1}^k$.*

*Proof.* Assume $A_e$ is rigid with respect to $A_f$. Let $x \in \operatorname{enull}(Q_e A_e Q_e^T)$, define $\hat{x} = Q_e^T x$. Since $A_e \hat{x} = A_e Q_e^T x = Q_e^T Q_e A_e Q_e^T x = 0$, $\hat{x}$ is in $\operatorname{null}(A_e)$. The vector $\hat{x}$ is also in $\operatorname{enull}(A_e)$, since

$$\hat{x} = Q_e^T x = Q_e^T \Xi_{Q_e A_e Q_e^T} x = Q_e^T Q_e \Xi_{A_e} Q_e^T x = \Xi_{A_e} Q_e^T x \ .$$

Let $\hat{y}$ be $\hat{x}$'s unique rigid mapping in $\operatorname{enull}(A_f)$. Define $y = Q_f \hat{y}$. We have that $Q_f A_f Q_f^T y = Q_f A_f Q_f^T Q_f \hat{y} = Q_f A_f \hat{y} = 0$. Therefore, $y \in \operatorname{null}(Q_f A_f Q_f^T)$. Since

$$y = Q_f \hat{y} = Q_f \Xi_{A_f} \hat{y} = Q_f \Xi_{A_f} Q_f^T Q_f \hat{y} = \Xi_{Q_f A_f Q_f^T} Q_f \hat{y} \ ,$$

$y$ is also in $\operatorname{enull}(Q_f A_f Q_f^T)$.

We still need to show that $x$ and $y$ coincide on $\mathcal{N}_{Q_e A_e Q_e^T} \cap \mathcal{N}_{Q_f A_f Q_f^T}$. Let $i \in \mathcal{N}_{Q_e A_e Q_e^T} \cap \mathcal{N}_{Q_f A_f Q_f^T}$. There exists an index $j$ such that $[Q_e]_{i,j} = [Q_f]_{i,j} = 1$. Therefore, $j \in \mathcal{N}_{A_e} \cap \mathcal{N}_{A_f}$ . By definition, $\hat{x}_j = \hat{y}_j$, so $(Q_e \hat{x})_i = (Q_f \hat{y})_i$. Therefore, $x_i = y_i$. This implies that $y$ is a mapping of $x$ to $\operatorname{enull}(Q_f A_f Q_f^T)$.

We now show that this mapping is unique. Let $u$ be a rigid mapping of $x$ into $\operatorname{enull}(Q_f A_f Q_f^T)$. Define $\hat{u} = Q_f^T u$. Let $j \in \mathcal{N}_{A_e} \cap \mathcal{N}_{A_f}$. There exists an index $i$ such that $[Q_e]_{i,j} = [Q_f]_{i,j} = 1$. Therefore, $i \in \mathcal{N}_{Q_e A_e Q_e^T} \cap \mathcal{N}_{Q_f A_f Q_f^T}$. By definition, $u_i = x_i$, so $(Q_f^T u)_j = (Q_e^T x)_j$. Therefore, $\hat{u}$ and $\hat{x}$ coincide on $\mathcal{N}_{A_e} \cap \mathcal{N}_{A_f}$. We have that

$$\hat{u} = Q_f^T u = Q_f^T \Xi_{Q_f A_f Q_f^T} u = Q_f^T Q_f \Xi_{A_f} Q_f^T u = \Xi_{A_f} Q_f^T u \ .$$

Be definition, $A_f \hat{u} = A_f Q_f^T u = Q_f^T Q_f A_f Q_f^T u = 0$. Therefore, $\hat{u} \in \operatorname{enull}(A_f)$. Since $A_e$ is rigid with respect to $A_f$, $\hat{u} = \hat{y}$. Therefore,

$$y = Q_f \hat{y} = Q_f \hat{u} = Q_f Q_f^T u = \Xi_{Q_f Q_f^T} u = u \ ,$$

so $y$ is unique. Therefore, $Q_e A_e Q_e^T$ is rigid with respect to $Q_f A_f Q_f^T$.

The other direction of the equivalence can be shown in a similar manner, so we omit its proof. $\square$

We are particularly interested in certain extensions, described by the following definition.

**Definition 8.8.** Let $\{A_i\}_{i=1}^k$ be a collection of $\mathbb{N}$-compatible symmetric positive semidefinite matrices, let $s$ be an extension mapping, and let $\{Q_i\}_{i=1}^k$ be a collection of extension matrices for this $s$. Let $G$ be the rigidity graph of $\{A_i\}_{i=1}^k$, and let $\hat{G}$ be the rigidity graph of $\{Q_i A_i Q_i^T\}_{i=1}^k$. If

- For every $j \in \mathcal{N}_A$, there is some $i$ such that $j \in \mathcal{N}_{A_i}$ and $[Q_i]_{j,j} = 1$,

- for every connected component $A_{i_1},\ldots,A_{i_m}$ of $G$, the matrices $Q_{i_1}A_{i_1}Q_{i_2}^T,\ldots,Q_{i_m}A_{i_m}Q_{i_m}^T$ are a connected component of $\hat{G}$ and vice versa, and
- for every connected component $A_{i_1},\ldots,A_{i_m}$ of $G$, there is at least one $j\in\{i_1,\ldots,i_m\}$ such that $Q_j$ is the $(n+\ell)$-by-$n$ identity matrix,

then we say that $\{Q_iA_iQ_i^T\}_{i=1}^k$ is a *fretsaw extension* of $\{A_i\}_{i=1}^k$. (The rationale behind the name fretsaw is explained below.) When the $Q_i$s are clear from the context, we use $\mathcal{F}(A)$ to denote $\sum_{i=1}^k Q_iA_iQ_i^T$; we call this matrix the *fretsaw extension of* $A=\sum_{i=1}^k A_i$.

