On the Exact Maximum Complexity of Minkowski Sums of Convex Polyhedra*

Efi Fogel† Dan Halperin† Christophe Weibel‡

Abstract

We present a tight bound on the exact maximum complexity of Minkowski sums of convex polyhedra in $\mathbb{R}^3$. In particular, we prove that the maximum number of facets of the Minkowski sum of two convex polyhedra with m and n facets respectively is bounded from above by $f(m,n) = 4mn - 9m - 9n + 26$. Given two positive integers m and n, we describe how to construct two convex polyhedra with m and n facets respectively, such that the number of facets of their Minkowski sum is exactly $f(m,n)$. We generalize the construction to yield a lower bound on the maximum complexity of Minkowski sums of many convex polyhedra in $\mathbb{R}^3$. That is, given k positive integers $m_1, m_2, \ldots, m_k$, we describe how to construct k convex polyhedra with corresponding number of facets, such that the number of facets of their Minkowski sum is $\sum_{1 \le i < j \le k}(2m_i - 5)(2m_j - 5) + \binom{k}{2} + \sum_{1 \le i \le k} m_i$. We also provide a conservative upper bound for the general case. The polyhedra models and an interactive program that computes their Minkowski sums and visualizes them can be downloaded from `http://www.cs.tau.ac.il/~efif/Mink`.

1 Introduction

Let P and Q be two compact convex polyhedra in $\mathbb{R}^d$. The Minkowski sum of P and Q is the convex polyhedron, polytope for short, $M = P \oplus Q = \{p + q \,|\, p \in P, q \in Q\}$.

Minkowski-sum computation constitutes a fundamental task in computational geometry. Minkowski sums are frequently used in areas such as robotics and motion planing [6, 8] and many additional domains, like solid modeling, design automation, manufacturing, assembly planning, virtual prototyping, etc., as Minkowski sums are closely related to proximity queries [7].

Various methods to compute the Minkowski sum of two polyhedra in $\mathbb{R}^3$ have been proposed. One common approach is to decompose each polyhedron into convex pieces, compute pairwise Minkowski sums of pieces of the two, and finally the union of the pairwise sums. Computing the Minkowski sum of two convex polyhedra remains a key operation. The combinatorial complexity of the sum can be as high as $\Theta(mn)$ when both polyhedra are convex.

Recently a few complete implementations of output-sensitive methods for computing exact Minkowski sums have need introduced: (i) a method based on Nef polyhedra embedded on the sphere [4], (ii) an implementation of Fukuda's algorithm by Weibel [2, 9], and (iii) a method based on the cubical Gaussian-map data structure [1]. These methods exploit efficient innovative techniques in the area of exact geometric-computing to minimize the time it takes to ensure exact results. However, even with the use of these techniques, the amortized time of a single arithmetic operation is large in comparison with a single arithmetic operation carried out on native number types, such as floating point. Thus, the constants involved in the expressions of these algorithm time complexities increases, which makes the question this paper attempts to answer, "What is the exact maximum complexity of Minkowski sums of polytopes in $\mathbb{R}^3$?", even more relevant.

Gritzmann and Sturmfels [5] formulated an upper bound on the number of features f_i^d of any given dimension i of the Minkowski sum of many polytopes in d dimensions. Fukuda and Weibel [3] obtained upper bounds on the number of edges and facets of the Minkowski sum of two polytopes in $\mathbb{R}^3$ in terms of the number of vertices of the summands: $f_2(P_1 \oplus P_2) \le f_0(P_1)f_0(P_2) + f_0(P_1) + f_0(P_2) - 6$. They also studied the properties of the Minkowski sums of perfectly centered polytopes and their polars, and provided a tight bound on the number of vertices of the sum of polytopes in any given dimension.

