Arrangements and Their Applications*

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Abstract

The arrangement of a finite collection of geometric objects is the decomposition of the space into connected cells induced by them. We survey combinatorial and algorithmic properties of arrangements of curves in the plane and of surfaces in higher dimensions. We present many applications of arrangements to problems in motion planning, visualization, range searching, molecular modeling, and geometric optimization. Some results involving planar arrangements of curves have been presented in a companion chapter in this book, and are extended in this chapter to higher dimensions.

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1 Introduction

The *arrangement* of a finite collection Γ of geometric objects in $\mathbb{R}^d$, denoted as $\mathcal{A}(\Gamma)$, is the decomposition of the space into relatively open connected cells of dimensions $0, \ldots, d$ induced by Γ, where each cell is a maximal connected set of points lying in the intersection of a fixed subset of Γ. Besides being interesting in their own right, because of the rich geometric, combinatorial, algebraic, and topological structures that arrangements possess, they also lie at the heart of several geometric problems arising in a wide range of applications including robotics, computer graphics, molecular modeling, and computer vision. Before proceeding further, we present a few such examples.

(a) Assume that we have a robot system B with d degrees of freedom, i.e., we can represent each placement of B as a point in $\mathbb{R}^d$, and we call the space of all placements the *configuration space* of B. Suppose the three-dimensional workspace of B is cluttered with polyhedral obstacles whose shapes and locations are known. B is allowed to move freely in a motion that traces a continuous path in the configuration space, but B has to avoid collision with the obstacles. For each combination of a geometric feature (vertex, edge, face) of an obstacle and a similar feature (face, edge, vertex) of B, define their *contact surface* as the set of all points in $\mathbb{R}^d$ that represent placements of B at which contact is made between these specific features. Let Γ be the set of all contact surfaces. Let Z be a point corresponding to a given initial *free* placement of B, i.e., a placement at which it does not intersect any obstacle. Then the set of all free placements of B that can be reached from Z via a collision-free continuous motion corresponds to the cell containing Z in the arrangement of the contact surfaces. Thus, the problem of determining whether there exists a collision-free path from an initial configuration I to a final configuration F is equivalent to determining whether I and F lie in the same cell of $\mathcal{A}(\Gamma)$. This close relationship between arrangements and motion planning has led to considerable work on arrangements; see, for example, [22, 52, 196, 205, 221, 222, 228, 232, 304]. If we want to compute the set of all placements reachable from the initial placement I, the *combinatorial complexity* of the cell in $\mathcal{A}(\Gamma)$ containing I, i.e., the total number of lower-dimensional faces appearing on its boundary, serves as a trivial lower bound for the running time. It turns out that in many instances one can design motion-planning algorithms whose performance almost matches this bound.

(b) A molecule can be modeled as an arrangement of spheres, where the radius of each sphere depends on the atom that it models and the position of each sphere depends on the molecular structure. In the *Van der Waals model*, a molecule is a family of possibly overlapping spheres, where the radius of each sphere is determined by the van der Walls radius of the corresponding atom in the molecule. Lee and Richards [229] proposed another model, called *solvent accessible* model, which is used to study the interaction between the protein and solvent molecules. Also in this model, a molecule is modeled as a sphere, but the balls representing the solvent molecules are shrunk to points and the balls representing atoms in the protein are inflated by the same amount [289]. Even though these models ignore various

properties of molecules, they have been useful in a variety of applications. Many problems in molecular modeling can be formulated as problems related to arrangements of spheres. For example, computing the "outer surface" of the molecule corresponds to computing the unbounded cell of the corresponding arrangement of spheres. See [135, 136, 200, 255] for more details on applications of arrangements in molecular biology.

(c) Arrangements are also attractive because of their relationship with several other structures. For example, using the *duality* transform, a point $p = (p_1, \ldots, p_d)$ in $\mathbb{R}^d$ can be mapped to the hyperplane $\sum_{i=1}^{d} x_i p_i = 1$, and vice versa. This facilitates the formulation of several problems related to point configurations in terms of arrangements of hyperplanes. See [132, 102] for a small sample of such problems. The Grassmann-Plücker relation transforms k-flats in $\mathbb{R}^d$ to hyperplanes or points in $\mathbb{R}^u$, for $u = \binom{d+1}{k+1} - 1$ [70, 208]; e.g., lines in $\mathbb{R}^3$ can be mapped to hyperplanes or points in $\mathbb{R}^5$. Therefore many problems involving lines in $\mathbb{R}^3$ have been solved using hyperplane arrangements in $\mathbb{R}^5$ [307, 279, 99, 127]. The well-known combinatorial structure *oriented matroids* of rank $k+1$ are closely related to arrangements of pseudo-hyperplanes in $\mathbb{R}^k$ [70, 290], and *zonotopes* in $\mathbb{R}^d$ correspond to hyperplane arrangements in $\mathbb{R}^{d-1}$ [132, 332]. Several applications of arrangements in singularity theory, algebraic group theory, and other fields of mathematics can be found in [266, 269].

Study of arrangements of lines and hyperplanes has a long, rich history. The first paper on this topic is perhaps by J. Steiner in 1826 [309], in which he obtained bounds on the number of cells in arrangements of lines and circles in the plane and of planes and spheres in $\mathbb{R}^3$. His results have since been extended in several ways [31, 32, 33, 82, 291, 326, 330, 331]. A summary of early work on arrangements of hyperplanes can be found in the monograph and the survey paper by Grünbaum [183, 184]. Most of the work on hyperplane arrangements until the 1980s dealt with the combinatorial structure of the entire arrangement or of a single cell in the arrangement (i.e., a convex polyhedron), and with the algebraic and topological properties of the arrangement [165, 267, 268, 269]. Various substructures and algorithmic issues of hyperplane arrangements, motivated by problems in computational and combinatorial geometry, have received attention mostly during the last fifteen years.

Although hyperplane arrangements possess an immensely rich structure, many applications (e.g., the motion-planning problem and molecular models described above) call for a systematic study of higher-dimensional arrangements of patches of algebraic surfaces. For more than a century, researchers in algebraic geometry have studied arrangements of algebraic surfaces, but their focus has largely been on algebraic and combinatorial issues rather than on algorithmic ones. Considerable progress has been made on all fronts during the last fifteen years.

It is beyond the scope of a survey paper, or even of a book, to cover all aspects of arrangements. In this chapter we will survey combinatorial and algorithmic problems on arrangements of (hyper)surfaces (or of surface patches) in real affine space $\mathbb{R}^d$. (Hyperplane

arrangements in complex space have also been studied; see, e.g., [69, 269].) We will assume that d is a small constant, that the surfaces are algebraic and their degree is bounded by a constant, and that any surface patch is a semialgebraic set defined by a Boolean combination of a constant number of polynomials of constant maximum degree. There has been some recent work on combinatorial and algorithmic issues involving arrangements of more general surfaces, known as semi-pfaffian sets, which include graphs of trigonometric, exponential, or logarithmic functions on bounded domains [175, 223]. We also note that a study of algebraic and topological problems on arrangements of algebraic surfaces can be found in [71]. In this survey we will mostly review the known results on the combinatorial complexity of various substructures of arrangements, the known algorithms for computing these substructures, and the geometric applications that benefit from these results. Many other combinatorial problems related to arrangements are discussed in [70, 166, 178, 272, 271, 306, 332]. An excellent source on combinatorial and algorithmic results on arrangements of hyperplanes is the book by Edelsbrunner [132]. The book by the authors [304] covers some of the topics discussed here in detail. Other survey papers on arrangements include [197, 190, 204, 302].

This survey is organized as follows. In Section 2 we define arrangements formally, state the assumptions we will be making in this survey, and discuss the known bounds on the complexity of the entire arrangement. Sections 3–10 discuss combinatorial complexities of various substructures of arrangements. Section 11 discusses several methods for representing arrangements. Section 12 describes algorithms for computing the entire arrangement, and Section 13 reviews algorithms for computing various substructures of arrangements. We discuss a few applications of arrangements in Section 14.

2 Preliminaries

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of n (hyper)surfaces or surface patches in $\mathbb{R}^d$. The set Γ induces a decomposition of $\mathbb{R}^d$ into connected cells (or faces), called the *arrangement* of Γ and denoted $\mathcal{A}(\Gamma)$, so that each cell is a maximal connected subset of the intersection of a fixed (possibly empty) subset of surface patches that avoids all other surface patches. Thus a d-dimensional cell is a maximal connected region that does not meet any surface patch of Γ. The *combinatorial complexity* of $\mathcal{A}(\Gamma)$ is the total number of cells, of all dimensions, in $\mathcal{A}(\Gamma)$. The combinatorial complexity of a k-dimensional cell C in $\mathcal{A}(\Gamma)$ is the number of cells of $\mathcal{A}(\Gamma)$ of dimension less than k that are contained in the boundary of C.