We note that the first fretsaw condition ensures that $\mathcal{N}_A\subseteq\mathcal{N}_{\mathcal{F}(A)}$.

Transforming an extended matrix $B=\sum Q_i^{(s)}A_i\, {Q_i^{(s)}}^T$ back to $A=\sum A_i$ is simple, as shown in the next lemma.

**Lemma 8.9.** *Let $A_1,A_2,\ldots,A_k$ be a collection of $n$-by-$n$ matrices. Let $P$ be a master extension matrix for some extension mapping $s$ and let $\{Q_i\}_{i=1}^k$ be a collection of extension matrices for this $s$. Denote $A=\sum_{i=1}^k A_i$ and $B=\sum_{i=1}^k Q_iA_iQ_i^T$. Then $A=P^TBP$.*

*Proof.* By Lemma 8.3, $P^TQ_i=I$, so

$$P^TBP=\sum_{i=1}^k P^TQ_iA_iQ_i^TP=\sum_{i=1}^k A_i=A\,.$$

□

**Definition 8.10.** Let $B$ be an $(n+\ell)$-by-$(n+\ell)$ matrix. Partition

$$B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}$$

such that $B_{11}$ is $n$-by-$n$ and $B_{22}$ is $\ell$-by-$\ell$. If $B_{22}$ is nonsingular, we denote $\mathrm{schur}_\ell(B)=B_{11}-B_{12}B_{22}^{-1}B_{21}$. When $\ell$ is clear from the context, we simply use $\mathrm{schur}(B)$.

The following theorem is the main structural result of this section. The theorem lists conditions that guarantee the preservation of the null space under a fretsaw extension.

**Theorem 8.11.** *Let $A=\sum_{i=1}^k A_i$, where $A_1,A_2,\ldots,A_k$ is a collection of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices, let $Q$ be the $(n+\ell)$-by-$n$ identity matrix, and let $\mathcal{F}(A)$ be a fretsaw extension of $A$. If the rigidity graph of $A_1,A_2,\ldots,A_k$ is connected, then*

1. *$QAQ^T$ and $\mathcal{F}(A)$ are mutually rigid.*
2. *For every $x\in\mathrm{enull}(QAQ^T)$, there exist a vector $y\in\mathrm{enull}(\mathcal{F}(A))$ such that $\Xi_{QAQ^T}y=x$.*
3. *$\mathrm{null}(\mathcal{F}(A))\subseteq\mathrm{null}(QAQ^T)$.*
4. *If $\mathrm{schur}(\mathcal{F}(A))$ exists, then $\mathrm{null}(A)=\mathrm{null}(\mathrm{schur}(\mathcal{F}(A)))$.*

*Proof.* Let $\{Q_i\}_{i=1}^k$ be the collection of the extension matrices used in $\mathcal{F}(A)$. By Lemma 8.6, the matrices in collections $\{Q_iA_iQ_i^T\}_{i=1}^k$ and $\{QA_iQ^T\}_{i=1}^k$ are compatible with $P^{(s)}\mathbb{N}$. By definition, the rigidity graph of $\{Q_iA_iQ_i^T\}_{i=1}^k$ is connected. By Lemma 8.7 and the assumption that the rigidity graph of $\{A_i\}_{i=1}^k$ is connected, the rigidity graph of $\{QA_iQ^T\}_{i=1}^k$ is also connected. By definition, the rigidity graph of $\{QA_iQ^T\}_{i=1}^k$ shares at least one vertex with the rigidity graph of $\{Q_iA_iQ_i^T\}_{i=1}^k$. Therefore, by Lemma 6.3, $QAQ^T$ and $\mathcal{F}(A)$ are mutually rigid.

The second part of the lemma follows the fact that $QAQ^T$ is rigid with respect to $\mathcal{F}(A)$ and that $\mathcal{N}_{QAQ^T} = \mathcal{N}_A \subseteq \mathcal{N}_{\mathcal{F}(A)}$.

The matrix $\mathcal{F}(A)$ is rigid with respect to $QAQ^T$. We also have that $\mathcal{N}_{QAQ^T} = \mathcal{N}_A \subseteq \mathcal{N}_{\mathcal{F}(A)}$. Therefore, by Lemma 3.5, $\text{null}(\mathcal{F}(A)) \subseteq \text{null}(QAQ^T)$. This proves the third part of the lemma.

Assume, without loss of generality, that $\mathcal{N}_A = \{1, \ldots, m\}$ for some $m$. Under this assumption, the fourth part of the lemma follows from Lemma 3.7 and the fact that $QAQ^T$ and $\mathcal{F}(A)$ are mutually rigid. $\square$

8.3. **Constructing a Fretsaw Extension from a Spanning Tree.** We now show a practical way to construct nontrivial fretsaw extensions. The extensions that we build here are essentially as sparse as possible: we can factor $\mathcal{F}(A)$ with no fill. Our simple spanning-tree extensions may not be effective preconditioners (the generalized eigenvalues may be large), but the construction shows that there are efficient algorithms for constructing nontrivial fretsaw extensions.

