Constructing Phylogenies from Quartets: Elucidation of Eutherian Superordinal Relationships*

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Abstract

In this work we present two new approaches for constructing phylogenetic trees. The input is a list of weighted quartets over n taxa. Each quartet is a subtree on four taxa, and its weight represents a confidence level for the specific topology. The goal is to construct a binary tree with n leaves such that the total weight of the satisfied quartets is maximized (an NP hard problem).

The first approach we present is based on geometric ideas. Using semidefinite programming, we embed the n points on the n-dimensional unit sphere, while maximizing an objective function. This function depends on Euclidean distances between the four points, and reflects the quartet topology. Given the embedding, we construct a binary tree by performing *geometric clustering*. This process is similar to the traditional neighbor joining, with the difference that the update phase retains geometric meaning: When two neighbors are joined together, their common ancestor is taken to be the center of mass of the original points.

The geometric algorithm runs in $poly(n)$ time, but there are no guarantees on the quality of its output. In contrast, our second algorithm is based on dynamic programming, and it is guaranteed to find the optimal tree (with respect to the given quartets). Its running time is a modest exponential, so it can be implemented for modest values of n.

We have implemented both algorithms, and ran them on real data for $n = 15$ taxa (14 mammalian orders and an outgroup taxon). The two resulting trees improve previously published trees and seem to be of biological relevance. On this dataset, the geometric algorithm produced a tree whose score is 98.2% of the optimal value on this input set (72.1% vs. 73.4%). This gives rise to the hope that the geometric approach will prove viable even for larger cases where the exponential, dynamic programming approach is no longer feasible.

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1 Introduction

Molecular data are the basis for current phylogenetic analysis of evolution. Since molecular data are not available evenly for all taxa of interest, molecular phylogeneticists must frequently decide on a trade off between the number of taxa and the amount of molecular data used in a study. If we restrict ourselves only to sequences that are common to all the taxa under study, we may end up ignoring the vast majority of data. If, on the other hand, we increase the amount of molecular data used in a study, we end up seriously restricting taxonomic sampling (i.e. study fewer taxa). An approach that tries to utilize all available data while avoiding the problem of taxonomic sampling is the *four taxon approach* suggested in [Gra93] and used in [GDG96, GDG96]. In this method, each 4-taxon tree (quartet) is accompanied by a measure of reliability or confidence (e.g., bootstrap value, likelihood) associated with the internal branch of the 4-taxon tree. This confidence value reflects two factors: how solid is the plurality of sequences supporting the conclusion (i.e., the strength of the phylogenetic signal), and what is the size of the sequence population used to build the tree. For each group of 4 taxa, there are three possible unrooted phylogenetic trees. In this type of studies, only the most supported topology is chosen.

Combinatorially, the justification for the quartet based approach is that if all $\binom{n}{4}$ quartets are given with the right topology, then the underlying tree is uniquely determined, and can be efficiently constructed. Of course, in reality there are errors, so a tree consistent with all given quartets may not exist[1]. So the goal is to construct a binary tree with n leaves such that the total weight of the satisfied quartets is maximized (an NP hard problem). Our contribution in this paper is in suggesting new algorithmic approaches to solve this optimization problem.

Our first algorithm has the following geometric intuition: We embed the n taxa as n points in $\mathcal{R}^n$, while trying to preserve distance relations that are implied by the input of four-taxon trees. To find such embedding[2], we employ semidefinite programming. Specifically, the points are embedded on the n-dimensional unit sphere. To construct a tree from the embedding, we employ an iterative technique that we call *geometric clustering*. The major difference between this technique and "traditional" neighbor joining [SN87] is in the phase where the metric is updated. At each iteration, we join the pair of points with minimum Euclidean distance, and take their center of mass as the common ancestor. We have implemented this algorithm on input of 15 taxa (14 mammalian orders and an outgroup taxon) that was derived from the HOVERGEN database [DMG94]. The resulting tree satisfies 72.1% of the weighted quartets. This is an improvement over all previously published trees (see Figure 1).

Even for a modest size like $n = 15$, the number of trees is too large to enable an exhaustive search for the best one. We have developed an exponential time,

[1]Indeed, determining if a given set of quartets is satisfiable is an NP complete problem [Ste92].

[2]A related notion of low distortion Euclidean embeddings for weighted trees has been studied in [Bou85, LMS98].

dynamic-programming algorithm, which finds an optimal tree. Its running time is $O(k \cdot 3^n)$, where k is the number of quartets in the input, $k \leq \binom{n}{4}$. This is a fairly modest exponential growth, and it enabled us to implement and run the algorithm on the same 15 taxa input set. The optimal tree satisfies 73.4% of the weighted quartets. It is interesting that the geometric algorithm, which is essentially a heuristic with no performance guarantees, came so close to the optimal result.

The remainder of this paper is organized as following: In Section 2 we survey some of the past research done in the area of phylogenetic reconstruction. In Section 3 we specify the problem, and also describe how the actual input was obtained. Section 4 explains the geometric algorithm, while Section 5 explains the dynamic programming algorithm. Section 6 describes the computational results, depicts the two output trees, and compares them to previous trees. Finally, Section 7 contains some concluding remarks.

2 Prior Work

Phylogeny reconstruction is an old problem; it has been investigated for over 150 years. Consequently, the related literature is vast. The algorithmic challenges that follow the fast developments in the bimolecular field have recently drawn the attention of computer scientists as well. In this chapter we merely quote the major computer-theoretic research issues related to phylogeny reconstruction, and give selected references.

In general, phylogeny methods are divided into *character-based* and *distance-based* methods. We will consider each in turn. An orthogonal classification identifies the *quartet method* as one possible algorithmic approach. Being in the general framework of this work, the existing quartet methods will also be listed. We also browse through the different techniques to evaluate the quality of a phylogeny. Another important aspect of this work is the usage of the *semidefinite programming* tool. We bring some related pointers in this regard as well.