We assume that Γ satisfies the following assumptions.

(A1) Each $\gamma_i \in \Gamma$ is a semialgebraic set of constant description complexity. The local dimension of every point in γ_i is $d - 1$.[1]

[1] A subset of $\mathbb{R}^d$ is called a *real semialgebraic set* if it is obtained as a finite Boolean combination of sets of the form $\{f = 0\}$ or $\{f > 0\}$ for d-variate polynomials f. A semialgebraic set has *constant description complexity* if it can be described in terms of a constant number of polynomials, with a constant bound on the degrees of the corresponding polynomials.

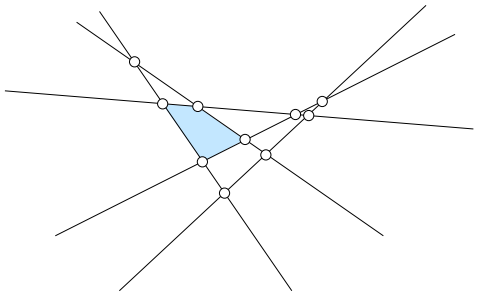

Figure 1: An arrangement of lines.

(A2) Each $\gamma_i \in \Gamma$ is of the form $(Q_i = 0) \wedge F_i(P_{i_1}\sigma_{i_1}0, P_{i_2}\sigma_{i_2}0, \ldots, P_{i_u}\sigma_{i_u}0)$. Here u is a constant; $\sigma_{i_j} \in \{\geq, \leq\}$; F_i is a Boolean formula; $Q_i, P_{i_1}, \ldots, P_{i_u} \in \mathbb{R}[x_1, \ldots, x_d]$; and the degrees of Q_i, P_{i_j} are at most b, for some constant b. Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$.

We will refer to semialgebraic sets satisfying (A1) and (A2) a *(hyper)surface patch* in $\mathbb{R}^d$. If γ_i is simply the zero set of Q_i, we will call γ_i a *(hyper)surface*. Using a stratification algorithm [71, 210], we can decompose each γ_i into a constant number connected surface patches so that the interior of each patch is *smooth* and each of them satisfies (A1) and (A2) with a different, possibly larger, value of b. We can also assume that each resulting patch is *monotone* in $x_1, \ldots, x_{d-1}$ (i.e., any line parallel to the x_d-axis intersects it in at most one point). In some cases, the resulting collection may also include *vertical* surface patches, namely, patches whose projection on the hyperplane $x_d = 0$ has dimension $\leq d-2$. However, in most of the presentation we will assume that no vertical patches exist.

An arrangement of hyperplanes is called *simple* if any d of the hyperplanes intersect in exactly one point, and no $d+1$ of them have a nonempty intersection. In a simple arrangement, a k-dimensional cell is contained in $d-k$ hyperplanes. We will also need a similar concept for arrangements of surface patches. An arrangement $\mathcal{A}(\Gamma)$ satisfying assumptions (A1) and (A2) is said to be in *general position* if the coefficients of the polynomials defining the surface patches in Γ and their boundaries are algebraically independent[2] over the rationals; otherwise, $\mathcal{A}(\Gamma)$ is called *degenerate*. This condition ensures that no degeneracy occurs among the surface patches, such as too many surface patches with a common point, tangencies or overlaps between different intersections of subsets of the surface patches, etc. We note that this definition of general position is quite strong (e.g., surfaces defined by polynomials with integer coefficients are not in general position). In all the applications only a much weaker version of general position is required, which rules out a specific list of forbidden degenerate situations. If $\mathcal{A}(\Gamma)$ is in general position, then any d surface patches of Γ intersect in at most s points for some constant s depending on d and b. By Bezout's theorem [208], $s \leq b^d$.

[2] A set $\{x_1, \ldots, x_k\}$ of real numbers is *algebraically independent* (over the rationals) if no k-variate polynomial with integer coefficients vanishes at $(x_1, x_2, \ldots, x_k)$.

If $\mathcal{A}(\Gamma)$ is degenerate, we can perturb the coefficients of the polynomials in $\mathcal{Q}$ by various infinitesimals so that the coefficients of the perturbed polynomials are in extensions of the reals that are fields of *Puiseux Series* in these infinitesimals, and so that the resulting surface patches are in general position. Moreover, it can be shown that, as far as worst-case bounds are concerned, the perturbation may reduce the combinatorial complexity of any cell of the arrangement by at most a constant factor [304, 303, 283]. Actually, in many cases the size of a substructure of Γ has maximum complexity when $\mathcal{A}(\Gamma)$ is in general position. This observation allows us to restrict our attention to arrangements in general position while investigating the combinatorial complexity of substructures of arrangements.

However, in order to achieve the general position defined above, the perturbation scheme has to introduce a different infinitismal for each coefficient, which makes any algorithm based on this perturbation scheme impractical. Fortunately, most of the algorithms involving arrangements either work for any degenerate arrangement or require a considerably weaker definition of general position, e.g., the intersection of any k surface patches is either empty or a $(d-k)$-dimensional set, all surface patches intersect "properly," etc. The perturbation scheme required by an algorithm depends on the degenerate situations that it wants to rule out. Several constructive perturbation schemes have been proposed that use only a few infinitismals [66, 146, 159, 160, 167]. Although these schemes cannot handle all the cases, they work for a wide range of applications. The paper by Seidel [300] contains a detailed discussion on "linear" perturbations and its applications in geometric algorithms. Many algorithms have also been proposed to handle degeneracies directly without resorting to perturbations; see e.g. [59, 83]. We will, nevertheless, use the strong definition of general position, defined above, in order to simplify the exposition, and refer the reader to original papers for specific *general-position assumptions* required by different algorithms.

In the light of the preceding discussion, and since we are mainly interested in asymptotic bounds, we will make the following additional assumptions on the surface patches in Γ, without any real loss of generality, whenever required.

(A3) Each surface patch in Γ is connected and monotone in $x_1, \ldots, x_{d-1}$, and its relative interior is smooth.

(A4) The surface patches in Γ are in general position.

(A5) Any d surface patches in Γ intersect in at most s points, for some constant s. (This is a consequence of the preceding assumptions, but is stated to introduce s explicitly.)

Generally, we will be stating assumptions (A1) and (A2), but most of the proofs and algorithms sketched in the survey will also make assumptions (A3)–(A5).

Assumptions (A1)–(A3) imply that we can regard each surface patch γ as the graph of a partially defined $(d-1)$-variate function $x_d = \gamma(x_1, \ldots, x_{d-1})$ of constant description complexity. We will refer to the projection of γ onto the hyperplane $x_d = 0$ as the *domain*, denoted γ^*, of γ (over which the function γ is defined). The boundary of γ^*, called the

domain boundary, is a collection of $O(1)$ $(d-2)$-dimensional surface patches in $\mathbb{R}^{d-1}$ satisfying assumptions corresponding to (A1)–(A2). Abusing the notation slightly, we will not distinguish between the surface patch γ and the underlying function $\gamma(x_1, \ldots, x_{d-1})$.

The most fundamental question in the combinatorial study of an arrangement $\mathcal{A}(\Gamma)$ of surfaces is to prove a bound on the combinatorial complexity, $f(\Gamma)$, of $\mathcal{A}(\Gamma)$.

In 1826, Steiner [309] studied the complexity of arrangements of lines and circles in $\mathbb{R}^2$ and of planes and spheres in $\mathbb{R}^3$. His results on arrangements of planes can be summarized as follows. Let Γ be a set of n planes in $\mathbb{R}^3$ so that Γ can be decomposed into k parallel families, containing $n_1, \ldots, n_k$ planes in each respective family, and the parallel families are in general position. Steiner proved the following bounds on the number of vertices, edges, two-dimensional faces, and three-dimensional cells of $\mathcal{A}(\Gamma)$.

σ_3	vertices,
$\sigma_2 + 3\sigma_3$	edges,
$-\sigma_2 + 3\sigma_3$	bounded edges,
$\sigma_1 + 2\sigma_2 + 3\sigma_3$	2-faces,
$\sigma_1 - 2\sigma_2 + 3\sigma_3$	bounded 2-faces,
$1 + \sigma_1 + \sigma_2 + \sigma_3$	3-cells,
$-1 + \sigma_1 - \sigma_2 + \sigma_3$	bounded 3-cells.