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ be a collection of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices. Let $G = (\mathcal{A}, E)$ be the rigidity graph of $\mathcal{A}$. Without loss of generality, we assume that $G$ is connected (otherwise we repeat the construction for each connected component of $G$). The construction builds the $Q_i$s by columns. This introduces a slight notational difficulty, since we do not know the number of rows in the $Q_i$s until the construction ends. We use the convention that the columns are tall enough ($nk$ is tall enough) and then chop the $Q_i$s to remove zero rows. We denote by $e_r$ the $r$th (long enough) unit vector.

We use a spanning tree $T$ of $G$ to define an extension $\mathcal{F}(A)$. We initialize a variable $r$ to $n$. This variable stores the index of the last nonzero row in the $Q_i$s. The algorithm iterates over the column indices $j \in \{1, \ldots, n\}$ (in any order). In iteration $j$, we construct column $j$ of $Q_1 \ldots, Q_k$.

We begin iteration $j$ by setting the $j$th column of $Q_1$ to $e_j$. This ensures that $Q_1$ is an identity matrix, so the second fretsaw condition is automatically satisfied.

We now construct the set $V^{(j)} = \{A_i \in \mathcal{A} | j \in \mathcal{N}_{A_i}\}$ of elements that are incident on the index $j$. We also construct the subgraph $G^{(j)} = (V^{(j)}, E^{(j)})$ of $T$ that is induced by $V^{(j)}$. We partition $G^{(j)}$ into its connected components and process each component separately. The ordering of the components is arbitrary, except that if $A_1 \in V^{(j)}$, then we process the component containing $A_1$ first. Let $\{A_{i_1}, A_{i_2}, \ldots, A_{i_m}\} \subseteq V^{(j)}$ be the vertices of the next component to be processed. If this component is the first component of $G^{(j)}$, then we set the $j$th columns of $Q_{i_1}, \ldots, Q_{i_m}$ to $e_j$. Otherwise, we increment $r$ and set the $j$th columns of $Q_{i_1}, \ldots, Q_{i_m}$ to $e_r$.

This process specifies the $j$th column of every $Q_i$ such that $j \in \mathcal{N}_{A_i}$. We complete the construction of the $Q_i$s by setting the $j$th column of every $Q_i$ such that $j \notin \mathcal{N}_{A_i}$ to $e_j$.

Sometimes the row/column indices of $A = \sum A_i$ have a natural grouping. For example, in problems arising in two-dimensional linear elasticity, each point in the geometry of the discrete structure is associated with two indices, an $x$-direction index, say $j_1$, and a $y$-direction index, say $j_2$. This usually implies that $G^{(j_1)}$ and $G^{(j_2)}$ are identical graphs. In such cases, we *extend* $A$ *consistently*:

