Exact Minkowski Sums of Convex Polyhedra*

Efi Fogel

Dan Halperin

School of Computer Science
Tel Aviv University, Israel
{efif,danha}@tau.ac.il

ABSTRACT

We present an exact implementation of an efficient algorithm that computes Minkowski sums of convex polyhedra in $\mathbb{R}^3$. Our implementation is complete in the sense that it does not assume general position, namely, it can handle degenerate input, and produces exact results. Our software also includes applications of the Minkowski-sum computation to answer collision and proximity queries about the relative placement of two convex polyhedra in $\mathbb{R}^3$. The algorithms use a dual representation of convex polyhedra, and their implementation is mainly based on the Arrangement package of CGAL, the Computational Geometry Algorithm Library. We compare our Minkowski-sum construction with a naïve approach that computes the convex hull of the pairwise sums of vertices of two convex polyhedra. Our method is significantly faster. The video demonstrates the techniques used on simple cases as well as on degenerate cases. The relevant programs, source code, data sets, and documentation are available at `http://www.cs.tau.ac.il/~efif/CD`. In particular this site contains a detailed report [3] on our algorithms and their implementation including the experimental comparison with the convex-hull approach.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*Geometrical problems and computations*

General Terms

Algorithms, Experimentation, Performance

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Keywords

Minkowski sums, arrangements, robustness and precision, CGAL

1. INTRODUCTION

Let P and Q be two convex polyhedra in $\mathbb{R}^3$. The Minkowski sum of P and Q is the convex polyhedron $M = P \oplus Q = \{p + q \,|\, p \in P, q \in Q\}$. Minkowski sums are ubiquitous in geometric computing and in particular they are useful for answering collision and proximity queries; see, e.g., the recent survey by Lin and Manocha [6].

We present an exact, complete, and robust implementation of an efficient algorithm to compute the Minkowski sum of *a set of convex polyhedra*. We use the Minkowski sums to detect collision, and compute the Euclidean separation distance between, and the directional penetration-depth of, two convex polyhedra in $\mathbb{R}^3$; see the accompanying paper [3] for more details. The algorithms use a dual representation of convex polyhedra, polytopes for short, named *Cubical Gaussian Map*. They are implemented on top of the CGAL library [1], and are mainly based on the Arrangement package of the library [4], although other parts, such as the Polyhedral-Surface package produced by L. Kettner [5], are used as well. The results obtained by this implementation are exact as long as the underlying number type supports the arithmetic operations $+$, $-$, $*$, and $/$ in unlimited precision over the rationals[1], such as the rational number type `Gmpq` provided by GMP — Gnu's Multi Precision library [2]. The implementation is complete and robust as it handles all degenerate cases, and guarantees exact results.

2. THE CUBICAL GAUSSIAN MAP

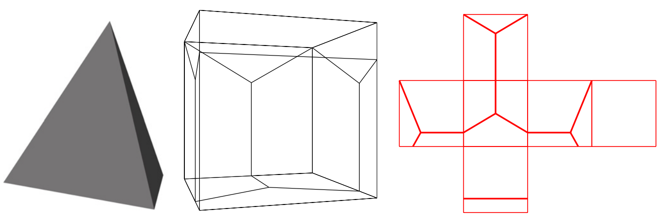
The *Gaussian Map* G of a compact convex polyhedron P in Euclidean three-dimensional space $\mathbb{R}^3$ is a set-valued function from P to the unit sphere $\mathbb{S}^2$, which assigns to each point p the set of outward unit normals to support planes to P at p. Thus, the whole of a facet f of P is mapped under G to a single point — the outward unit normal to f. An edge e of P is mapped to a geodesic segment $G(e)$ on $\mathbb{S}^2$, whose length is easily seen to be the exterior dihedral angle at e. A vertex v of P is mapped by G to a spherical polygon $G(v)$, whose sides are the images under G of edges incident to v and whose angles are the angles supplementary to the planar angles of the facets incident to v; that is, $G(e_1)$ and $G(e_2)$ meet at angle $\pi - \alpha$ whenever e_1 and e_2 meet at angle

[1]Commonly referred to as a *field* number type.