Here $\sigma_1 = \sum_{i=1}^{k} n_i = n$, $\sigma_2 = \sum_{i<j} n_i n_j$, and $\sigma_3 = \sum_{i<j<k} n_i n_j n_k$. In particular, if $n_i = 1$ for $1 \leq i \leq k$, i.e., Γ is a set of $n = k$ planes in general position, then $\mathcal{A}(\Gamma)$ has $\binom{n}{3}$ vertices, $\binom{n}{2} + 3\binom{n}{3}$ edges, $n^2 + 3\binom{n}{3}$ 2-faces, and $1 + n + \binom{n}{2} + \binom{n}{3}$ 3-cells. Later Roberts [291] extended Steiner's formula to count the number faces in arbitrary arrangements of planes (allowing all kinds of degeneracies) in $\mathbb{R}^3$, using the inclusion-exclusion principle. Brousseau [81] used a plane-sweep argument to count the number of faces in arrangements of planes in $\mathbb{R}^3$. (A similar argument was used by Hadwiger [194] to derive Euler's formula for convex polytopes.) His method was later extended by Alexanderson and Wetzel [33].

Buck [82] was the first to bound the combinatorial complexity of hyperplane arrangements in higher dimensions. In more recent work, Zaslavsky [330, 331] studied hyperplane arrangements; he used the Möbius inversion formula and lattice theory to count the number of cells of all dimensions in (possibly degenerate) hyperplane arrangements. Let Γ be a set of n hyperplanes in $\mathbb{R}^d$. Let $\varphi_k(\Gamma)$ denote the number of k-cells in $\mathcal{A}(\Gamma)$. Zaslavsky [330] and Las Vergnas [227] proved that for non-simple arrangements, $\varphi_k(\Gamma)$, depends on the underlying matroid structure. There are several results on bounding $\varphi_k(\Gamma)$ in nonsimple hyperplane arrangements. For example, Fukuda *et al.* [173] proved that the mean number of $(k-1)$-cells bounding a k-cell in an arrangement of n hyperplanes is less than $2k$, which immediately implies that $\phi_k(\Gamma) \leq \binom{d}{k}\varphi_d(\Gamma)$. See [173, 252, 292] for some other results of this type.

In summary, the following theorem gives a bound on the combinatorial complexity of hyperplane arrangements.

Theorem 2.1 (Buck [82]) *Let Γ be a set of n hyperplanes in $\mathbb{R}^d$. For any $0 \leq k \leq d$,*

$$\varphi_k(\Gamma) \leq \binom{n}{d-k} \sum_{i=0}^{k} \binom{n-d+k}{i}.$$

The maximum is attained when $\mathcal{A}(\Gamma)$ is simple.

Proof: We will prove the theorem for simple arrangements. Let $\varphi_k(n,d)$ denote the number of k-cells in a simple d-dimensional arrangement of n hyperplanes. Let $\mathcal{A}(\Gamma)$ be a simple arrangement of a set Γ of n hyperplanes in $\mathbb{R}^d$. Fix a subset $R \subseteq \Gamma$ of $d-k$ hyperplanes, and let $\Pi = \bigcap_{h \in R} h$; Π is a k-flat. Set $\Gamma|_\Pi = \{h \cap \Pi \mid h \in H \setminus R\}$. The k-cells of $\mathcal{A}(\Gamma|_\Pi)$ are the same as the k-cells of $\mathcal{A}(\Gamma)$ that lie in Π. Since $\mathcal{A}(\Gamma|_\Pi)$ is a simple k-dimensional arrangement and there are $\binom{n}{d-k}$ subsets of Γ of size $d-k$, we obtain

$$\varphi_k(\Gamma) = \binom{n}{d-k} \varphi_k(n-d+k, k).$$

By Euler's relation for cell complexes in affine space (see e.g., [132]),

$$\sum_{k=0}^{d} (-1)^k \varphi_k(\Gamma) = (-1)^d,$$

therefore

$$\sum_{k=0}^{d} (-1)^k \binom{n}{d-k} \varphi_k(n-d+k, k) = (-1)^d.$$

The above equality can be rewritten as

$$\sum_{k=0}^{d} (-1)^k \binom{n}{k} \varphi_{d-k}(n-k, d-k) = 1. \tag{2.1}$$

We claim that

$$\varphi_d(n,d) = \sum_{i=0}^{d} \binom{n}{i}, \tag{2.2}$$

which will complete the proof of the theorem.

Since (2.1) is a recurrence with $\varphi_0(n,0) = 1$, there is a unique solution to the recurrence.

By induction on d and substituting (2.2) in (2.1), we obtain

$$\begin{aligned}\sum_{k=0}^{d}(-1)^k\binom{n}{d-k}\left[\sum_{i=0}^{d-k}\binom{n-k}{i}\right] &= \sum_{k=0}^{d}\sum_{i=0}^{d-k}(i-1)^k\binom{n}{k+i}\binom{k+i}{k} \\ &= \sum_{i=0}^{d}\sum_{k=0}^{i}(-1)^k\binom{n}{i}\binom{i}{k} \\ &= 1+\sum_{i=1}^{d}\binom{n}{i}\sum_{k=0}^{i}(-1)^k\binom{i}{k} \quad = 1.\end{aligned}$$

This completes the proof of the theorem. □

For arrangements $\mathcal{A}(\Gamma)$ of a set Γ of surfaces satisfying assumptions (A1) and (A2), obtaining a bound on $f(\Gamma)$ is not easy. If surfaces patches are in general position, in the sense defined above, it is easy to argue that $f(\Gamma) = O(n^d)$. However, it is not easy to argue that the arrangements have maximum complexity when the surface patches are in general position (this is due to the complicated algebraic structures that can arise in degenerate settings). Heintz *et al.* [211] proved that $f(\Gamma) = (nb)^{O(d)}$. A lower bound of $\Omega((nb/d)^d)$ is not difficult to prove. Warren [323] had proved that the number of d-dimensional cells in an arrangement of n hypersurfaces, each of degree $\leq b$, in $\mathbb{R}^d$ is $O((nb/d)^d)$. This bound also follows from the results by Milnor [259], Petrovskiĭ and Oleĭnik [281], and Thom [317]. Using a perturbation argument, Pollack and Roy [283] generalized Warren's result and proved that the number of cells of all dimensions in an arrangement of n hypersurfaces is $O((nb/d)^d)$. An easy consequence of their result is the following theorem.

Theorem 2.2 *Let Γ be a set of n surface patches in $\mathbb{R}^d$ satisfying assumptions (A1) and (A2). Then*

$$f(\Gamma) = \left(\frac{O(nb)}{d}\right)^d.$$

A recent result by Basu *et al.* [65] can be used to extend the above theorem as follows. Let Σ be a k-dimensional algebraic surface of degree at most b. Then the number of cells in the subdivision of Σ induced by Γ is at most $O(n^k b^d)$.

Improved bounds on the complexity of the arrangement can be proved in some special cases. For example, if Γ is a set of n $(d-1)$-simplices in $\mathbb{R}^d$ that form the boundaries of k convex polytopes, then $f(\Gamma) = O(n^{\lfloor d/2 \rfloor}k^{\lceil d/2 \rceil})$ [44]. See [121] for improved bounds in a few other cases. A concept closely related to the combinatorial complexity of arrangements is the number of *realizable sign patterns* of a family of polynomials. Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$ be a set of d-variate polynomials as defined above, and let Γ be the family of the zero-sets

of the polynomials in $\mathcal{Q}$. We can define $\sigma_i(\mathbf{x})$, for a point $\mathbf{x} \in \mathbb{R}^d$, as follows.

$$\sigma_i(\mathbf{x}) = \begin{cases} -1 & Q_i(\mathbf{x}) < 0, \\ 0 & Q_i(\mathbf{x}) = 0, \\ +1 & Q_i(\mathbf{x}) > 0. \end{cases}$$

Since $\sigma_i(\mathbf{x})$ remains the same for all points $\mathbf{x}$ in a single cell of $\mathcal{A}(\Gamma)$, we can define the *sign sequence* for each cell $\sigma(C) = \langle \sigma_1(\mathbf{x}), \sigma_2(\mathbf{x}), \ldots, \sigma_n(\mathbf{x}) \rangle$ for any point $\mathbf{x} \in C$. A sign sequence σ is *realized* by $\mathcal{A}(\Gamma)$ if there is a cell $C \in \mathcal{A}(\Gamma)$ with $\sigma = \sigma(C)$. A well-studied question in algebraic geometry is to bound the number of sign sequences that can be realized by a set of polynomials [34]. Obviously, $f(\Gamma)$ is an upper bound on this quantity.