2.1 Character-Based methods

A character-based method considers qualitative characters of the input taxa. Any such character is a partition of the input set according to the value each taxon takes. Each equivalence class defined thus is called a *character state*. For example, a DNA sequence is composed of the nucleotides A, C, T and G. Sequences of different species are aligned in a *multiple sequence alignment* process which inserts gaps into the sequences so as to maximize the resemblance of the sequences to each other when laid out side by side. Thus, each position of the aligned sequence (called *site*) is a character with five states (A, C, T, G, or the gap symbol -). Input for such a method will typically be a $|S| \times |C|$ matrix, where S is the taxa set and C is the character set. Each entry denotes a state which a particular taxon exhibits for a given character.

Given sequences for each taxon, a natural approach is to build a tree with the internal nodes, as well as the external ones, labeled by the sequences; the labels at the leaves are given as input, and the internal labels are computed. The optimization criterion is the sum of Hamming distances between neighboring sequences. Methods that try to minimize it are called *maximum parsimony* methods. The general maximum parsimony problem is NP-hard [Day83, FG82, Gus84], but finding the internal labeling for a given topology can be done efficiently [Fit78, Har73]. This leads to algorithms which scan many different topologies and output the most parsimonious one.

Another approach tries to construct a tree such that for each character, the node-sets that correspond to any character state form a connected sub-graph. Maximizing the number of characters for which this is true is called the *maximum compatibility* problem. This problem, too, is NP-hard [DS86, PW96, War]. The point of intersection between maximum parsimony and maximum compatibility is called the *perfect phylogeny* problem. It decides whether all of the characters are compatible. Perfect phylogeny was shown to be NP-hard by equivalence to the *triangulating colored graphs* problem [BFW92, Ste92]. Although hard in the general case, polynomial-time algorithms for perfect phylogeny exist in cases where some of the input parameters (e.g., the number of states) are fixed [War].

2.2 Distance-Based methods

A distance based phylogeny method would typically take as input a symmetric zero-diagonal $|S| \times |S|$ matrix. Here, too, the goal is to produce a phylogeny whose induced metric represents the input data in the best way possible. If the input distance matrix M is realizable by a tree and its induced path lengths, then M is said to be *additive*. The special case where all leaves have the same distance from the root is called an *ultrametric*. Ultrametric trees correspond to the biological theory that substitutional events in different species occur at the same rate (this is the "molecular clock" assumption, nowadays popularly discredited).

Given an additive metric, constructing the tree is easy [WSSB77]. The problem is that real-life input is erroneous. Some of the errors are inherent in the assumed model of evolution. Therefore, we seek phylogenies with induced metric which approximates the "best possible" tree metric under some criterion. Again, most of the problems in this domain are NP-hard.

One popular heuristic approach is *agglomerative clustering*. Here, at each iteration two nodes are joined into one father node. Examples include the popular *Neighbor Joining* method [FM67], the *Fitch-Margoliash* method [FM67], and BIONJ [Gas97].

As it turns out, approximating ultrametrics under the L_∞ criterion is doable in polynomial time [GR69, SS63, FKW95, War]. Consequently, there are algorithms that use an approximated ultrametric to produce an approximation of the additive metric [ABF+96, CF97].

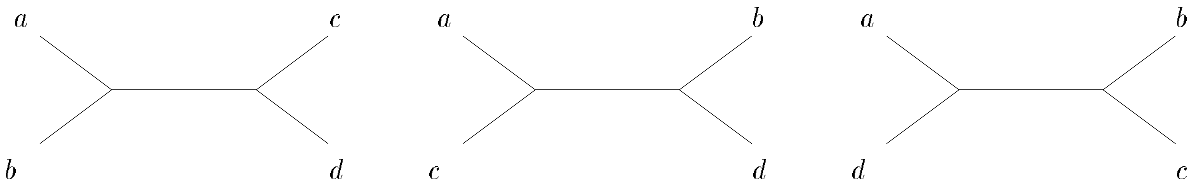


Figure 1: The three possible quartets for the species set $\{a, b, c, d\}$

2.3 Quartet Methods

The fact that small phylogenies are easier to infer than large ones leads to another approach. The key idea is to consider small subsets of taxa, one at a time, and infer the phylogenies for these subsets. All methods use subsets of size four. The next stage combines the multiple topologies (called *quartets*) into a single phylogeny. Given a complete set of correct quartets, the true tree can be constructed in polynomial time. However, doing so when the individual topologies may contain errors is NP-hard [Ste92]. Published quartet methods include the so-called "naive" method, [War, p. 25], the Buneman tree [Bun71, BG98], split decomposition [War], the short quartet method [ESSW97], neighbor-joining variants [Gra96, GDG96, GGD97], and quartet puzzling [SH96]. See also [BD86, Fit81]. In general, these heuristics deal with unweighted quartets (i.e., each quartet has the same significance). Some also require as input the full set of $\binom{n}{4}$ quartets. In principle, the quartet puzzling algorithm can be extended to the unweighted general case, but no experiments in this direction have been done.

3 Problem Description

3.1 Specifications

The problem is defined over a set of n taxa, numbered $1, \ldots, n$. A *quartet* is a quadruple of taxa, with an associated topology — a partition of the four taxa into two pairs of taxa (*e.g.* $\{a, b\}$ and $\{c, d\}$). This subdivision expresses the most likely topology induced by the underlying n taxa phylogeny. Such a quartet is denoted $ab|cd$. Figure 1 shows the three possible quartets over a given set of 4 species. The input to the problem consists of a set of k such quartets. We denote the associated taxa for the j-th quartet by $a_j b_j | c_j d_j$. No two quartets share the same set of four taxa. Each input quartet is accompanied by a positive weight, denoted by C_j, which represents the confidence in the quartet topology.