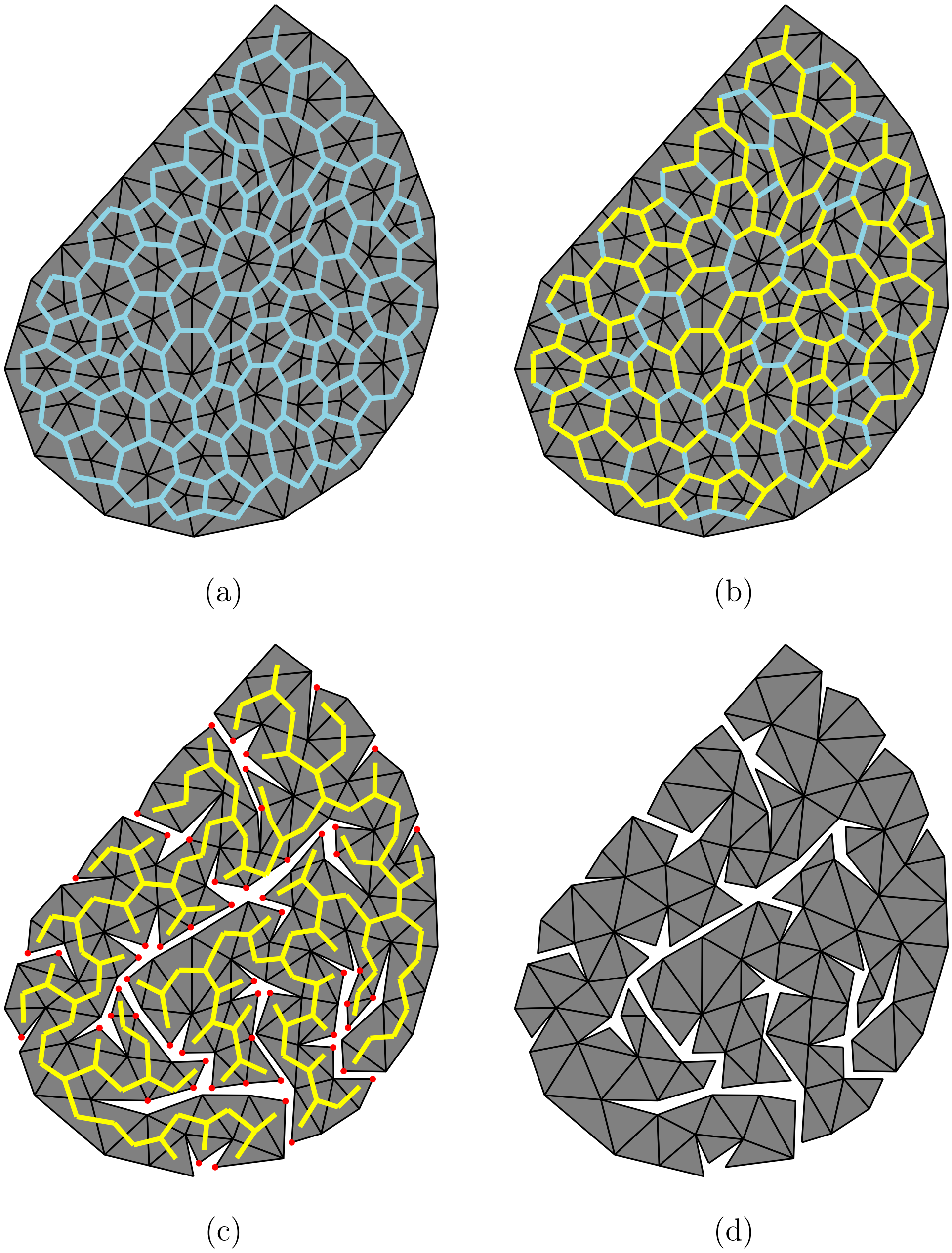


Figure 8.1. The construction of a spanning-tree fretsaw-extended matrix. (a) The elements of the original structure and its rigidity graph in blue; elements are mutually rigid iff they share a side. (b) A spanning tree $T$ (yellow) of the rigidity graph. (c) The structure induced by the spanning tree; duplicated nodes are marked by red circles. In the illustration, the triangles have been slightly shrunk to show how rigidity relationships have been severed, but the element matrices are only permuted, so they still model the original triangles. (d) The fretsaw-extended structure.

we order the connected components of $G^{(j_1)}$ and $G^{(j_2)}$ consistently, which means that $[Q_i]_{:,j_1} = e_{j_1}$ if and only if $[Q_i]_{:,j_2} = e_{j_2}$.

Figure 8.1 illustrates the construction of a spanning-tree fretsaw-extended matrix for a structure consisting of linear elastic elements in two dimensions. The figure explains the rationale behind the term fretsaw. A fretsaw is a fine-toothed saw held under tension, designed for cutting thin slits in flat materials, such as sheets of plywood. When applied to two dimensional elastic structures, like the one shown in Figure 8.1, the spanning-tree fretsaw construction appears to cut the original structure like a fretsaw.

Once the $Q_i$s are constructed, we form $\mathcal{F}(A) = \sum_{i=1}^{k} Q_i A_i Q_i^T$. The next theorem shows that the $Q_i$s are extension matrices for some $s$ and that $\mathcal{F}(A)$ is a fretsaw extension of $A$.

**Theorem 8.12.** *Let $\{A_i\}_{i=1}^{k}$ be a collection of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices, let $s$ be a spanning-tree extension mapping, and let $\{Q_i\}_{i=1}^{k}$ be the extension matrices for this extension. Then $\mathcal{F}(A) = \sum_{i=1}^{k} Q_i A_i Q_i^T$ is a fretsaw extension of $A$.*

*Proof.* We first show that the $Q_i$s are indeed valid extension matrices for a single $s$. We can construct an $s$ step-by-step along the construction of the $Q_i$s. We initialize $s = \langle\rangle$. When we increment the value $r$, we concatenate the current index $j$ to the end of $s$. By definition, $Q_i$ contains a nonzero in $(r, j)$ if and only if $s$ contains the value $j$ in its $(r-n)$th position. Therefore, the $Q_i$s are consistent with $P^{(s)}$. Moreover, by definition, every column $j$ of a matrix $Q_i$ is either $e_j$ or $e_r$ where $r > n$ and $r$ is unique in among the columns of $Q_i$. Therefore, the $Q_i$s are valid extension matrices.

Let $G$ be the rigidity graph of $\{A_i\}_{i=1}^{k}$, and let $\hat{G}$ be the rigidity graph of $\{Q_i A_i Q_i^T\}_{i=1}^{k}$. We show now that every connected component $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$in $G$, the matrices $Q_{i_1} A_{i_1} Q_{i_1}^T, Q_{i_2} A_{i_2} Q_{i_2}^T, \ldots, Q_{i_m} A_{i_m} Q_{i_m}^T$ form a connected component in $\hat{G}$ and vice versa. Let $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ be a connected component of $G$. Let $T$ be the spanning tree used to create $Q_{i_1}, Q_{i_2}, \ldots, Q_{i_m}$. Let $A_p$ and $A_q$ be to matrices adjacent in $T$. For every index $j \in \mathcal{N}_{A_p} \cap \mathcal{N}_{A_q}$, $A_p$ and $A_q$ belong to the same connected component of $G^{(j)}$. Therefore, $Q_p$ and $Q_j$ coincide on their $j$th column. Therefore, by Lemma 8.7, $Q_p A_p Q_p^T$ and $Q_q A_q Q_q^T$ are mutually rigid. Therefore, $Q_{i_1} A_{i_1} Q_{i_1}^T, Q_{i_2} A_{i_2} Q_{i_2}^T, \ldots, Q_{i_m} A_{i_m} Q_{i_m}^T$ is a connected component in $\hat{G}$. In a similar manner if $Q_{i_1} A_{i_1} Q_{i_1}^T, Q_{i_2} A_{i_2} Q_{i_2}^T, \ldots, Q_{i_m} A_{i_m} Q_{i_m}^T$ form a connected component in $\hat{G}$, by Lemma 8.7, $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ is a connected component in $G$.