3 Lower Envelopes

Definitions and preliminary results. In chapter DS-?? we reviewed lower envelopes of arcs in the plane and showed the relationship between such envelopes and Davenport–Schinzel sequences, which eventually led to the derivation of tight or almost tight bounds on the complexity of these structures. In this section we study lower envelopes of surface patches in higher dimensions. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of surface patches in $\mathbb{R}^d$ satisfying assumptions (A1)–(A3). If we regard each surface patch as the graph of a partially defined function, the *lower envelope* of Γ, denoted $L(\Gamma)$ (or L for brevity), is defined as the graph of the partially defined function

$$L_\Gamma(\mathbf{x}) = \min_{1 \le i \le n} \gamma_i(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{d-1};$$

$\gamma_i(\mathbf{x})$ is set to $+\infty$ if $\mathbf{x} \notin \gamma_i^*$. The *upper envelope* $U(\Gamma)$ of Γ is defined as the graph of the partially defined function

$$U_\Gamma(\mathbf{x}) = \max_{1 \le i \le n} \gamma_i(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{d-1};$$

$\gamma_i(\mathbf{x})$ is set to $-\infty$ if $\mathbf{x} \notin \gamma_i^*$. We can extend the definitions of lower and upper envelopes even if Γ satisfies only (A1) and (A2). We can decompose each γ_i into $O(1)$ connected patches, each of which is monotone in $x_1, \ldots, x_{d-1}$ directions and satisfies (A1) and (A2). Let Γ' denote the resulting set of surface patches. We define $L(\Gamma) = L(\Gamma')$ and $U(\Gamma) = U(\Gamma')$.

$L(\Gamma)$ induces a partition of $\mathbb{R}^{d-1}$ into maximal connected ($(d-1)$-dimensional) regions so that $L(\Gamma)$ is attained by a fixed (possibly empty) subset of Γ over the interior of each such region. The boundary of such a region consists of points at which $L(\Gamma)$ is attained by at least two of the surface patches or by the relative boundary of at least one surface. Let $\mathcal{M}(\Gamma)$ denote this subdivision of $\mathbb{R}^{d-1}$, which we call the *minimization diagram* for the collection Γ. A *face* of $\mathcal{M}(\Gamma)$ is a maximal connected region over which $L(\Gamma)$ is attained by the same set of functions and/or relative boundaries of function graphs in Γ. Note that if a face $f \in \mathcal{M}(\Gamma)$ lies on the boundary of the domain of a surface in Γ, then f

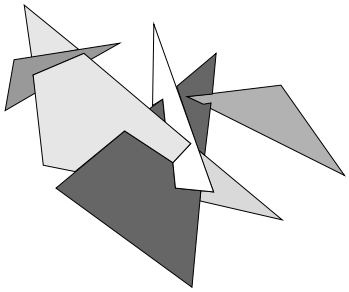

Figure 2: Lower envelope of triangles in $\mathbb{R}^3$, as viewed from below.

may not correspond to any face of $L(\Gamma)$. However, if f lies in the relative interior of the domains of all the relevant surface patches, f is the projection of a face $\hat{f}$ of $L(\Gamma)$. The *combinatorial complexity* of $L(\Gamma)$, denoted $\kappa(\Gamma)$, is the number of faces of all dimensions in $\mathcal{M}(\Gamma)$. For an infinite family $\mathbf{G}$ of surface patches satisfying assumptions (A1)–(A2), we define $\kappa(n, d, \mathbf{G}) = \max \kappa(\Gamma)$, where the maximum is taken over all subsets Γ of $\mathbf{G}$ of size n. If $\mathbf{G}$ is the set of all surface patches satisfying (A1)–(A3) or if $\mathbf{G}$ is obvious from the context, we will simply write $\kappa(n, d)$. The *maximization diagram* is defined as the subdivision of $\mathbb{R}^{d-1}$ induced, in the same manner, by the upper envelope $U(\Gamma)$ of Γ.

As discussed in Chapter DS-??, the complexity of the lower envelope of n arcs in the plane, each pair of which intersects in at most s points, is at most $\lambda_{s+2}(n)$, the maximum length of an (n, s)-Davenport–Schinzel sequence (see also [304]). Extending to higher dimensions, it was conjectured that the complexity of the lower envelopes of a family of n surface patches satisfying (A1)–(A2) is $O(n^{d-2}\lambda_q(n))$ for a constant $q \geq 0$. If Γ is a set of n hyperplanes in $\mathbb{R}^d$, then the Upper Bound Theorem implies that the complexity of $L(\Gamma)$ is $\Theta(n^{\lfloor d/2 \rfloor})$ [332]. Let Δ be the set of all $(d-1)$-simplices in $\mathbb{R}^d$. Extending the lower-bound construction by Wiernik and Sharir [327] to higher dimensions, one can prove that $\kappa(n, d, \Delta) = \Omega(n^{d-1}\alpha(n))$. This suggests we cannot hope to aim for an $o(n^{d-1})$ bound on $\kappa(n, d)$ for general surface patches. At the end of this section we will discuss some cases in which better bounds on $\kappa(n, d)$ can be proved.

Using a divide-and-conquer approach, Pach and Sharir [273] proved that, for a set Γ of n simplices in $\mathbb{R}^d$, the number of $(d-1)$-dimensional faces in $\mathcal{M}(\Gamma)$ is $O(n^{d-1}\alpha(n))$. Roughly speaking, they divide Γ into subsets Γ_1, Γ_2, each of size at most $\lceil n/2 \rceil$, and bound the number of $(d-1)$-dimensional faces of $\mathcal{M}(\Gamma_1), \mathcal{M}(\Gamma_2)$ recursively. They prove that the number of $(d-1)$-dimensional faces in $\mathcal{M}(\Gamma)$ is $|\mathcal{M}(\Gamma_1)| + |\mathcal{M}(\Gamma_2)| + O(n^{d-1}\alpha(n))$, thereby obtaining the claimed bound. Edelsbrunner [133] extended their result to give the same asymptotic bound for the number of faces of all dimensions. Simpler proofs for this bound were proposed by Sharir and Agarwal [304] and Tagansky [314]. Roughly speaking, both proofs proceed by induction on d, and they bound the change in the complexity of the minimization diagram as a simplex is inserted into Γ.

The main complexity bound. All the aforementioned proofs rely crucially on the fact that if Γ is a set of surface patches in general position, then any triple of surface patches intersect in at most one point. These proofs do not extend to the case when a tripe intersects in two or more points. Halperin and Sharir [202] proved a near-quadratic bound on $\kappa(n,3)$ for the case when $s \leq 2$. Sharir [303] extended their approach to higher values of s and d. Their results are stated in the following theorem.

Theorem 3.1 (Halperin and Sharir [202]; Sharir [303]) *Let Γ be a set of n surface patches in $\mathbb{R}^d$ satisfying assumptions (A1)-(A2). Then $\kappa(n,d) = O(n^{d-1+\varepsilon})$, for any $\varepsilon > 0$. The constant of proportionality depends on ε, d, b (and s).*

Proof: We will sketch the proof for a set of bivariate surface patches in $\mathbb{R}^3$ satisfying assumptions (A1)–(A5) with $s = 2$, i.e., a triple of surface patches intersect in at most two points. For a pair of surface patches $\gamma_i, \gamma_j \in \Gamma$, let β_{ij} denote the intersection arc $\gamma_i \cap \gamma_j$. If β_{ij} is not x_1-monotone, we decompose it at its x_1-extremal points; each intersection arc is thereby decomposed into $O(1)$ pieces. If any of these points appears on the lower envelope, we regard it as a vertex on the envelope and its projection as a vertex on the minimization diagram.

Since Γ is in general position, it suffices to bound the number of vertices in $\mathcal{M}(\Gamma)$. Indeed, a higher dimensional face f of $\mathcal{M}(\Gamma)$ must be incident to a vertex v of $\mathcal{M}(\Gamma)$, and we can charge f to v. By the general-position assumption, each vertex is charged only a constant number of times. For a subset $R \subseteq \Gamma$, let $\phi^*(R)$ denote the number of vertices in $\mathcal{M}(R)$; set $\phi^*(r) = \max \phi^*(R)$, where the maximum is taken over all subsets of Γ of size r.

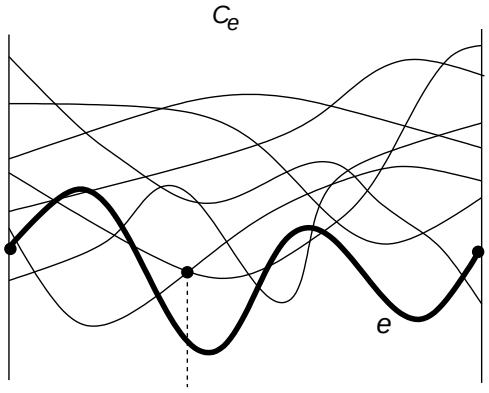

Figure 3: Vertical cylinder C_e and the vertical cross-section Γ_e of Γ.