The output is an unrooted tree with n leaves which correspond to the input taxa. Usually, one of the input taxa is an *outgroup*, namely an external taxon, not belonging to the same family. Using the outgroup taxon, the tree can be (uniquely) rooted such that the outgroup is an immediate descendent of the root. Once the tree is rooted, the branching of its nodes is viewed as a description of the evolution of the n taxa. If all internal nodes of a tree have out-degree 2, the tree is called *bifurcating*. Otherwise, the tree is said to be *multifurcating*.

Given a tree and a quadruple $\{a, b, c, d\}$, we can compute the quartet topology induced by T, using the following procedure. First, all leaves but a, b, c and d are deleted from the tree. Edges adjacent to these leaves are also removed. Next, internal nodes with degree two are contracted and deleted, so their two adjacent nodes become connected. This process is repeated until no internal nodes of degree two are left. It is easy to see that there are four possible induced topologies for the quadruple — three quartets and the star topology (which can be induced only by a multifurcating tree). Given a specific quartet and the induced four taxa subtree, we say that the quartet is *unresolved* if the tree induces the star topology on the four taxa. Otherwise, the quartet is either *satisfied* (if the topology induced by the tree equals the quartet's topology), or *violated*.

Given a tree T and a set of quarters Q, we would like to know how well does T represent Q. To do this, we find the subset of quartets $S \subset Q$ that are satisfied by T, and the subset of quartets $U \subset Q$ that are unresolved by T. We now define the *score* of the tree as follows:

$$\sum_{s \in S} C_s + \frac{1}{3} \sum_{u \in U} C_u \ .$$

That is, we add the confidence weights of the satisfied quartets, plus one third of the weights of the unresolved quartets. This latter term was chosen because there are three possible pairings for every quadruple. Therefore this term equals the expected increase to the tree score that will result from a random bifurcation of the tree (performed at nodes with more than two descendents). In a variant of our method this factor is zeroed, so unresolved quartets do not contribute to the score. We remark that in the history of phylogeny many multifurcating trees were published. However, modern algorithms, including ours, always produce a bifurcating tree.

An upper bound on the score of any tree is $\sum_{q \in Q} C_q$. This upper bound can be achieved only if there exists a tree that satisfies all quartets. We usually state the score of a tree as a fraction of the upper bound, although it is possible that no tree of 100% score exists.

We are now ready to formulate the problem precisely. Given a set of quartets Q and associated confidence scores, find a tree T with maximum score[3]. This problem is NP–hard even if all the confidence weights have values 0 or 1 [Ste92]. Exhaustive search methods seem infeasible even for modest values of n (say $n = 15$), because the number of unrooted bifurcating trees [Fel78] with n leaves is

$$\frac{(2n-5)!}{2^{n-3}(n-3)!} \ .$$

For $n = 15$ the number of such trees is just below 8×10^{12} .

[3]One may consider a similar problem where for every 4-tuple of taxa, all three quartet topologies are given in the input (with different weights). Small modifications of both the geometric and the exact algorithm will solve this variant as well.

3.2 Actual Input

We briefly describe how the four-taxon trees, which constitute the actual input to our algorithms, were obtained. The following 14 mammalian taxa were used: CAR = Carnivora (carnivores), CET = Cetartiodactyla (even-hoofed ungulates and cetaceans), CHI = Chiroptera (bats), EDE = Edentata (sloths and armadillos), HYR = Hyracoidea (hyraxes), HYS = Hystricognathi (guinea pigs and porcupines), INS = Insectivora (hedgehogs and shrews), LAG = Lagomorpha (rabbits and hares), PER = Perissodactyla (horses), PRI = Primates (humans, apes and monkeys), PRO = Proboscidea (elephants), SCA = Scadentia (tree shrews), SCI = Sciurognathi (rats, mice and squirrels), and SIR = Sirenia (sea cows and dugongs). As an outgroup (OUT) to the above eutherian mammalian taxa we used chicken or marsupial.

These 15 taxa were subject to the following process: Protein sequences have been collected from the HOVERGEN [DMG94] database. The sequences were aligned using the CLUSTAL W program [THG94]. Genetic distances were computed as in [Kim83]. Four-taxon phylogenetic trees were inferred by using maximum parsimony [Fel89]. Reliability of the internal branch in each 4-taxon tree was assessed by 1000 bootstrap replicates [Fel85]. This list of $\binom{15}{4} = 1365$ quartet topologies and associated weights was the input to our algorithms.

4 The Geometric Algorithm

Our geometric algorithm assigns each taxa a point in $\mathcal{R}^n$. This is done by applying the semidefinite programming method, to be described shortly. The embedding is built using "hints" from the input set. For example, if the quartet $ab|cd$ is in the list, we try to place a and b close to each other, but a and d far apart. Once the points are embedded in $\mathcal{R}^n$, a clustering heuristic converts the embedding into a tree.

4.1 Semidefinite Programming

We briefly explain the paradigm of SDP — semidefinite programming (see [GLS87] for a thorough explanation). We describe how SDP can be augmented by the incomplete Cholesky decomposition method to find embeddings of points in $\mathcal{R}^n$, subject to certain constraints.

Definition 1: *For positive integers m and n, a semidefinite program is defined over a collection of n^2 real variables $\{x_{ij}\}_{i=1,j=1}^{n,n}$. The input consists of a set of mn^2 real numbers $\{a_{ij}^{(k)}\}_{i=1,j=1,k=1}^{n,n,m}$, a vector of m real numbers $\{b^{(k)}\}_{k=1}^{m}$ and a vector of n^2 real numbers $\{c_{ij}\}_{i=1,j=1}^{n,n}$. The objective is to find $\{x_{ij}\}_{i=1,j=1}^{n,n}$ so as to*

$$\begin{array}{rl} \textit{maximize} & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \textit{subject to} & \\ \forall k \in \{1,\ldots,m\} & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}x_{ij} \le b^{(k)}. \\ \textit{and} & \textit{The matrix } X = \{x_{ij}\} \textit{ is symmetric} \end{array}$$

and positive semidefinite.