α. In other words, $G(P)$ is combinatorially dual to P, and metrically it is the unit sphere $\mathbb{S}^2$.

An alternative and practical definition follows. A direction in $\mathbb{R}^3$ can be represented by a point $u \in \mathbb{S}^2$. Let P be a polytope in $\mathbb{R}^3$, and let V denote the set of its boundary vertices. For a direction u, we define the *extremal point* in direction u to be $\lambda_V(u) = \arg\max_{p \in V} \langle u, p \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product. The decomposition of $\mathbb{S}^2$ into maximal connected regions, so that the extremal point is the same for all directions within any region forms the Gaussian map of P.

Similarly, the *Cubical Gaussian Map* (CGM) C of a polytope P in $\mathbb{R}^3$ is a set-valued function from P to the six faces of the unit cube whose edges are parallel to the major axes and are of length two. The decomposition of the unit-cube faces into maximal connected regions, so that the extremal point is the same for all directions within any region forms the CGM of P. Observe that, a single edge e of P is mapped to a chain of at most three connected segments that lie in three adjacent cube-faces respectively, and a vertex v of P is mapped to at most five abutting convex polygons that lie in five adjacent cube-faces respectively. Figure 1 shows the CGM of a tetrahedron.



(a) (b) (c)

Figure 1: (a) A tetrahedron, (b) the CGM of the tetrahedron, and (c) the CGM unfolded. Thick lines indicate real edges.

Te CGM is unique up to the scaling of the polytope. Therefore we extend each face of the CGM with the coordinates of its dual (original) vertex.

While using the CGM increases the overhead of some operations sixfold, and introduces degeneracies that are not present in the case of alternative representations, it simplifies the construction and manipulation of the representation, as the partition of each cube face is a planar map of segments, a well known concept that has been intensively experimented with in recent years. We use the CGAL planar-map [4] data structure (which is part of the arrangement package) to maintain the planar maps. The construction of the six planar maps from the polytope features and their incident relations amounts to the insertion of segments that are pairwise disjoint in their interiors into the planar maps, an operation that can be carried out efficiently, especially when one or both endpoints are known, and we take advantage of it. Computing the Minkowski sum, which we describe in the next section, amounts to the computation of the overlay of six pairs of planar maps, an operation well supported by the data structure as well.

3. EXACT MINKOWSKI SUMS OF CONVEX POLYHEDRA

The overlay of two planar subdivisions $\mathcal{S}_1$ and $\mathcal{S}_2$ is a planar subdivision $\mathcal{S}$ such that there is a face f in $\mathcal{S}$ if and only if there are faces f_1 and f_2 in $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively such that f is a maximal connected subset of $f_1 \cap f_2$. The overlay of the Gaussian maps of two polytopes P and Q identifies all the pairs of features of P and Q respectively that have common supporting planes, as they occupy the same space on the unit sphere, thus, identifying all the pairwise features that contribute to the boundary of the Minkowski sum of P and Q. A facet of the Minkowski sum is either a facet f of Q translated by a vertex of P supported by a plane parallel to f, or vice versa, or it is a facet parallel to two parallel planes supporting an edge of P and an edge of Q respectively. A vertex of the Minkowski sum is the sum of two vertices of P and Q respectively supported by parallel planes. A similar argument holds for the cubical Gaussian map with the unit cube replacing the unit sphere. More precisely, a single map that subdivides the unit sphere is replaced by six planar maps, and the computation of a single overlay is replaced by the computation of six overlays of corresponding pairs of planar maps. A vertex is attached to each planar-map face, which is the sum of two vertices attached to the two overlapping faces of the two CGMs of the two input polytopes respectively.

The algorithm that we implemented is output-sensitive. It runs in $O((n+k)\log n)$ time, where n is the total number of faces of the input operands, and k is the number of faces of the Minkowski sum. As we show in the accompanying paper [3], the algorithm efficiently computes the sum of several polytopes at once.

As mentioned above we use the Minkowski sum computation to answer a variety of collision detection and proximity queries [3].

4. REFERENCES

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