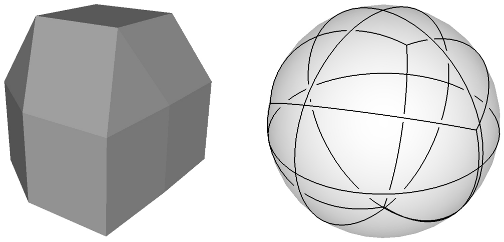
2 The Upper Bound

The *Gaussian Map* $G = G(P)$ of a compact convex polyhedron P in $\mathbb{R}^3$ is a set-valued function from P to the unit sphere $\mathbb{S}^2$, which assigns to each point p the set of outward unit normals to support planes to P at p. The overlay of the Gaussian maps of two polytopes P and Q respectively identifies all pairs of features of P and Q respectively that have common supporting

*This work has been supported in part by the IST Programme of the EU as Shared-cost RTD (FET Open) Project under Contract No IST-006413 (ACS - Algorithms for Complex Shapes), by the Israel Science Foundation (grant no. 236/06), by the Hermann Minkowski–Minerva Center for Geometry at Tel Aviv University, and by the Swiss National Science Foundation Project 200021-105202, "Polytopes, Matroids and Polynomial Systems".

†School of Computer Science, Tel-Aviv University, Israel. `{efif,danha}@post.tau.ac.il`

‡Mathematics Institute, EPFL, Laussane, Switzerland. `christophe.weibel@epfl.ch`



(a) (b)

Figure 1: (a) The Minkowski sum of a tetrahedron and a cube and (b) the Gaussian map of the Minkowski sum.

planes, as they occupy the same space on the unit sphere, thus, identifying all the pairwise features that contribute to the boundary of the Minkowski sum of P and Q. A facet of the Minkowski sum is either a facet f of Q translated by a vertex of P supported by a plane parallel to f, or vice versa, or it is a facet parallel to two parallel planes supporting an edge of P and an edge of Q respectively. A vertex of the Minkowski sum is the sum of two vertices of P and Q respectively supported by parallel planes.

The number of facets of the Minkowski sum M of two polytopes P and Q with m and n facets respectively is equal to the number of vertices of the Gaussian map of M. A vertex in the Gaussian map of M is either due to a vertex in the Gaussian map of P, due to a vertex in the Gaussian map of Q, or due to an intersection between an edge of the Gaussian map of P and an edge of the Gaussian map of Q. Thus, the exact complexity $f(m,n)$ of M can be upper bounded by the expression $g(m,n)+m+n$, where $g(m,n)$ is the number of edge intersections in the Gaussian map of M.[1]

Corollary 1 *The maximum exact number of edges in a Gaussian map $G(P)$ of a polytope P with m facets is $3m-6$. The exact number of faces in such a Gaussian map is $2m-4$.*

The above can be obtained by a simple application of Euler's formula for planar graphs to the Gaussian maps $G(P)$. It can be trivially used to bound the exact number of facets of the Minkowski sum of two polytopes. We can plug the bound on the number of dual faces, which is the number of primal vertices, in the expression introduced by Fukuda and Weibel, (see Section 1), to obtain: $f(m,n) \leq (2m-4)\cdot(2n-4)+(2m-4)+(2n-4)-6 = 4mn-6m-6n+2$. We can improve the bound, but first we need to bound the number of faces in $G(M)$.

Lemma 2 *Let G_1 and G_2 be two Gaussian maps, and let G be their overlay. Let f_1, f_2, and f denote the number of faces of G_1, G_2, and G respectively. Then, the number of faces f cannot exceed $f_1 \cdot f_2$.*

[1] The exact complexity is strictly equal to the given expression, only when no degeneracies occur.

This lemma is similar to the one where convex planar maps replace the Gaussian maps, the proof of which appears in several flavors in the literature. We are ready to tackle the upper bound.