A symmetric matrix X is positive semidefinite when there exists a real matrix V such that $V^T \cdot V = X$. Given a positive semidefinite matrix X, such V can be found in polynomial time using incomplete Cholesky decomposition. The semidefinite program itself is solvable in time polynomial in the size of its input [GLS87].

The matrix V can be used to interpret the solution obtained by the semidefinite programming problem geometrically. Interpret the columns of the $n \times n$ matrix V as n vectors $v_1, \ldots, v_n$ in $\mathcal{R}^n$. Now, the variables x_{ij} of the matrix X simply correspond to $\langle v_i, v_i \rangle$, the inner product of v_i and v_j. Thus a linear constraint on the x_{ij}'s is simply a linear constraint on the inner products of the vectors v_i's. The objective function and the constraints are simply linear combinations of the inner products. Hence we can use SDP to solve any problem of the form:

> Find n vectors $v_1, \ldots, v_n \in \mathcal{R}^n$ so as to maximize the quantity $\sum_{i,j} c_{ij} \langle v_i, v_j \rangle$, subject to the constraints $\sum_{i,j} a_{ij}^{(k)} \langle v_i, v_j \rangle \leq b^{(k)}$.

In what follows we will use the last interpretation of SDP to find an embedding of n points which represent the n taxa.

4.2 Geometric embedding of taxa

We will generally denote the embedding of taxon i by the point v_i in $\mathcal{R}^n$. Given a list of k quartets $a_j b_j | c_j d_j$ and confidence values C_j ($j = 1, \ldots, k$), we solve the following semidefinite program:

$$\begin{array}{ll} \text{maximize:} & \\ & \sum_{1 \leq j \leq k} C_j(\langle a_j, b_j \rangle + \langle c_j, d_j \rangle) \\ -0.5 & \sum_{1 \leq j \leq k} C_j(\langle a_j, c_j \rangle + \langle a_j, d_j \rangle + \langle b_j, c_j \rangle + \langle b_j, d_j \rangle) \end{array}$$

such that

$$\langle v_i, v_i \rangle \quad = \quad 1 \qquad (1 \leq i \leq n)$$

This formulation embodies many local constraints, such as "taxa a and b should be close together", or "taxa a and d should be far apart", into a single expression. This approach is similar to the one used in [GW95] to approximate MAX CUT. Consider the quartet $ab|cd$ with confidence level C. Its maximum contribution to the objective function occurs when a and b are placed at the same point, while both c and d are placed at the antipodean point on the unit sphere. In this case $\langle a, b \rangle = \langle c, d \rangle = 1$, while $\langle a, c \rangle = \langle a, d \rangle = \langle b, c \rangle = \langle b, d \rangle = -1$. The overall contribution of the quartet will be $4C$ with this embedding. The worst embedding places a and c together, and b and d at the antipodean point. This will contribute $-4C$ to the objective function. The semidefinite program will therefore look for a global embedding which maximizes "good" quartet placements and avoids "bad" ones.

We have experimented with this approach, and it turned out that small variants of it are helpful in improving the final result (the produced tree). One such variant is ignoring quartets with low confidence in the objective function. Only quartets with confidence level above a certain threshold (90% for example) are included. One possible explanation why this may prove helpful is that low confidence quartets probably carry most of the inconsistencies, and their inclusion may lower the value of the objective function. We have also discovered that by imposing additional constraints we get (small) improvements in the score of the tree. For example, we force the points to maintain some small pairwise distance by adding the $\binom{n}{2}$ constraints

$$\langle v_i, v_j \rangle \quad \leq \quad 1 - \varepsilon \qquad (1 \leq i < j \leq n)$$

for some positive ε (say $\varepsilon = 0.25$). This "represses" the SDP's tendency to find embeddings where points are very close to each other. This tendency may seem harmless, but it has a negative effect on the tree building method, which we describe next.

4.3 Geometrical Clustering

Having solved the SDP problem, one seeks a tree that reflects the geometric data. For instance, if two points reside in the same geometric vicinity, they should have a common ancestor which is not too much high up the resulting tree. To this end, we employ a simple neighbor joining clustering heuristic (see [SN87]). The program is initialized with n clusters, each containing a single point. An invariant kept by the algorithm is that each cluster has a point in $\mathcal{R}^n$ associated with it. This is the case at initialization, and remains true as new clusters are formed. At each step of the algorithm the number of clusters is decreased by one, by removing two clusters and adding one. This defines a tree, as the newly-added node represents the father of the two deleted nodes in the output tree. The selection of the clusters to be removed is done by calculating the pairwise distances between clusters, and selecting the pair having the shortest Euclidean distance. The point associated with the new cluster is the center of mass of the points of the removed clusters (the "mass" of a point being the number of taxa it represents). When the number of clusters reaches one, the tree is completed. The resulting tree is rooted, but we disregard the rooting, since the input is inherently unrooted. To root the tree we use external information – the identity of the outgroup taxon. We root the tree by forcing the outgroup to be an immediate descendent of the root. We remark that this heuristic is a special case of the general neighbor joining approach. The main difference is that since clusters are associated with points in Euclidean n space, the center of mass of the nearest neighbor pair is a natural choice for the new cluster. This choice again induces Euclidean distances between the new cluster and the "old" ones.