We call a vertex of $\mathcal{M}(\Gamma)$ a *boundary* vertex if it lies on the boundary $\partial\gamma_i^*$ of the domain of a surface γ_i; otherwise we call it an *inner* vertex. The number of boundary vertices is $O(n\lambda_q(n))$, where q is a constant depending on b, the maximum degree of surface patches and their boundaries. Indeed, let C_e be the vertical cylinder erected on an edge e of the

boundary $\partial\gamma_i^*$, i.e., $C_e = e \times \mathbb{R}$. Define $\Gamma_e = \{\gamma \cap C_e \mid \gamma \in \Gamma \setminus \{\gamma_i\}\}$, which is a collection of $O(n)$ arcs; see Figure 3. Each arc in Γ_e satisfies assumptions (A1)–(A3) (with $d = 2$, and with larger, but still constant, parameters b and s). It is easily seen that a boundary vertex of $\mathcal{M}(\Gamma)$ appearing on e is a vertex of $\mathcal{M}(\Gamma_e)$. If the arcs in Γ_e intersect in at most $q-2$ points, $O(\lambda_q(n))$ boundary vertices lie on e. Summing over all $O(n)$ edges of domain boundaries of Γ, we obtain the desired bound on the number of boundary vertices.

We call an inner vertex *regular* if it is not an x_1-extremal vertex of any of the three intersection curves. The number of irregular vertices is obviously $O(n^2)$. For a subset $R \subseteq \Gamma$, let $\phi(R)$ denote the number of regular (inner) vertices in $\mathcal{M}(R)$, and let $\phi(r) = \max_{|R|=r} \phi(R)$. The above discussion implies that

$$\phi^*(\Gamma) \leq \phi(\Gamma) + O(n\lambda_q(n)).$$

Next, we derive a recurrence for $\phi(\Gamma)$, which will solve to $O(n^{2+\varepsilon})$. Fix a regular vertex v of $\mathcal{M}(\Gamma)$. let $\hat{v}$ be the corresponding vertex of $L(\Gamma)$ (since v is an inner vertex, $\hat{v}$ is well defined). Suppose $\hat{v}$ is one of the two intersection points of three surface patches, say, $\gamma_1, \gamma_2, \gamma_3$. Assume, without loss of generality, that if $|\gamma_1 \cap \gamma_2 \cap \gamma_3| = 2$, then the x_1-coordinate of the other intersection point of γ_1, γ_2, and γ_3 is larger than that of $\hat{v}$. Since $\hat{v}$ is a regular vertex, one of the three pairwise-intersection curves β_{ij}, say β_{12}, lies above $L(\Gamma)$ in the halfspace $x_1 < x_1(v)$ in a sufficiently small neighborhood of $\hat{v}$. We mark on β_{12} the intersection points of β_{12} with other surface patches of Γ and the points that lie above the boundaries of other surface patches in Γ.

We fix a parameter $t = t(\varepsilon)$ and follow β_{12} in the $(-x_1)$-direction, starting from $\hat{v}$, until one of the following three events occurs:[3]

(C1) we reach the left endpoint of β_{12};

(C2) β_{12} appears on $L(\Gamma)$; or

(C3) we crossed t marked points on β_{12}.

We call v a vertex of type (Ci), for $i = 1, 2, 3$, if we first reach an event of type (Ci). If (C1) occurs, we charge v to the left endpoint of β_{12}. Since each endpoint is charged at most twice, the total number of regular vertices of type (C1) is $O(n^2)$. If (C2) occurs, then we must have passed above the boundary of γ_3 while following β_{12} because β_{12} lies strictly above γ_3 in the halfspace $x_1 < x_1(v)$. Let w be the marked point on β_{12} that lies above $\partial\gamma_3^*$ and that we have visited. We charge v to w. Suppose w lies above an edge e of $\partial\gamma_3^*$. We can define C_e and Γ_e as above; then w is a vertex of $\mathcal{A}(\Gamma_e)$. Since (C2) occurred before (C3), at most t marked points lie on β_{12} between v and w, which implies that less than t arcs of Γ_e lie below w. As shown in [301], the number of vertices of $\mathcal{A}(\Gamma_e)$ that lie above at most

[3] If the x_1-coordinate of the other intersection point of γ_1, γ_2, and γ_3 were smaller than that of $\hat{v}$, we would have traced β_{12} in the $(+x_1)$-direction.

t arcs is $O(t\lambda_q(n))$. Summing over all edges of domain boundaries, the number of marked points on intersection arcs to which a vertex of type (C2) is charged is $O(nt\lambda_q(n))$. Since each marked point is charged $O(1)$ times, the number of type (C2) vertices is $O(nt\lambda_q(n))$.

Finally, if (C3) occurs, then we charge $1/t$ to each marked point on β_{12} that we visited. Each marked point will be charged only $O(1/t)$ units, and each such marked point lies above at most t surface patches of Γ. Theorem 6.1 in Section 6 implies that the number of such marked points, summed over all intersection curves, is $O(t^3\phi^*(n/t))$. The total number of vertices of type (C3) is thus

$$O(1/t) \cdot O(t^3\phi^*(n/t)) = O(t^2\phi(n/t) + n\lambda_q(n)).$$

Hence, we obtain the following recurrence for $\phi(n)$:

$$\phi(n) \leq At^2\phi\left(\frac{n}{t}\right) + Btn\lambda_q(n),$$

where A and B are constants (depending on b). The solution of the above recurrence is

$$\phi(n) = O(tn^{1+\log_t A}\lambda_q(n)).$$

If $t = t(\varepsilon)$ is chosen sufficiently large, then $\phi(n) = O(n^{2+\varepsilon})$. This proves the theorem for $d = 3, s = 2$.

For $s > 2$, Sharir [301] introduces the notion of *index* of a regular vertex. The index of a vertex $v \in \bigcap_{i=1}^{3} \gamma_i$ is the number of points of $\bigcap_{i=1}^{3} \gamma_i$ whose x_1 coordinates are less than that of v. For $0 \leq j < s$, let $\phi^{(j)}(\Gamma)$ be the number of regular vertices in $L(\Gamma)$ of index j. Then $\phi(\Gamma) = \sum_{j=0}^{s-1} \phi^{(j)}(\Gamma)$.

Modifying the above argument slightly, Sharir derived a system of recurrences for the quantities $\phi^{(j)}(\Gamma)$, for $j < s$. There are three main differences. First, the tracing of β_{12} is always done in the decreasing x_1-direction. Second, the value of the parameter t now depends on j and is denoted by t_j, Third, there is one more stopping criterion:

(C4) β_{12} intersects γ_3; let z be the (first) intersection point.

Using the fact that the index of z is $\leq j-1$ and that at most t_j surface patches lie below z, Sharir derives the following recurrence for $\phi^{(j)}(n) = \max_{|\Gamma|=n} \phi^{(j)}(\Gamma)$.

$$\phi^{(j)}(n) \leq A_j t_j^2 \phi^*\left(\frac{n}{t_j}\right) + B_j\left(t_j^3\phi^{(j-1)}\left(\frac{n}{t_j}\right) + nt\lambda_q(n)\right).$$

By expanding this system of recurrences and by choosing the values of t_j carefully, Sharir proved that the solution of this system satisfies

$$\phi^*(n) = O(n^{2+\varepsilon}).$$

The theorem is proved in higher dimensions by induction on d, using a similar charging scheme. See the original paper by Sharir for details. □

Open Problem 1 *Let Γ be a set of n surface patches in $\mathbb{R}^d$ satisfying assumptions (A1) and (A2). Is $\kappa(n,d) = O(n^{d-2}\lambda_q(n))$, where q is a constant depending on d and b?*

Bounds in special cases. As noted above, sharper bounds are known on the complexity of lower envelopes in some special cases; see [294, 304]. For example, if Γ is a set of pseudo-planes in $\mathbb{R}^3$, i.e., each triple of surfaces intersects in at most one point and the intersection of a pair of surfaces in Γ is a single (closed or unbounded) Jordan curve, then $\kappa(\Gamma) = O(n)$. On the other hand, if Γ is a set of pseudo-spheres, i.e., each triple intersects in at most two points and the intersection curve of any pair is a single Jordan curve, then $\kappa(\Gamma) = O(n^2)$. If Γ is a family of hypersurfaces in $\mathbb{R}^d$, a sharper bound on $\kappa(\Gamma)$ can be proved using the so-called *linearization* technique. Here is a sketch of this technique.