Theorem 3 *Let P and Q be two polytopes in $\mathbb{R}^3$ with m and n facets respectively, and let $f(m,n)$ denote the number of facets of their Minkowski sum $M = P \oplus Q$. Then, $f(m,n) \leq 4mn-9m-9n+26$. The maximum complexity is attained only when the number of edges of each of P and Q is maximal for the given number of facets.*

Proof. Let v_1, e_1, f_1 and v_2, e_2, f_2 denote the number of vertices, edges, and faces of $G(P)$ and $G(Q)$ respectively. Recall that $v_1 = m$, $v_2 = n$, and $v = f(m,n)$, where v denotes the number of vertices of $G(M)$. The number of edges and faces of $G(M)$ is similarly denoted as e and f respectively. Assume that P and Q are two polytopes, such that the number of facets $f(m,n)$ of their Minkowski sum is maximal. First, we need to show that vertices of $G(P)$, vertices of $G(Q)$, and intersections between edges of $G(P)$ and edges of $G(Q)$ do not coincide. Assume to the contrary that some do. Then, one of the polytopes P or Q or both can be slightly rotated to escape this degeneracy, but this would increase the number of vertices $v = f(m,n)$, contradicting the fact that $f(m,n)$ is maximal. Therefore, the number of vertices v is exactly equal to $v_1+v_2+v_x$, where v_x denotes the number of intersections of edges of $G(P)$ and edges of $G(Q)$ in $G(M)$.

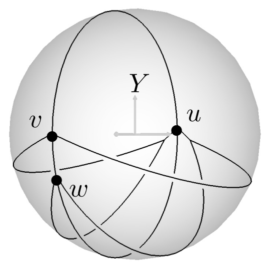
The total count of degrees of all vertices of $G(M)$ is twice the number of edges e of $G(M)$ on one hand, as each edge contributes two to this count. On the other hand, it is equal to the sum of degrees of all vertices of $G(P)$, vertices of $G(Q)$, and intersection vertices. Each edge of $G(P)$ and each edge of $G(Q)$ contributes exactly two to the count of degrees of the original vertices, and the degree of each new intersection is exactly four. Thus, we have $2e_1+2e_2+4v_x = 2e$. Applying Euler's formula and Lemma 2 yields $v_x \leq f_1f_2+v_1+v_2-2-e_1-e_2$.

Corollary 1 sets an upper bound on the number of edges e_1. Thus, e_1 can be expressed in terms of ℓ_1, a non-negative integer, as follows: $e_1 = 3v_1-6-\ell_1$. Applying Euler's formula, the number of facets can be expressed in terms of ℓ_1 as well: $f_1 = e_1-2-v_1 = 2v_1-4-\ell_1$. Similarly, we have $e_2 = 3v_2-6-\ell_2$ and $f_2 = 2v_2-4-\ell_2$ for some non-negative integer ℓ_2. $G(P)$ consists of a single connected component. Therefore, the number of edges e_1 must be at least v_1-1. This is used to obtain an upper bound on ℓ_1 as follows: $v_1-1 \leq e_1 = 3v_1-6-\ell_1$, which implies $\ell_1 \leq 2v_1-5$, and similarly $\ell_2 \leq 2v_2-5$.

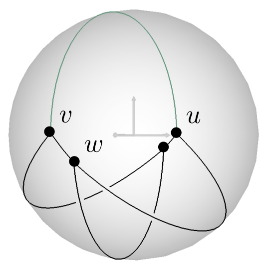
Plugging all this in the above inequality results with $v_x \leq 4v_1v_2-10v_1-10v_2+26$, and since $f(m,n) = v_1+$

v_2+v_x, we conclude that $f(m,n) \le 4v_1v_2 - 9v_1 - 9v_2 + 26$. The maximum complexity can be reached when $h(\ell_1, \ell_2)$ diminishes. This occurs when $\ell_1 = \ell_2 = 0$. That is, when the number of edges of $G(P)$ and $G(Q)$, (respectively P and Q), is maximal. □