5 The Dynamic Programming Algorithm

Since the underlying problem is NP-hard we can not hope to give a polynomial time algorithm to solve it optimally. However, in this section we show how an optimal phylogenetic tree can be found, using a dynamic programming algorithm, with a modest exponential time. The method is applicable to instances with modest size (say $n \leq 21$). In particular, we have used it to find the optimal phylogenetic tree for the 15 taxa input set. We emphasize that even for 15 taxa, it is not feasible to exhaustively check all quartets for every possible bifurcating tree. (The number of unrooted trees for $n = 15$ is approximately $\approx 8 \cdot 10^{12}$, the number of quartets is approximately $1,300$, so the number of steps would be more than 10^{16}).

5.1 Definitions

The following discussion deals with *rooted* bifurcating trees. For a node v, its left and right children will be denoted by v_ℓ and v_r, respectively. Given a rooted tree T and a node v in it we denote by $T(v)$ the subtree of T rooted at v. We denote by $L(T)$ the set of leaves (*i.e.*, taxa) of the tree T. For a pair of nodes u, v the *least common ancestor* of u and v, $\texttt{lca}(u, v)$ is defined as an ancestor p of both u and v such that no node in $T(p)$ other than p is an ancestor of both u and v (see Figure 2).

Definition 2 *Given a quartet $q = ab|cd$ and a tree T, the* quartet *least common ancestor of q, $\texttt{qlca}(q)$ is defined as a node p that is the $\texttt{lca}$ of two or more pairs of elements from $\{a, b, c, d\}$, and no node in $T(p)$ except p is the $\texttt{lca}$ of two or more pairs of elements from $\{a, b, c, d\}$.*

For the implementation of our algorithm and the proof of its correctness, we will use an equivalent definition.

Definition 3 *Given a quartet $q = ab|cd$ and a tree T, the $\texttt{qlca}$ of q is a node p such that*

1. $|L(T(p)) \cap \{a, b, c, d\}| \geq 3$.
2. *For any child s of p, $|L(T(s)) \cap \{a, b, c, d\}| \leq 2$.*

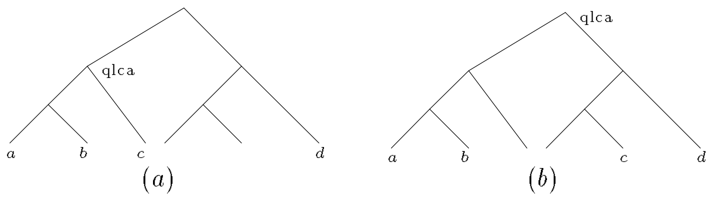


Figure 2: Two possible arrangements of a $\texttt{qlca}$ for quartets over $\{a, b, c, d\}$

Observation 1 *Every quartet q has a unique $\mathtt{qlca}(q)$.*

Proof: Assume, towards a contradiction, that the quartet q has two distinct $\mathtt{qlca}(\cdot)$ in T, p_1 and p_2. By definition 2 the two points cannot have an ancestor-descendant relationship. Therefore, the two must belong to two separate subtrees. Definition 3 then implies that each $T(p_i)$ contains at least 3 different taxa from q, so q must have at least 6 taxa — a contradiction. $\square$

Lemma 2 *Given a tree T and a quartet q, the subtree rooted at $\mathtt{qlca}(q)$ determines whether q is satisfied in the tree T.*

Proof: Let $q = ab|cd$ and $v = \mathtt{qlca}(q)$. We look at v's children v_ℓ, v_r and the subtrees rooted at them $T(v_\ell)$, $T(v_r)$. At least one of these subtrees contains exactly two taxa e, f from $\{a, b, c, d\}$. The quartet q is satisfied by T iff the pair $\{e, f\}$ is either $\{a, b\}$ or $\{c, d\}$. $\square$

Corollary 3 *Given a quartet $q = ab|cd$ and a tree T, let $v = \mathtt{qlca}(q)$. Then T satisfies q if and only if at least one of the following holds:*

1. $\{a, b\} \subseteq L(T(s))$.
2. $\{c, d\} \subseteq L(T(s))$.

Where s is either v's left child ($s = v_\ell$) or v's right child ($s = v_r$).

We remark that the node in T where q's satisfaction is determined can in fact "be pushed" even further down, to every u which is on a path between an $\mathtt{lca}$ v of two taxa from u and $\mathtt{qlca}(q)$ above v. There could be either one such node u, if we are in case (a) of Figure 2, or more nodes in case (b) of Figure 2. We postpone the "recognition event" to $\mathtt{qlca}(q)$ because this choice facilitates the use of dynamic programming.

5.2 The Algorithm

Let Q be a fixed set of input quartets. Let T be a rooted tree, and v a node in T. We denote by $\mathrm{SAT}_Q(T(v))$ the set of quartets $q \in Q$ such that q is satisfied by T, and $\mathtt{qlca}(q)$ is a node in $T(v)$. Let $\mathrm{TOP}_Q(T(v)) \subset \mathrm{SAT}_Q(T(v))$ be the set of quartets in Q that have v as their $\mathtt{qlca}$ and are satisfied by T. The following equality describes a partition of $\mathrm{SAT}_Q(T)$ to three disjoint subsets

$$\mathrm{SAT}_Q(T(v)) = \mathrm{TOP}_Q(T(v)) \cup \mathrm{SAT}_Q(T(v_\ell)) \cup \mathrm{SAT}_Q(T(v_r)) \ . \tag{1}$$

(The equality follows from Lemma 2 and Observation 1.) For a set $A \subseteq Q$ of quartets, let $\mathtt{sum}(A) \stackrel{\mathrm{def}}{=} \sum_{q \in A} C_q$ denote the sum of their weights. The *score* of the subtree $T(v)$ (with respect to Q) is defined as

$$\mathtt{score}_Q(T(v)) \stackrel{\mathrm{def}}{=} \mathtt{sum}(\mathrm{SAT}_Q(T(v))) \ .$$