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of hypersurfaces of degree at most b, i.e., each γ_i is the zero set of a d-variate polynomial Q_i of degree at most b. Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$. We say that Γ admits a *linearization* of dimension k if, for some $p > 0$, there exists a $(d+p)$-variate polynomial

$$g(\mathbf{x}, \mathbf{a}) = \psi_0(\mathbf{a}) + \psi_1(\mathbf{a})\varphi_1(\mathbf{x}) + \psi_2(\mathbf{a})\varphi_2(\mathbf{x}) + \cdots + \psi_{k-1}(\mathbf{a})\varphi_{k-1}(\mathbf{x}) + \varphi_k(\mathbf{x}),$$

for $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{a} \in \mathbb{R}^p$, so that, for each $1 \le i \le n$, we have $Q_i(\mathbf{x}) = g(\mathbf{x}, \mathbf{a}_i)$ for some $\mathbf{a}_i \in \mathbb{R}^p$. Here each $\psi_j(\mathbf{a})$, for $0 \le j \le k$, is a p-variate polynomial, and each $\varphi_j(\mathbf{x})$, for $1 \le j \le k+1$, is a d-variate polynomial. It is easily seen that such a polynomial representation always exists for $p \le d^{b+1}$—let the φ's be the monomials that appear in at least one of the polynomials of $\mathcal{Q}$, and let $\psi_j(\mathbf{a}) = a_j$ (where we think of $\mathbf{a}$ as the vector of coefficients of the monomials).

We define a transform $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ that maps each point in $\mathbb{R}^d$ to the point

$$\varphi(\mathbf{x}) = (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \ldots, \varphi_k(\mathbf{x}));$$

the image $\varphi(\mathbb{R}^d)$ is a d-dimensional algebraic surface Σ in $\mathbb{R}^k$. For each function $Q_i(\mathbf{x}) = g(\mathbf{x}, \mathbf{a}_i)$, we define a k-variate linear function

$$h_i(\mathbf{y}) = \psi_0(\mathbf{a}_i) + \psi_1(\mathbf{a}_i)y_1 + \cdots \psi_{k-1}(\mathbf{a}_i)y_{k-1} + y_k.$$

Let $H = \{h_i = 0 \mid 1 \le i \le n\}$ be a set of n hyperplanes in $\mathbb{R}^k$. Let ξ be a vertex of $L(\Gamma)$. If ξ is incident to $\gamma_1, \ldots, \gamma_d$, then $Q_1(\xi) = \cdots = Q_d(\xi) = 0$ and $Q_{d+1}(\xi)\sigma_{d+1}0, \ldots, Q_n(\xi)\sigma_n 0$, where $\sigma_i \in \{>, <\}$. By construction, $Q_i(\xi) = h_i(\varphi(\xi))$. Let $Q \in \mathbb{R}[x_1, \ldots, x_d]$ be a d-variate polynomial. If we regard Q as a univariate polynomial in x_d and the coefficient of the leading term in Q is a positive constant, then we call Q a *positive polynomial*. If all Q_i's are positive, then, by the definition of lower envelopes, $Q_i(\xi) < 0$ for every $i > d$. Hence, $h_1(\varphi(\xi)) = \cdots h_d(\varphi(\xi)) = 0$ and $h_{d+1}(\varphi(\xi)) < 0, \ldots, h_n(\varphi(\xi)) < 0$. That is, $\varphi(\xi)$ is a vertex of $L(H) \cap \Sigma$. Since each h_i is a hyperplane in $\mathbb{R}^k$ and the degree of Σ depends only on d and b, the Upper Bound Theorem for convex polyhedra (see McMullen and Shephard [251] and Ziegler [332]) implies that the number of vertices in $\Sigma \cap L(H)$ is $O(n^{\lfloor k/2 \rfloor})$. Hence, we can conclude the following.

Theorem 3.2 *Let Γ be a collection of n hypersurfaces in $\mathbb{R}^d$, of constant maximum degree b. If Γ admits a linearization of dimension k and each surface Γ is the zero set of a positive polynomial, then $\kappa(\Gamma) = O(n^{\lfloor k/2 \rfloor})$, where the constant of proportionality depends on k, d, and b.*

We illustrate the linearization technique by giving an example. A sphere in $\mathbb{R}^d$ with center $(a_1, \ldots, a_d)$ and radius a_{d+1} can be regarded as the zero set of the polynomial $g(\mathbf{x}, \mathbf{a})$, where

$$\begin{aligned} g(\mathbf{x}, a_1, \ldots, a_{d+1}) &= [a_1^2 + \cdots + a_d^2 - a_{d+1}^2] - [2a_1 \cdot x_1] - [2a_2 \cdot x_2] - \cdots - \\ &\quad [2a_d \cdot x_d] + [x_1^2 + \cdots x_d^2] \end{aligned}$$

Thus, setting

$$\begin{aligned} \psi_0(\mathbf{a}) &= \sum_{i=1}^{d} a_i^2 - a_{d+1}^2, \quad \psi_1(\mathbf{a}) = 2a_1, \quad \cdots \quad \psi_d(\mathbf{a}) = 2a_d, \quad \psi_{d+1}(\mathbf{a}) = -1, \\ \varphi_1(\mathbf{x}) &= x_1, \quad \cdots \quad \varphi_d(\mathbf{x}) = x_d, \quad \varphi_{d+1}(\mathbf{x}) = \sum_{i=1}^{d} x_i^2, \end{aligned}$$

we obtain a linearization of dimension $d + 1$. We can therefore conclude the following:

Corollary 3.3 *Let Γ be a set of n spheres in $\mathbb{R}^d$. Then $\kappa(\Gamma) = O(n^{\lceil d/2 \rceil})$.*

The overlay of minimization diagrams. Motivated by several applications, researchers have studied the complexity of the overlay of two minimization diagrams. That is, let Γ and Γ' be two families of surface patches satisfying assumptions (A1)–(A3); set $n = |\Gamma| + |\Gamma'|$. The *overlay* of $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$ is the decomposition of $\mathbb{R}^{d-1}$ into maximal connected regions so that each region lies within a fixed pair of faces of $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$. It is conjectured that the complexity of the overlay of the two diagrams is also close to $O(n^{d-1})$. Although this conjecture is obviously true for the minimization diagrams of arcs in the plane, it is not intuitive even in $\mathbb{R}^3$ because the overlay of two planar maps with m edges each may have $\Omega(m^2)$ vertices. Edelsbrunner *et al.* [141] proved an $O(n^{d-1}\alpha(n))$ upper bound if Γ and Γ' are sets of a total of n simplices in $\mathbb{R}^d$.

Agarwal *et al.* [21] proved that the overlay of two minimization diagrams, defined for a total of n surface patches, in $\mathbb{R}^3$ has $O(n^{2+\varepsilon})$ complexity, for any $\varepsilon > 0$. Note that in $\mathbb{R}^3$, each vertex of the overlay is a vertex of $\mathcal{M}(\Gamma)$, a vertex of $\mathcal{M}(\Gamma')$, or an intersection point of an edge of $\mathcal{M}(\Gamma)$ with an edge of $\mathcal{M}(\Gamma')$. The proof in [21] establishes an upper bound on the number of intersection points by generalizing the proof technique of Theorem 3.1.

Open Problem 2 *What is the complexity of the overlay of two minimization diagrams in $\mathbb{R}^4$?*

The following problem is closely related to the overlay of minimization diagrams. Let Γ, Γ' be two sets of surface patches in $\mathbb{R}^d$ satisfying (A1)–(A2). Regarding each surface as the graph of a partially defined function, define

$$\mathcal{S}(\Gamma, \Gamma') = \left\{ \mathbf{x} \;\middle|\; L_{\Gamma'}(x_1, \ldots, x_{d-1}) \leq x_d \leq U_{\Gamma}(x_1, \ldots, x_{d-1}) \right\},$$

i.e., $\mathcal{S}(\Gamma, \Gamma')$ is the set of points lying above all surface patches of Γ' and below all surface

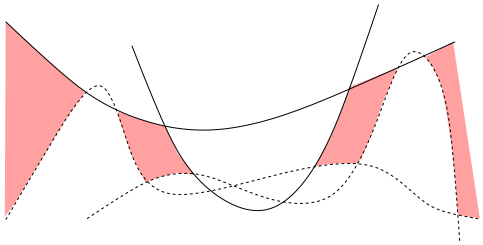

Figure 4: $\mathcal{S}(\Gamma, \Gamma')$; solid arcs are in Γ, and dashed arcs are in Γ'.

patches of Γ; see Figure 4. It can be shown that the combinatorial complexity of $\mathcal{S}(\Gamma, \Gamma')$ is proportional to the complexity of the overlay of the minimization diagram of Γ and the maximization diagram of Γ'. The result of Agarwal *et al.* [21] implies that $\mathcal{S}(\Gamma, \Gamma') = O(n^{2+\varepsilon})$ in 3-space. In general, the complexity of the overlay of the minimization diagram of Γ and the maximization diagram of Γ' may be larger than that of $\mathcal{S}(\Gamma, \Gamma')$. As an application, which also illustrates this discrepancy, consider the following example. Let $S = \{S_1, \ldots, S_n\}$ be a set of n spheres in $\mathbb{R}^3$. A line in $\mathbb{R}^3$ can be parameterized by four real parameters. We can therefore define the set of lines tangent to a sphere S_i and lying above (resp. below) S_i as a surface patch γ_i (resp. γ_i') in $\mathbb{R}^4$. Define $\Gamma = \{\gamma_i \mid 1 \leq i \leq n\}$ and $\Gamma' = \{\gamma_i' \mid 1 \leq i \leq n\}$. If the lines are parameterized carefully, Agarwal *et al.* [10] showed that $\mathcal{S}(\Gamma, \Gamma')$ is the set of lines intersecting all the spheres of S and that the combinatorial complexity of $\mathcal{S}(\Gamma, \Gamma')$ is $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$. However, a construction of Pellegrini [278] implies that the combinatorial complexity of the overlay of the two diagrams can be $\Omega(n^4)$.