There are two additional properties that need to be verified in order to show that $\mathcal{F}(A)$ is a fretsaw extension. By definition, the construction ensures that there is at least one $Q_{i_1}$ for every connected component which is an $(n+l)$-by-$n$ identity matrix. The second property that we need to show is that $\mathcal{N}_A \subseteq \mathcal{N}_{\mathcal{F}(A)}$. Let $j \in \mathcal{N}_A$. Let $A_p$ be a matrix in the connected component of $G^{(j)}$ that was processed first. By definition, column $j$ of $Q_p$ contains $e_j$. Therefore, $j \in \mathcal{N}_{Q_p A_p Q_p^T}$. Lemma 2.6, ensures that $j \in \mathcal{N}_{\mathcal{F}(A)}$. □

8.4. **Perfect Elimination Orderings for Spanning-Tree Fretsaw Extensions.** The spanning-tree fretsaw construction is motivated by an elimination ordering that guarantees that all the fill occurs within the nonzero structure of the element matrices. If $[A_e]_{i,j} \neq 0$ for all $i, j \in \mathcal{N}_A$, this ordering is a no-fill ordering of $\mathcal{F}(A)$.

**Lemma 8.13.** *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ be a collection of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices. Let $\mathcal{F}(A)$ be a spanning-tree fretsaw extension of $A = \sum_{i=1}^{k} A_i$. There is a permutation matrix $\Psi$ such that all the* *nonzeros $L_{i,j}$ of the Cholesky factor $L$ of $\Psi\mathcal{F}(A)\Psi^T$ satisfy $i,j \in \mathcal{N}_{\Psi Q_e A_e Q_e^T \Psi^T}$ for some $e$ ($e$ depends on $i$ and $j$).*

*Proof.* We root the spanning tree $T$ of the rigidity graph at $A_1$ and take $\phi$ to be a postorder of this rooted tree. That is, $\phi$ is an ordering of the element matrices in which the leaves of the rooted tree appear first, followed by parents of leaves, etc. We construct an elimination ordering $\psi$ incrementally. Initially, $\psi = \langle\rangle$ is an empty ordering. Let $A_e$ be the next unprocessed element matrix in $\phi$, and let $A_f$ be the parent of $A_e$ in the rooted tree (if it has a parent). Let $\{i_1, \ldots, i_m\}$ be the indices in

$$\mathcal{N}_{Q_e A_e Q_e^T} \setminus \left(\mathcal{N}_{Q_f A_f Q_f^T} \bigcup \psi\right)$$

(with $\psi$ taken to be a set of already ordered indices). We concatenate $\langle i_1, \ldots, i_m\rangle$ to $\psi$ in an arbitrary order. The permutation matrix $\Psi$ is the matrix that corresponds to $\psi$. That is, $\Psi_{i,\psi_i} = 1$.

Now that we have specified $\Psi$, we show that it limits fill as claimed.

*Claim A.* Let $i \in \{1, \ldots, n, n+1, \ldots, n+\ell\}$. Let $j$ be the column in $P^{(s)}$ such that $P^{(s)}_{i,j} \neq 0$ (recall that every row in $P^{(s)}$ contains exactly one nonzero). We denote by $G^{(j)}_i$ the connected component of $G^{(j)}$ in which $j$ is mapped to $i$. The graph $G^{(j)}_i$ is a connected subgraph of $G^{(j)}$ which is an induced subgraph of a rooted tree. Therefore, $G^{(j)}_i$ is itself a rooted tree. We claim that $i$ is added to $\psi$ during the processing of the root $A_h$ of $G^{(j)}_i$.

*Proof of Claim A.* We first show that if $i$ is added to $\psi$, then it is added during the processing of the root of $G^{(j)}_i$.Let $A_e$ be the element during the processing of which we add $i$ to $\psi$. We first show that $A_e \in G^{(j)}_i$. Clearly, $i \in \mathcal{N}_{Q_e A_e Q_e^T}$. Therefore, $j \in \mathcal{N}_{A_e}$. By the definition of $G^{(j)}$, $A_e \in G^{(j)}$, and by the definition of $G^{(j)}_i$, $A_e \in G^{(j)}_i$. Now suppose for contradiction that $A_e$ is not the root of $G^{(j)}_i$. Then $A_e$ has a parent $A_f$ in $G^{(j)}_i$. Because $A_f$ is in $G^{(j)}_i$, $i \in \mathcal{N}_{Q_f A_f Q_f^T}$, so the algorithm would not have added $i$ to $\psi$ during the processing of $A_e$. Therefore, $A_e$ *is* the root of $G^{(j)}_i$.

To complete the proof of Claim A, we show that $i$ is added to $\psi$. Suppose for contradiction that it is not. When we process the root $A_h$ of $G^{(j)}_i$, $i \notin \psi$. But $i$ cannot be in $\mathcal{N}_{Q_f A_f Q_f^T}$, where $A_f$ is the parent of $A_h$ in the global rooted tree. If it was, then $j \in \mathcal{N}_{A_f}$, so $A_f$ would be in $G^{(j)}$, and because it is connected to $A_h$, it must also be in $G^{(j)}_i$. But $A_h$ is the root of $G^{(j)}_i$, so $i \notin \mathcal{N}_{Q_f A_f Q_f^T}$. Therefore, $i$ is added to $\psi$. This concludes the proof of Claim A.