By Equation (1)

$$\mathtt{score}_Q(T(v)) = \mathtt{sum}(\mathrm{TOP}_Q(T(v))) + \mathtt{score}_Q(T(v_\ell)) + \mathtt{score}_Q(T(v_r)) \ . \tag{2}$$

Let S be a set of three or more taxa. Denote by $\texttt{opt\_score}_Q(S)$ the maximum score with respect to Q among all trees that have S as their set of leaves [4]. We denote by $\texttt{opt\_tree}_Q(S)$ a tree which attains the maximum score. For every proper partition of S into two subsets S_1 and S_2, let $T(S_1, S_2)$ denote a tree whose left subtree equals $\texttt{opt\_tree}_Q(S_1)$ and its right subtree equals $\texttt{opt\_tree}_Q(S_2)$. By equation (2), we have

$$\texttt{score}_Q(T(S_1, S_2)) = \texttt{sum}(\mathrm{TOP}_Q(T(S_1, S_2))) + \texttt{opt\_score}_Q(S_1) + \texttt{opt\_score}_Q(S_2) .$$

This implies that

$$\texttt{opt\_score}_Q(S) = \tag{3}$$
$$\max_{S_1 \cup S_2 = S} \Big(\texttt{sum}(\mathrm{TOP}_Q(T(S_1, S_2))) + \texttt{opt\_score}_Q(S_1) + \texttt{opt\_score}_Q(S_2) \Big) .$$

Let $\langle S_1, S_2 \rangle$ be a partition of S which attains the maximum, then $\texttt{opt\_tree}_Q(S)$ is defined as $T(S_1, S_2)$.

Equation (3) yields a recursive algorithm to compute the optimal tree (with respect to the given list of weighted quartets, Q): Given Q and S, go over all partitions $\{S_1, S_2\}$ of S, and choose a partition which maximizes

$$\texttt{sum}(\mathrm{TOP}_Q(T(S_1, S_2))) + \texttt{opt\_score}_Q(S_1) + \texttt{opt\_score}_Q(S_2) .$$

This partition defines which taxa belong to the left subtree, and which to the right one. Apply the procedure recursively until each subtree has size smaller than 3. An optimal tree may be constructed by means of backtracking the partitioning steps.

The drawback of this recursive algorithm is that the score of each set S is recomputed whenever S is encountered as a subset in a partition. (Thus if S is of size i, its score will be computed 2^{n-i} times.) In order to avoid this wasteful repetition, we now employ the dynamic programming paradigm. We make a record of computed $\texttt{opt\_score}_Q(S)$ values, so that we will not have to recompute them. To do this, we scan the subsets $S \subseteq \{1, 2, \ldots, n\}$ by increasing size of S. This guarantees that in the computation for a set S, all subsets of S are already scanned over.

5.3 The Code

Let $\{1\}, \ldots, \{n\}, \{1, 2\}, \ldots, \{n-1, n\}, \ldots, \{1, \ldots, n\}$ be an ordering of all $2^n - 1$ non-empty subsets of $\{1, \ldots, n\}$, in which every set appears after all its subsets. For each non-empty subset S, we compute the optimal score of a phylogenetic tree for S. In addition we store for each S a partition $\langle S_1, S_2 \rangle$ that induces the score $\texttt{opt\_score}_Q(S)$. In the backtracking phase, this partition is used to reconstruct an optimal phylogenetic tree.

Optimal score computation

Go over the sets $S \subseteq \{1, \ldots, n\}$ in order,

[4] By the definition, trees with one or two leaves do not contain any `qlca`, so for sets S of size 1 or 2 we define $\texttt{opt\_score}_Q(S) = 0$.

- If $|S| \leq 2$, then $\texttt{opt\_score}_Q(S) = 0$.
- Compute the score of each S-partition, $\langle S_1, S_2 \rangle$, which equals

$$\texttt{sum}(\mathrm{TOP}_Q(T(S_1, S_2))) + \texttt{opt\_score}_Q(S_1) + \texttt{opt\_score}_Q(S_2) \ .$$

- Let $\texttt{opt\_score}_Q(S)$ be a maximal score, and let $\texttt{opt\_tree}_Q(S)$ be a partition for which the maximal score is obtained.

The optimal score for the whole set is stored in the entry $\texttt{opt\_score}_Q(\{1, \ldots, n\})$.

Tree Reconstruction

We construct a phylogenetic tree T with an optimal score as follows:

- The root of T is determined by an optimal partition $\langle S_1, S_2 \rangle$ of $S = \{1, \ldots, n\}$.
- Recursively build a phylogenetic tree for S_1 and S_2.

5.4 Algorithm Complexity

The elementary step in the algorithm is checking whether a quartet q is satisfied by a given partition of S, $\langle S_1, S_2 \rangle$. We notice that each of the four taxa in q satisfies exactly one of the following three properties with respect to the partition $\langle S_1, S_2 \rangle$:

1. The taxon is in S_1.
2. The taxon is in S_2.
3. The taxon is in neither.

We can therefore identify with q a characteristic vector which describes the presence of its taxa in S_1 and S_2. The vector `112N`, for example, means the two elements of first pair are in S_1, while one of the elements of the second pair is in S_2, and the other does not belong to the set of taxa currently in discussion. By corollary 3, we conclude that the root of $T(S_1, S_2)$ is $\texttt{qlca}(q)$ and that it satisfies q. All we need, then, is to construct at the preprocessing stage, a lookup table of size 3^4. The entries in this table are all the possible characteristic vectors (4-tuples over $\{1, 2, N\}$). By setting a convention that the first and second elements in the vector represent the elements of the first pair in the quartet, a single table may be used for all input quartets.