4 Single Cells

Lower envelopes are closely related to other substructures in arrangements, notably *cells* and *zones*. The lower envelope is a portion of the boundary of the bottommost cell of the arrangement, though the worst-case complexity of $L(\Gamma)$ can be larger than that of the bottommost cell of $\mathcal{A}(\Gamma)$.) In two dimensions, it was shown in [191] that the complexity of a single face in an arrangement of n arcs, each pair of which intersect in at most s points, is $O(\lambda_{s+2}(n))$, and so has the same asymptotic bound as the complexity of the lower envelope of such a collection of arcs. The prevailing conjecture is that the complexity of a single cell in an arrangement of n surface patches in $\mathbb{R}^d$ satisfying the assumptions (A1) and (A2) is

close to $O(n^{d-1})$. The Upper Bound Theorem implies that the complexity of a single cell in arrangement of hyperplanes in $\mathbb{R}^d$ is $O(n^{\lfloor d/2 \rfloor})$, and the linearization technique described in Section 3 implies that the complexity of a single cell in an arrangement of n spheres is $O(n^{\lceil d/2 \rceil})$. However, the lower-bound construction for lower envelopes implies a lower bound of $\Omega(n^{d-1}\alpha(n))$ for a complexity of a single cell already for arrangements of simplices.

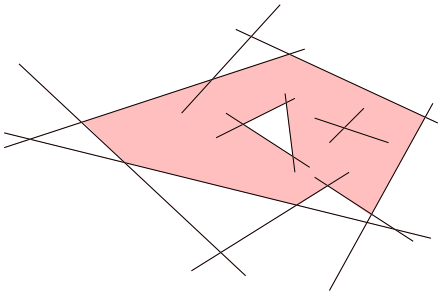

Figure 5: A single cell in an arrangement of segments.

Pach and Sharir [273] were the first to prove a subcubic upper bound on the complexity of a single cell in arrangements of triangles in $\mathbb{R}^3$. This bound was improved by Aronov and Sharir [50] to $O(n^{7/3})$, and subsequently to $O(n^2 \log n)$ [52]. The latter approach extends to higher dimensions; that is, the complexity of a single cell in an arrangement of n $(d-1)$-simplices in $\mathbb{R}^d$ is $O(n^{d-1} \log n)$. A simpler proof was given by Tagansky [314]. These approaches, however, do not extend to nonlinear surfaces even in $\mathbb{R}^3$.

Halperin [196] proved near-quadratic bounds on the complexity of a single cell in arrangement of certain classes of n bivariate surface patches, which arise in motion-planning applications. One of the more significant results in this direction is by Halperin and Sharir [205], who proved such a bound on the complexity of a single cell in an arrangement of the contact surfaces that arise in a rigid motion of a convex polygon amidst convex polygons in the plane, i.e., the surfaces that represent the placements of the polygon at which it touches one of the obstacles. The proof borrows ideas from the proof of Theorem 3.1.

A near optimal bound on the complexity of a single cell in the arrangement of an arbitrary collection of surface patches in $\mathbb{R}^3$ satisfying assumptions (A1) and (A2) was finally proved by Halperin and Sharir [203]:

Theorem 4.1 (Halperin and Sharir [203]) *Let Γ be a set of surface patches in $\mathbb{R}^3$ satisfying assumptions (A1) and (A2). For any $\varepsilon > 0$, the complexity of a single cell in $\mathcal{A}(\Gamma)$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε and on the maximum degree of the surface patches and of their boundaries.*

The proof proceeds along the same lines as the proof of Theorem 3.1. However, they establish the following two additional results to "bootstrap" the recurrences that the proof derives. Let C be the cell of $\mathcal{A}(\Gamma)$ whose complexity we want to bound.

(a) There are only $O(n^2)$ vertices v of the cell C that are locally x-extreme (that is, there is a neighborhood N of v and a connected component C' of the intersection of N with the interior of C, such that v lies to the left (in the x-direction) of all points of C', or v lies to the right of all these points).

(b) There are only $O(n^{2+\varepsilon})$ vertices on *popular faces* of C, that is, 2-faces f for which C lies locally near f on both sides of f.

Property (a) is proved by an appropriate decomposition of C into $O(n^2)$ subcells, in the style of a *Morse decomposition* of C (see [260]), so that each subcell has at most two points that are locally x-extreme in C. Property (b) is proved by applying the machinery of the proof of Theorem 3.1, where the quantity to be analyzed is the number of vertices of popular faces of C, rather than all inner vertices. Once these two results are available, the proof of Theorem 3.1 can be carried through, with appropriate modifications, to yield a recurrence for the number of vertices of C, whose solution is $O(n^{2+\varepsilon})$. We refer the reader to the original paper for more details.

It looks plausible that this proof can be extended to higher dimensions, to yield a bound of $O(n^{d-1+\varepsilon})$ on the complexity of a single cell in an arrangement of n surface patches in $\mathbb{R}^d$ satisfying assumptions (A1) and (A2). For this, appropriate extensions of both properties (a) and (b) have to be established. The extension of (a) appears to require topological considerations related to Morse theory, and the extension of (b) requires an inductive argument, in which bounds on the number of vertices of popular faces of all dimensions need to be derived, using induction on the dimension of the faces. Unfortunately, a complete proof is not yet available.

The linearization technique in the previous section can be extended to bound the complexity of a cell as well, namely, one can prove the following.

Theorem 4.2 *Let Γ be a collection of n hypersurfaces in $\mathbb{R}^d$, of constant maximum degree b. If Γ admits a linearization of dimension k, then the combinatorial complexity of a cell of $\mathcal{A}(\Gamma)$ is $O(n^{\lfloor k/2 \rfloor})$, where the constant of proportionality depends on k, d, and b.*

5 Zones

Let Γ be a set of n surfaces in $\mathbb{R}^d$. The *zone* of a surface σ (not belonging to Γ), denoted as $zone(\sigma; \Gamma)$, is defined to be the set of d-dimensional cells in $\mathcal{A}(\Gamma)$ that intersect σ. The complexity of $zone(\sigma; \Gamma)$ is defined to be the sum of complexities of the cells of $\mathcal{A}(\Gamma)$ that belong to $zone(\sigma; \Gamma)$, where the complexity of a cell in $\mathcal{A}(\Gamma)$ is the number of faces of all dimensions that are contained in the closure of the cell.

The complexity of a zone was first studied by Edelsbrunner *et al.* [147]; see also [102]. The 'classical' zone theorem [132, 149] asserts that the maximum complexity of the zone

of a hyperplane in an arrangement of n hyperplanes in $\mathbb{R}^d$ is $\Theta(n^{d-1})$, where the constant of proportionality depends on d. The original proof given by Edelsbrunner *et al.* [147] had some technical problems. A correct, and simpler, proof was given by Edelsbrunner *et al.* [149]. Their technique is actually quite general and can also be applied to obtain several other interesting combinatorial bounds involving arrangements. For example, the proof by Aronov and Sharir for the complexity of a single cell in arrangements of simplices [52] used a similar approach. Other results based on this technique can be found in [4, 48, 49]. We therefore describe the technique, as applied in the proof of the zone theorem:

Theorem 5.1 (Edelsbrunner, Seidel, and Sharir [149]) *The maximum complexity of the zone of a hyperplane in an arrangement of n hyperplanes in $\mathbb{R}^d$ is $\Theta(n^{d-1})$.*

This result is easy to prove for $d = 2$; see Chapter DS-??. For a set Γ of n hyperplanes in $\mathbb{R}^d$ and another hyperplane b, let $\tau_k(b;\Gamma)$ denote the total number of k-faces contained on the boundary of cells in $zone(b;\Gamma)$; each such k-face is counted once for each cell that it bounds. Let

$$\tau_k(n,d) = \max \tau_k(b;\Gamma)\,,$$

where the maximum is taken over all hyperplanes b and all sets Γ of n hyperplanes in $\mathbb{R}^d$. The maximum complexity of $zone(b;\Gamma)$ is at most $\sum_{k=0}^{d} \tau_k(n,d)$. Thus the following lemma immediately implies the upper bound in Theorem 5.1.