*Claim B.* Just before $A_f$ is processed, $\psi$ is exactly the set

$$\begin{aligned}\psi \;=\; & \left\{i : i \in \mathcal{N}_{Q_e A_e Q_e^T} \text{ for some } A_e \text{ that appears before } A_f \text{ in } \phi\right\} \\ & \setminus \left(\mathcal{N}_{Q_f A_f Q_f^T} \cup \left\{i : i \in \mathcal{N}_{Q_g A_g Q_g^T} \text{ for some } A_g \text{ that appears after } A_f \text{ in } \phi\right\}\right) .\end{aligned}$$

*Proof of Claim B.* The claim follows by induction from the process of constructing $\psi$ and from the fact that $\phi$ is a postorder of the rooted tree.

*Claim* C. If $L_{\hat{r},\hat{c}} \neq 0$ then $\hat{r}, \hat{c} \in \mathcal{N}_{\Psi Q_e A_e Q_e^T \Psi^T}$ for some $e$.

*Proof of Claim* C. If $L_{\hat{r},\hat{c}} \neq 0$, then either $\left[\Psi\mathcal{F}(A)\Psi^T\right]_{\hat{r},\hat{c}} \neq 0$ or there is some $\hat{\imath} < \hat{r}, \hat{c}$ such that $L_{\hat{r},\hat{\imath}} \neq 0$ and $L_{\hat{c},\hat{\imath}} \neq 0$. The first condition cannot violate Claim B, because if $\left[\Psi\mathcal{F}(A)\Psi^T\right]_{\hat{r},\hat{c}} \neq 0$ then there is some $e$ such that $\left[\Psi Q_e A_e Q_e^T \Psi^T\right]_{\hat{r},\hat{c}} \neq 0$, so $\hat{r}, \hat{c} \in \mathcal{N}_{\Psi Q_e A_e Q_e^T \Psi^T}$.

If for some $\hat{r}$ and $\hat{c}$ we have $L_{\hat{r},\hat{c}} \neq 0$ because of the second condition, then let $\hat{\imath}$ be the minimal index such that $L_{\hat{r},\hat{\imath}} \neq 0$ and $L_{\hat{r},\hat{\imath}} \neq 0$ *for all such* $(\hat{r},\hat{c})$ *pairs.* This definition of $\hat{\imath}$ guarantees that $\left[\Psi\mathcal{F}(A)\Psi^T\right]_{\hat{r},\hat{\imath}} \neq 0$ and $\left[\Psi\mathcal{F}(A)\Psi^T\right]_{\hat{c},\hat{\imath}} \neq 0$. Define $i = \psi_{\hat{\imath}}$, $r = \psi_{\hat{r}}$ and $c = \psi_{\hat{c}}$.

Let $A_f$ be the element during the processing of which $i$ was added to $\psi$. Because $\hat{\imath} < \hat{r}, \hat{c}$, when $i$ is added to $\psi$, $r$ and $c$ are not yet in $\psi$. We claim that $r$ and $c$ are in $\mathcal{N}_{Q_f A_f Q_f^T}$; if true, Claim C holds. Suppose for contradiction that $r \notin \mathcal{N}_{Q_f A_f Q_f^T}$. By Claim B, $r \in \mathcal{N}_{A_g}$ for some $A_g$ that appears after $A_f$ in $\phi$. Because $\left[\Psi\mathcal{F}(A)\Psi^T\right]_{\hat{r},\hat{\imath}} \neq 0$ and because $A_g$ is symmetric, we must also have $i \in \mathcal{N}_{A_g}$. But this implies that $i$ cannot be added to $\psi$ when $A_f$ is processed (by Claim B). This concludes the proof of Claim C and of the entire proof. □

8.5. **Quantitative analysis.** Lemma 8.11 showed that if the rigidity graph of a finite-element matrix $A = \sum A_e$ is connected and if $\mathrm{schur}(\mathcal{F}(A))$ exists for a fretsaw extension $\mathcal{F}(A)$, then $A$ and $\mathrm{schur}(\mathcal{F}(A))$ have the same range and null space. We now strengthen this result and show that the generalized eigenvalues of $(\mathrm{schur}(\mathcal{F}(A)), A)$ are bounded from above by 1. We note that $\mathrm{schur}(\mathcal{F}(A))$ can be implicitly used as a preconditioner; in the preconditioning step of an iterative linear solver, we can solve a linear system whose coefficient matrix is $\mathcal{F}(A)$, not $\mathrm{schur}(\mathcal{F}(A))$ [2, 4]. In particular, the previous section showed that we can factor a spanning-tree fretsaw-extension $\mathcal{F}(A)$ with essentially no fill.