Checking satisfiability of a given quartet q as described takes 5 memory accesses (4 to construct the characteristic vector and 1 for the table lookup). In case the quartet is satisfied, an additional memory access is required to retrieve its weight, and one floating point addition. So for k input quartets, the value $\texttt{sum}(\mathrm{TOP}_Q(T(S_1, S_2)))$ can be computed by performing up to $6k$ memory accesses and k floating point additions. Viewing the 6 memory accesses and the 1 addition as one basic step, this means that finding $\texttt{opt\_score}_Q(S)$ for S of size i and Q of size k takes $2^i \cdot k$ basic steps. This computation is held for every set $S \subset \{1, \ldots, n\}$, and therefore the total number of steps in computing the optimal tree is

$$\sum_{i=0}^{n} \binom{n}{i} 2^i k = k3^n \ .$$

In terms of memory, for each subset of $\{1, \ldots, n\}$ we need an entry with its optimal score and optimal partition. This means $\Theta(2^n)$ memory requirements.

5.4.1 Ramifications

We now turn to describe some improvements of the algorithm. Even though none of these improvements has a significant effect on the asymptotic behavior of the algorithm, they all make constant factor improvements to the actual running time. These seemingly minor improvements of reducing the running time by constant factors make the difference between the processing of real-life datasets in reasonable time and resorting to toy problems.

The first observation is that each partition $\langle S_1, S_2 \rangle$ of S is processed twice; the second time is when S_1 and S_2 exchange roles. Therefore, the result will not change if we only scan those partitions where S_1 precedes S_2 in lexicographic order. This gives a factor 2 saving.

A common practice in phylogenetic studies is to include in the dataset one "external" taxon, which is known not to be affiliated with any other taxon in the data. This taxon, labeled "outgroup", is later used to root the resulting tree by placing the new root at the edge that connects the outgroup with the rest of the tree[5]. When we know in advance which taxon is the outgroup, we consider at the top-level only the partition that places the outgroup in one subtree, and every other taxon in the other. This means that an optimal tree should be found just for the set of $n-1$ "internal" taxa. Since the dependence of the run time on n, the number of taxa is proportional to 3^n, this reduction of n by 1 yields a factor 3 saving.

In the general case, the given input quartets are kept in a list which is sequentially scanned for each partition $\langle S_1, S_2 \rangle$. But when all possible $k = \binom{n}{4}$ input quartets are present, the performance can be improved further yet. Consider a partition of a set S. Instead of checking satisfiability of every one of the $\binom{n}{4}$ quartets, we can inhibit the test for those quartets which are known *a priory* to be unresolved by S. In particular, it suffices to inspect only those quartets which have three or more elements in S. To implement this, we prepare (in a preprocessing stage) a four-dimensional array of size n^4, where each dimension is indexed by taxa. The entries in the array are the input quartets and their weights. Given S, construct a list of the taxa in S. To scan all quartets which contain 4 taxa from S, enumerate all 4-tuples of elements in the list. For each 4-tuple the access to the appropriate quartet in the array takes constant time, and the elementary step takes constant time as well. For the quartets having exactly 3 elements from S use a similar enumeration. Once a relevant quartet is accessed, we process it (with respect to $\langle S_1, S_2 \rangle$) as before.

[5] In fact, there are $n-1$ rooted trees at the top level that correspond to the same unrooted tree (and thus have the same score). Recall that satisfaction of quartets is unaffected by re-rooting.

To analyze the amount of saving, we first notice that if we consider a random set $S \subset \{1, \ldots, n\}$ (each element chosen independently with probability 1/2) and a random quartet q, the probability that S contains three or more elements from q is 5/16. But while our dynamic programming algorithm does go over all sets S, it does not weigh them uniformly: Larger sets are weighted more heavily (to be precise, a subset of size ℓ has probability $2^l/3^n$). Since larger sets will contain 3 or more elements from a given quartet more frequently, the overall saving effect will be less dramatic. However, it is still meaningful. We argue that we achieve a factor 11/27 saving, compared to going over all quartets for each S. For a set S with ℓ taxa, the number of quartets having all 4 taxa in S is $\binom{\ell}{4}$. The number of quartets having 3 taxa in S is $\binom{\ell}{3}(n-\ell)$. So overall, the number of quartets that will be inspected at all stages of the dynamic programming algorithm equals

$$\sum_{l=0}^{n} \binom{n}{\ell} 2^{\ell} \left[\binom{\ell}{4} + \binom{\ell}{3}(n-\ell) \right]$$

We now compare this to the number of quartets inspected using the "all quartets" approach.

Lemma 4

$$\sum_{l=0}^{n} \binom{n}{l} 2^{l} \left[\binom{l}{4} + \binom{l}{3}(n-l) \right] = \frac{16}{27} \binom{n}{4} 3^n$$

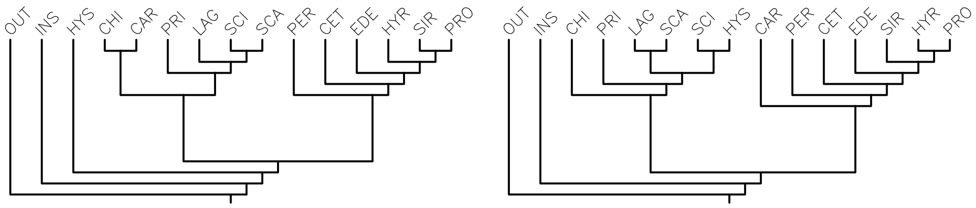
We omit the proof of this combinatorial fact from the paper. The interested reader may find it in [Pel98]. It is interesting to note that incorporating this improvement into the algorithm did improve the actual running time by about 30%.

6 Computational Results

In this section we describe the results of executing our algorithms on real data. We implemented both algorithms, as well as supporting software. The geometric algorithm was implemented using the semidefinite package SDPA [FKN96]. The dynamic programming algorithm was coded in `C++`. Both of them, as well as additional software (e.g., to assign scores to trees and to conduct large-scale experiments), are available on the net [Pel98].