3 The Lower Bound

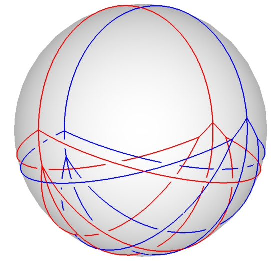


Given two integers $m \ge 4$ and $n \ge 4$, we describe how to construct two polytopes in $\mathbb{R}^3$ with m and n facets respectively, such that the number of facets of their Minkowski sum is exactly $4mn - 9m - 9n + 26$. More precisely, given i, we describe how to construct a skeleton of a polytope P_i with i facets, $3i-6$ edges, and $2i-4$ vertices, and prove that the number of facets of the Minkowski sum of P_m and P_n properly adjusted and oriented is exactly $4mn - 9m - 9n + 26$. The figures above and below depict the Gaussian map of P_5 and P_4 respectively.



We use the subscript letter i in all notations X_i to identify some object X with the polytope P_i. For example, we give the Gaussian map $G(P_i)$ of P_i a shorter notation G_i First, we examine the structure of the Gaussian map G_i. Let V_i denote the set of vertices of G_i. Recall that the number of vertices, edges, and faces of G_i is i, $3i-6$, and $2i-4$ respectively. The unit sphere, where G_i is embedded on, is divided by the plane $y=0$ into two hemispheres $H^- \subset \{(x,y,z) \,|\, y \le 0\}$ and $H^+ \subset \{(x,y,z) \,|\, y > 0\}$. One vertex v_i is located exactly at the pole $(0,0,1)$. Another vertex w_i lies in H^- very close to v_i. A third vertex u_i is located very close to the opposite pole $(0,0,-1)$. It is the only vertex (out of the i vertices) that lies in H^+. All the remaining $i-3$ vertices in $V' = V_i \setminus \{u_i, v_i, w_i\}$ are concentrated near the pole $(0,0,-1)$ and lie in H^-. The edge $\overline{uv}_i$ is the only edge whose interior is entirely contained in H^+. Every vertex in V' is connected by two edges to v_i and w_i respectively. These edges together with the edge $\overline{uw}_i$ form a set of $2i-5$ edges, denoted as E'. The length of all edges in E' is almost π, due to the near proximity of u_i, v_i, and w_i to the respective poles.

It is easy to verify that if the polytope P_i is not degenerate; namely, its affine hull is 3-space, then any edge of G_i is strictly less than π long. Bearing this in mind, the main difficulty in arriving at a tight-bound construction is to force all edges but one of the Gaussian map of one polytope to intersect all edges but one of the Gaussian map of the other polytope, and on top of that force the pair of excluded edges, one from each Gaussian map, to intersect as well. As shown below, this is the best one can do in terms of intersections.



The number of facets in the Minkowski sum of P_m and P_n is maximal, when the number of vertices in the overlay of G_m and G_n is maximal. This occurs, for example, when one of G_m and G_n is rotated 90° about the Y axis, as depicted on the right for the case of $m = n = 5$. In this configuration, each edge of the $2m-5$ edges in E'_m intersects each edge of the $2n-5$ edges in E'_n. These intersections occur in H^-. In addition, the edge $\overline{uv}_m$ intersects the edge $\overline{uv}_n$ near the pole $(0,1,0)$. Counting all these intersections results with $(2m-5)(2n-5)+1 = 4mn - 10m - 10n + 26$. Adding the original vertices of G_m and G_n yields the desired result.

All the vertices of P_i lie on the boundary of a cylinder the axis of which coincides with the Z axis. When P_i is looked at from $z = \infty$, two facets are visible, and when looked at from $z = -\infty$, the remaining $i-2$ facets are visible. The precise details that govern the construction of $P_i, i \ge 4$, which match the description of G_i above, are omitted due to lack of space.