**Lemma 8.14.** *Let $A_1, A_2, \ldots, A_k$ be a collection of $\mathbb{N}$-compatible symmetric positive semidefinite $n$-by-$n$ matrices. Let $A = \sum_{i=1}^k A_i$. Let $\mathcal{F}(A)$ be a fretsaw extension of $A$. Then if* $\mathrm{schur}(\mathcal{F}(A))$ *exists, and $\lambda$ is a finite generalized eigenvalue of the pencil* $(\mathrm{schur}(\mathcal{F}(A)), A)$*, then $\lambda \leq 1$.*

*Proof.* We partition $\mathcal{F}(A)$ into

$$\mathcal{F}(A) = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} ,$$

where $B_{11}$ is $n$-by-$n$ and $B_{22}$ is $\ell$-by-$\ell$. By the assumption that $\mathrm{schur}(\mathcal{F}(A))$ exists, $B_{22}$ is symmetric positive definite. In this notation, $\mathrm{schur}(\mathcal{F}(A)) = B_{11} - B_{12}B_{22}^{-1}B_{12}^T$. Let $P$ be the $(n+\ell)$-by-$n$ master extension matrix corresponding to the fretsaw extension $\mathcal{F}(A)$ and let $Q$ be the $(n+\ell)$-by-$n$ identity matrix.

Let $\lambda_{\max}$ be the maximal finite generalized eigenvalue of the pencil $(\mathrm{schur}(\mathcal{F}(A)), A)$ and let $x$ be the corresponding eigenvector. We let

$$\hat{x} = \begin{bmatrix} x \\ -B_{22}^{-1}B_{21}x \end{bmatrix}$$

and multiply it by $\mathcal{F}(A)$,

$$\begin{aligned} \mathcal{F}(A)\hat{x} &= \begin{bmatrix} B_{11}x + B_{12}(-B_{22}^{-1}B_{21}x) \\ B_{21}x + B_{22}(-B_{22}^{-1}B_{21}x) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{schur}(\mathcal{F}(A))x \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_{\max}Ax \\ 0 \end{bmatrix} \\ &= \lambda_{\max}QAQ^T\hat{x} \, . \end{aligned}$$

Multiplying both sides by $\hat{x}^T$, we obtain $\hat{x}^T\mathcal{F}(A)\hat{x} = \lambda_{\max}\hat{x}^TQAQ^T\hat{x}$.

We now show that $\hat{x}^T\mathcal{F}(A)\hat{x} \le \hat{x}^T QAQ^T\hat{x}$. For a length-$\ell$ vector $y$, define the function $f(y) = \begin{bmatrix} x^T & y^T \end{bmatrix} \mathcal{F}(A) \begin{bmatrix} x \\ y \end{bmatrix}$. We note that $f(-B_{22}^{-1}B_{21}x) = \hat{x}^T\mathcal{F}(A)\hat{x}$. For an arbitrary $y$,

$$\begin{aligned}
\begin{bmatrix} x^T & y^T \end{bmatrix} \mathcal{F}(A) \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x^T & y^T \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= x^T B_{11} x + y^T B_{12}^T x + x^T B_{12} y + y^T B_{22} y \\
&= x^T \left(B_{11} - B_{12}B_{22}^{-1}B_{12}^T\right) x + x^T B_{12}B_{22}^{-1}B_{12}^T x \\
&\quad + y^T B_{12}^T x + x^T B_{12} y + y^T B_{22} y \\
&= x^T \left(B_{11} - B_{12}B_{22}^{-1}B_{12}^T\right) x \\
&\quad + \left(y + B_{22}^{-1}B_{12}^T x\right)^T B_{22} \left(y + B_{22}^{-1}B_{12}^T x\right) .
\end{aligned}$$

Because $B_{22}$ is positive definite, $f(y)$ is minimized at $y = -B_{22}^{-1}B_{12}^T x$.

By Lemma 8.9, $\hat{x}^T QAQ^T\hat{x} = \hat{x}^T Q(P^T\mathcal{F}(A)P)Q^T\hat{x} = x^T P^T\mathcal{F}(A)Px$. By the definition of a master extension matrix, the vector $Px$ contains the vector $x$ in its first $n$ coordinates, so $Px = \begin{bmatrix} x^T & z^T \end{bmatrix}$ for some $z$ and $x^T P^T\mathcal{F}(A)Px = f(z)$. Since $-B_{22}^{-1}B_{12}^T x$ minimizes $f$,

$$\hat{x}^T QAQ^T\hat{x} = x^T P^T\mathcal{F}(A)Px = f(z) \ge f(-B_{22}^{-1}B_{12}^T x) = \hat{x}^T\mathcal{F}(A)\hat{x}$$

This implies that $\lambda_{\max} \le 1$ and concludes the proof of the lemma. □

## 9. Numerical Examples

In this section we present experimental results that indicate that fretsaw-tree sparsifications can be used as preconditioners. We do not claim that they are particularly effective. Our only goal in this section is to demonstrate that fretsaw-tree sparsifications can be used computationally as preconditioners. The results presented in this section also suggest that the qualitative convergence behavior of fretsaw-extension preconditioners is similar to that of Vaidya's precondioners when applied to weighted Laplacians [7].

Figure 9.1 shows convergence results for an iterative solver (preconditioned conjugate gradients) with a fretsaw-tree preconditioner. The figure shows results for two different physical two-dimensional problems that we discretized on the same triangulated mesh. One problem was a Poisson problem and the other a linear-elasticity problem, both with constant coefficients and with Neumann (natural) boundary conditions. In each case, we constrained one or three unknowns belonging to a single triangle to transform the coefficient matrix into a non-singular one.