The version that produced the best geometric tree used the following parameters: Only quartets with confidence level 90% or higher were included in the objective function. There were 452 such quartets, which means that about two thirds of the input quartets were ignored.[6] In addition, we imposed the

[6] To ensure that no taxon is severly underrepresented in this reduced quartet set, we counted the number of times that each taxon appears in the input. There are $4 \times 452 = 1808$ appearances, and under a uniform representation distribution one would expect each taxon to appear in 1808/15 of them, or about 120 times. Our calculations show that the least frequent taxon (INS) appears in 82 of the input quartets (4.5%), while the most frequent taxon (PRO) appears in 157 quartets (8.7%).



geometric tree   optimal tree

Figure 3: Phylogenetic trees obtained by the geometric and optimal methods

Tree	Score		Satisfied
	with Compensation	without Compensation	Quartets
Gregory (1910)	56.1%	47.8%	44.8%
Simpson (1945)	55.2%	49.2%	46.2%
Mckenna (1975)	54.0%	46.9%	44.0%
Novacek (1982)	55.6%	43.1%	39.6%
Shoshani (1986)	61.1%	bifurcating	57.7%
Novacek *et. al.* (1988)	53.0%	36.2%	33.3%
Novacek (1992)	58.1%	48.7%	45.3%
Graur (1997)	71.0%	bifurcating	66.7%
Puzzling	71.6%	bifurcating	68.1%
Geometric	72.1%	bifurcating	68.7%
Optimal	73.4%	bifurcating	70.2%

Table 1: Comparison of published trees. Trees with year numbers are from [Gra93]. "Puzzling" was generated by the algorithm in [SH96], and our quartet data.

"distance constraint" $\langle v_i, v_j \rangle \leq 0.75$ for each pair of points. The execution took less than 5 seconds on a Sun Ultra-4 machine. The resulting tree has score 72.1%.

The dynamic-programming algorithm took into account all 1365 weighted quartets. The execution took some 7 minutes on a Sun Ultra-4 machine (see Table 2). The resulting *optimal* tree is unique and has score 73.4%. The two trees are shown in Figure 3.

We computed the score (with respect to the same four-taxon input) of other phylogenetic trees that were published in the literature (references in [Gra93]). Table 1 lists these scores. It should be noted that most of the earlier trees are multifurcating. For these trees, the contribution of each unresolved quartet is computed either as 1/3 of its confidence ("with compensation") or as 0 ("without compensation"). As seen, our algorithms produce the best trees with respect to the quartet satisfaction criterion on our data-set. We remark the quartet data was not available during the construction of most of the trees (therefore their optimization criteria were different). The exceptions, beside

n	time
15	7 min.
16	28 min.
17	109 min.
18	7 hr.
19	29 hr.
20	128 hr.

Table 2: Run times for the "exact" algorithm with all $\binom{n}{4}$ quartets in the input. For $n = 15, 16, 20$ the times are measured on a \$30K machine, in 1997 prices (Sun Ultra-4 at 300 MHz). For other values of n they are estimated.

our algorithms, are the quartet-puzzling tree and Graur's tree from 1997. The quartet puzzling implementation was obtained from [SH96] and modified to construct a tree from our (unweighted) quartet data. After 1000 puzzling trees are constructed, a consensus tree is computed using the CONSENSE program in [Fel89]. This output consensus tree is what we call the "puzzling" tree. The whole process takes about 10 seconds on the same Sun machine. As seen, the performance of this method falls just a little short of our geometric algorithm. Graur's algorithm uses a different approach and applies a neighbor-joining algorithm on a non-metric distance matrix, defined by pairwise agglomeration within quartets [Oph97].

7 Discussion

We believe that the methods presented in this paper are significant from a biological and evolutionary point of view. They allow us to combine partial phylogenetic trees into a tree containing all the taxa of interest. This enables the use of most available sequences, unlike "traditional" reconstruction methods. The databases of sequences are updated periodically with the accumulation of new sequence data. Typically such updates still leave many sequences unknown for many taxa, but they will tend to produce more reliable quartet information. Our methods should thus enable a periodical reassessment of phylogenetic theories.

Obviously the biological components of this study will have to be refined and updated in the future. Most importantly, the performance of different tree making methods and confidence measures will have to be assessed against real data and simulation results. Another point of interest will be the assignment of confidence values to the generated tree edges. Our algorithms give no indication that some edges in the generated tree are not evident from the data. These may in fact be furcations which are arguable from a biological point of view. A direction for further research and experimentation would be to find an assignment of support values, given the data and the produced trees.

From the algorithmic point of view, this paper raises a number of interesting open problems. It would be nice to prove any performance guarantee for the "geometric" reconstruction method. In fact, we are not aware of any efficient algorithm for reconstructing trees from weighted quartets whose output is guaranteed to be above one third of the maximum. (A tree chosen at random is expected to satisfy one third of the quartets). For realistic datasets, any decrease of the exponent's base for an "exact" algorithm will be significant. (Currently the dynamic programming algorithm takes 5 days for datasets with $n = 20$ on a \$30K machine in 1997 prices.)

The fact that the score of the optimal tree is only 73.4% indicates that the input is not very reliable. However, even with this limited quality data, both the geometric and the optimal trees make biological sense. We note, in particular, the position of Insectivora as an outgroup to the other eutherian taxa [KGA95], and the monophyly of the extended superordinal taxon Paenungulata, which includes Edentata [SCM+97]. Our phylogenies are bifurcated and resolved, unlike most mammalian phylogenies in the literature that are multifurcated and unresolved. We are currently compiling an updated dataset of mammalian protein sequences, with which we hope to resolve the 100-year-old problem of mammalian ordinal phylogeny "not vaguely and generally, but with all possible precision in place, weight, and measure" (Sir William Herschel, 1738–1822).

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