4 Maximum Complexity of Minkowski Sums of Many Polytopes

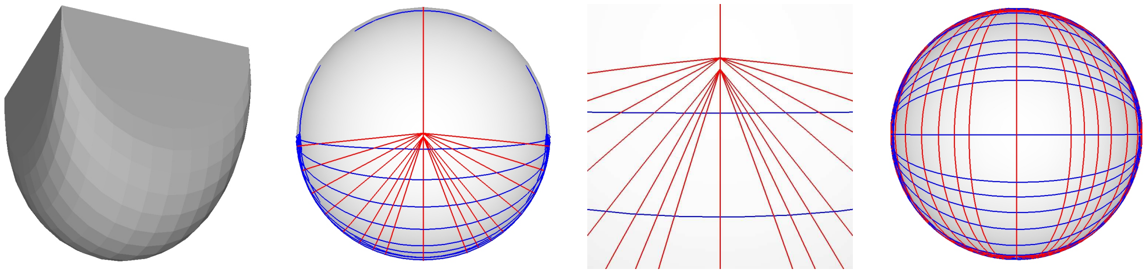
In this section we discuss the bounds on the exact complexity of the Minkowski sum many polytopes generalizing some of the arguments presented above.

Conjecture 4 *Let $P_1, P_2, \ldots, P_k$ be a set of k polytopes in $\mathbb{R}^3$, such that the number of facets of P_i is m_i for $i = 1, 2, \ldots, k$. The exact maximum complexity of the Minkowski sum $P_1 \oplus P_2 \oplus \ldots \oplus P_k$ is $\sum_{1 \le i < j \le k} (2m_i - 5)(2m_j - 5) + \binom{k}{2} + \sum_{i=1}^{k} m_i$.*

In the following sections we establish the lower bound, but provide only a conservative upper bound, which leaves a gap between the two bounds.

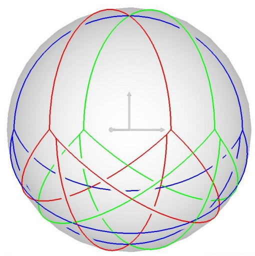
4.1 The Lower Bound

Given k positive integers $m_1, m_2, \ldots, m_k$, such that $m_i \ge 4$, we describe how to construct k polytopes in $\mathbb{R}^3$ with corresponding number of facets, such that the number of facets of their Minkowski sum is exactly $\sum_{1 \le i < j \le j} (2m_i - 5)(2m_j - 5) + \binom{k}{2} + \sum_{i=1}^{k} m_i$. More precisely, given i, we describe how to construct a skeleton of a polytope P_i with i facets, $3i-6$ edges, and $2i-4$ vertices, and prove that the number of facets of the Minkowski sum $M = P_1 \oplus P_2 \oplus \ldots \oplus P_k$ of the k polytopes properly adjusted and oriented is exactly the expression above. We use the same construction described in Section 3.



(a) (b) (c) (d)

Figure 2: (a) The Minkowski sum $M_{11,11} = P_{11} \oplus P'_{11}$, where P'_{11} is P_{11} rotated 90° about the Y axis. (b) The Gaussian map of $M_{11,11}$ looked at from $z = \infty$. (c) A scaled up view of the Gaussian map of $M_{11,11}$ looked at from $z = \infty$. (d) The Gaussian map of $M_{11,11}$ looked at from $y = -\infty$.



The number of facets in the Minkowski sum of $P_i, i = 1, 2, \ldots, k$ is maximal, when the number of vertices in the overlay of $G_i, i = 1, 2, \ldots, k$ is maximal. This occurs, for example, when G_i is rotated $180^\circ i/k$ about the Y axis for $i = 1, 2, \ldots, k$, as depicted on the right for the case of $m_1 = m_2 = m_3 = 4$. In this configuration, all the $2m_i - 5$ edges in E'_i intersect all the $2m_j - 5$ edges in E'_j, for $1 \leq i < j \leq k$. These intersections occur in H^-. In addition, the edge $\overline{uv}_{m_i}$ intersects the edge $\overline{uv}_{m_j}$ for $1 \leq i < j \leq k$. These intersection points lie in H^+ near the pole $(0, 1, 0)$. Counting all these intersections results with $\sum_{1 \leq i < j \leq j} (2m_i - 5)(2m_j - 5) + \binom{k}{2}$. Adding the original vertices of $G(P_i), i = 1, 2, \ldots, k$, yields the bound asserted in Conjecture 4.