Each graph shows convergence results for three conjugate-gradients solvers: with no preconditioning, with no-fill incomplete-Cholesky preconditioning (denoted *cholinc(0)* in the graphs), and with fretsaw-tree preconditioning. The fretsaw trees for the two problems are different, of course, because the rigidity graphs are different. We chose to compare the fretsaw-tree preconditioner with a no-fill incomplete-Cholesky preconditioner because both are equally sparse.

The results show that fretsaw trees can be used as preconditioners. The experiments are too limited to fully judge them, but the experiments do indicate that they are not worse than another no-fill preconditioner. Two other observations on the graphs are (1) the fretsaw is better than incomplete Cholesky on the Poisson problem, but the two are comparable on the linear-elasticity problem,

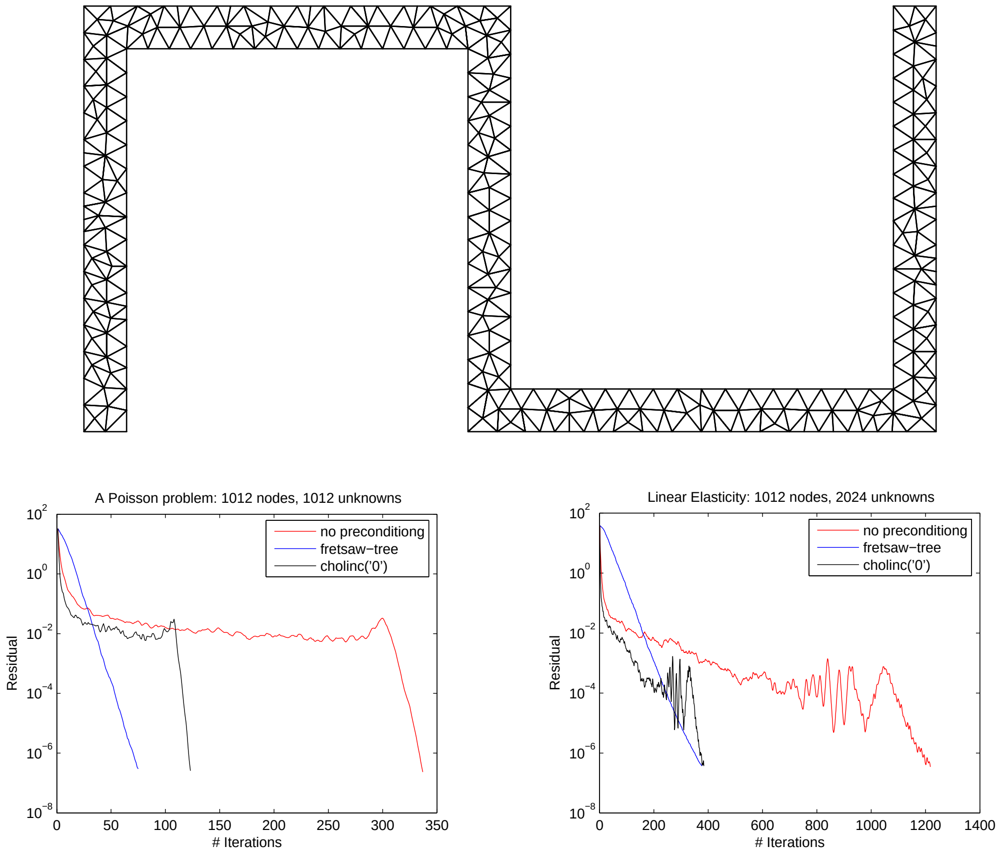


FIGURE 9.1. A triangularization of a two-dimensional domain (top) and convergence plots for two problems discretized on this domain. The triangularization used in the plots is finer than the one shown in the top part of the figure. The graph on the left shows the convergence of iterative solvers on a discretization of a Poisson problem, and the graph on the right shows convergence on a linear-elasticity problem, both with constant coefficients.

and (2) the steady linear convergence behavior of the fretsaw trees is similar to the convergence behavior of Vaidya's preconditioners on weighted Laplacians [7].

## 10. Concluding Remarks

To keep the paper readable and of reasonable length, we have omitted from it several topics, which we plan to cover in other papers.

- Element matrices that represent boundary conditions. In much of this paper, we have assumed that all the element matrices are compatible with $\mathbb{N}$. This means, in particular, that the element matrix is singular. In many practical computations, boundary conditions are added to remove the singularity. We kept the discussion focused on singular matrices to reduce clutter. We plan to explore the handling of boundary conditions in a future paper.
- Fretsaw constructions other than spanning-tree fretsaw extensions. Previous work on combinatorial preconditioners indicates, both theoretically and experimentally, that tree and tree-like preconditioners are not effective; augmented trees and other constructions usually work better. We

have developed augmented-spanning-tree fretsaw extension algorithms for Laplacians, but this construction is beyond the scope of this paper.

In addition, there are several interesting problems that we have not yet solved. The most interesting one is proving lower bounds on the generalized eigenvalues of $(\mathrm{schur}(\mathcal{F}(A)), A)$ and finding fretsaw constructions that ensure that this bound is not too small. A particularly interesting question is whether this can be done by assigning weights to the edges of the rigidity graph.

Another question is what other results from spectral graph theory can be generalized to finite-element matrices as defined in this paper, and whether the rigidity graph, perhaps weighted, would be useful in such generalizations.

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