4.2 An Upper Bound

We apply a similar technique to the one used in Section 2 to obtain an upper bound. First, we extend Lemma 2.

Lemma 5 *Let $G_1, G_2, \ldots, G_k$ be a set of k Gaussian maps, and let G be their overlay. Let f_i denote the number of faces of G_i, and let f denote the number of faces of G. Then, the number of faces f of G cannot exceed $\sum_{1 \leq i < j \leq k} f_i \cdot f_j$.*

The proof of the lemma above is a simple generalization of the proof of lemma 2. Secondly, we count the total degrees of vertices in $G(M)$. Let $P_1, P_2, \ldots, P_k$ be k polytopes in $\mathbb{R}^3$ with $m_1, m_2, \ldots, m_k$ facets respectively. Let $G(P_i)$ denote the Gaussian map of P_i, and let v_i, e_i, and f_i denote the number of vertices, edges, and faces of $G(P_i)$ respectively. Let v_x denote the number of intersections of edges of $G(P_i)$ and edges of $G(P_j)$, $i \neq j$ in $G(M)$. Starting with $\sum_{i=1}^{k} e_i + 2v_x = e$, and applying Lemma 5 and Theorem 3 we get $v_x \leq \sum_{1 \leq i < j \leq k} (2v_i - 4)(2v_j - 4) - 2\sum_{i=1}^{k} v_i + 6k - 2$.

For example, the complexity of the Minkowski sum of k tetrahedra is $v_x + \sum_{i=1}^{k} v_i$, and by the inequality above it is bounded from above by $8k^2 - 6k - 2$. The construction described in the previous section yields a configuration of k tetrahedra, the Minkowski sum of which is $5k^2 - k$. For $k = 2$ both expressions evaluate to 18.

References

[1] E. Fogel and D. Halperin. Exact and efficient construction of Minkowski sums of convex polyhedra with applications. In *Proc. 8th Workshop Alg. Eng. Exper. (Alenex'06)*, 2006.

[2] K. Fukuda. From the zonotope construction to the Minkowski addition of convex polytopes. *Journal of Symbolic Computation*, 38(4):1261–1272, 2004.

[3] K. Fukuda and C. Weibel. On f-vectors of Minkowski additions of convex polytopes. *Discrete and Computational Geometry*, 2006. To appear.

[4] M. Granados, P. Hachenberger, S. Hert, L. Kettner, K. Mehlhorn, and M. Seel. Boolean operations on 3D selective Nef complexes: Data structure, algorithms, and implementation. In *Proc. 11th Annu. Euro. Sympos. Alg.*, volume 2832 of *LNCS*, pages 174–186. Springer-Verlag, 2003.

[5] P. Gritzmann and B. Sturmfels. Minkowski addition of polytopes: Computational complexity and applications to Gröbner bases. *SIAM J. Disc. Math*, 6(2):246–269, 1993.

[6] D. Halperin, L. Kavraki, and J.-C. Latombe. Robotics. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry, 2nd Edition*, chapter 48, pages 1065–1093. CRC, 2004.

[7] M. C. Lin and D. Manocha. Collision and proximity queries. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry, 2nd Edition*, chapter 35, pages 787–807. CRC, 2004.

[8] M. Sharir. Algorithmic motion planning. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry, 2nd Edition*, chapter 47, pages 1037–1064. CRC, 2004.

[9] C. Weibel. Minkowski sums. `http://roso.epfl.ch/cw/poly/public.php`.