Lemma 5.2 *For each d and $0 \le k \le d$,*

$$\tau_k(n,d) = O(n^{d-1})\,,$$

where the constants of proportionality depend on d and k.

Proof: We use induction on d. As just noted, the claim holds for $d = 2$. Assume that the claim holds for all $d' < d$, let Γ be a set of n hyperplanes in $\mathbb{R}^d$, and let b be some other hyperplane. Without loss of generality, we can assume that the hyperplanes in $\Gamma \cup \{b\}$ are in general position. We define a *k-border* to be a pair (f, C), where f is a k-face incident to a (full-dimensional) cell C of $\mathcal{A}(\Gamma)$. Thus $\tau_k(b;\Gamma)$ is the total number of k-borders (f, C) for which $C \in zone(b;\Gamma)$.

We pick a hyperplane $\gamma \in \Gamma$ and count the number of all k-borders (f, C) in $zone(b;\Gamma)$ such that f is not contained in γ. If we remove γ from Γ, then any such k-border is contained in a k-border (f', C') of $zone(b;\Gamma \setminus \{\gamma\})$ (i.e., $f \subseteq f'$ and $C \subseteq C'$). Our strategy is thus to consider the collection of k-borders in $zone(b;\Gamma \setminus \{\gamma\})$ and to estimate the increase in the number of k-borders as we add γ back to Γ. Observe that we do not count k-borders that lie in γ.

Let $\Gamma|_\gamma = \{\gamma' \cap \gamma \mid \gamma' \in \Gamma \setminus \{\gamma\}\}$; the set $\Gamma|_\gamma$ forms a $(d-1)$-dimensional arrangement of $n-1$ hyperplanes within γ. Let (f, C) be a k-border of $zone(b;\Gamma \setminus \{\gamma\})$, and consider what happens to it when we reinsert γ. The following cases may occur:

$\gamma \cap C = \emptyset$: In this case the k-border (f, C) gives rise to exactly one k-border in $zone(b; \Gamma)$, namely itself.

$\gamma \cap C \neq \emptyset$, $\gamma \cap f = \emptyset$: Let γ^+ be the open half-space bounded by γ that contains f, and let $C^+ = C \cap \gamma^+$. If C^+ intersects b, then (f, C) gives rise to one k-border in $zone(b; \Gamma)$, namely (f, C^+) (this is the case for the edge $f = e$ in Figure 6); otherwise it gives rise to no k-border in $zone(b; \Gamma)$.

$\gamma \cap f \neq \emptyset$: Let γ^+ and γ^- be the two open half-spaces bounded by γ and let $C^+ = C \cap \gamma^+$ and $C^- = C \cap \gamma^-$. If the closure of only one of C^+ and C^- intersects b, say, C^+, then (f, C) gives rise to only one k-border in $zone(b; \Gamma)$, namely $(f \cap \gamma^+, C^+)$ (this is the case for the edge $f = e'$ in Figure 6). If both C^+ and C^- intersect b, then (f, C) gives rise to two k-borders in $zone(b; \Gamma)$, namely $(f \cap \gamma^+, C^+)$ and $(f \cap \gamma^-, C^-)$ (this is the case for the edge $f = e''$ in Figure 6). In this case, however, we can charge uniquely this increase in the number of k-borders to $(f \cap \gamma, C \cap \gamma)$, which, as easily seen, is a $(k-1)$-border in $zone(b \cap \gamma; \Gamma|_\gamma)$.

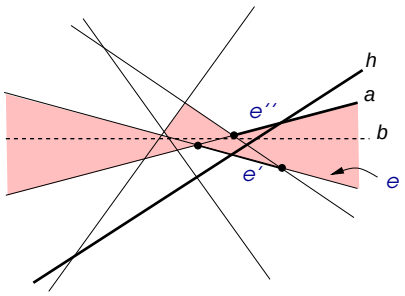

Figure 6: Inserting γ into $zone(b; \Gamma \setminus \{\gamma\})$.

If we repeat this process over all k-borders of $zone(b; \Gamma \setminus \{\gamma\})$, we obtain that the total number of k-borders (f, C) in $zone(b; \Gamma)$, for f not contained in γ, is at most

$$\begin{aligned} \tau_k(b; \Gamma \setminus \{\gamma\}) + \tau_{k-1}(b \cap \gamma; \Gamma|_\gamma) &\leq \tau_k(n-1, d) + \tau_{k-1}(n-1, d-1) \\ &= \tau_k(n-1, d) + O(n^{d-2}), \end{aligned}$$

where the last inequality follows from the induction hypothesis. Repeating this analysis for all hyperplanes $\gamma \in \Gamma$, summing up the resulting bounds, and observing that each k-border of $zone(b; \Gamma)$ is counted exactly $n - d + k$ times, we obtain

$$\tau_k(n, d) \leq \frac{n}{n-d+k} \left(\tau_k(n-1, d) + O(n^{d-2}) \right).$$

Edelsbrunner *et al.* [149] showed that this recurrence solves to $O(n^{d-1})$ for $k \geq 2$. Using Euler's formula for cell complexes, one can show that $\tau_k(n, d) = O(n^{d-1})$ for $k = 0, 1$ as well. This completes the proof of the theorem. For the lower bound, it is easily checked that

the complexity of the zone of a hyperplane b in an arrangement of n hyperplanes in $\mathbb{R}^d$ in general position is $\Omega(n^{d-1})$. In fact, the complexity of the cross-section of the arrangement within b is already $\Omega(n^{d-1})$. □

The above technique can be extended to bound the quantity $\sum_{C\in\mathcal{A}(\Gamma)} |C|^2$, where Γ is a set of hyperplanes, C ranges over all d-dimensional cells of the arrangement, and $|C|$ denotes the number of lower-dimensional faces incident to C. For $d \leq 3$, an easy application of the zone theorem (see Edelsbrunner [132]) implies that $\sum_C |C|^2 = O(n^d)$; this bound is obviously tight if the lines or planes of Γ are in general position. For $d > 3$, the same application of the zone theorem yields only $\sum_C |C| f_C = O(n^d)$, where f_C is the number of hyperplanes of Γ meeting the boundary of C. Using the same induction scheme as in the proof of Theorem 5.1, Aronov *et al.* [48] showed that

$$\sum_{C\in\mathcal{A}(\Gamma)} |C|^2 = O(n^d \log^{\lfloor \frac{d}{2} \rfloor - 1} n).$$

It is believed that the right bound is $O(n^d)$. Note that such a result does not hold for arrangements of simplices or of surfaces because the complexity of single cell can be $\Omega(n^{d-1})$.

The zone theorem for hyperplane arrangements can be extended as follows.

Theorem 5.3 (Aronov, Pellegrini, and Sharir [49]) *Let Γ be a set of n hyperplanes in $\mathbb{R}^d$. Let σ be a p-dimensional algebraic hypersurface of some fixed degree, or the relative boundary of any convex set with affine dimension $p+1$, for $0 \leq p \leq d$. The complexity of the zone $(\sigma; \Gamma)$ is $O(n^{\lfloor (d+p)/2 \rfloor} \log^{\gamma} n)$, where $\gamma = d + p \pmod 2$, and the bound is almost tight (up to the logarithmic factor) in the worst case.*

In particular, for $p = d-1$, the complexity of the zone is $O(n^{d-1} \log n)$, which is almost the same as the complexity of the zone of a hyperplane in such an arrangement.

The proof proceeds along the same lines of the inductive proof of Theorem 5.1. However, when a hyperplane $\gamma \in \Gamma$ is removed and then reinserted, and splits a face f of $zone(\sigma; \Gamma \setminus \{\gamma\})$ into two subfaces, both lying in $zone(\sigma; \Gamma)$, the charging scheme used in the proof of Theorem 5.1 becomes inadequate, because $f \cap \gamma$ need not belong to the zone of $\sigma \cap \gamma$ in the d-dimensional cross-section of $\mathcal{A}(\Gamma)$ along γ. What is true, however, is that $f \cap \gamma$ is a face incident to a *popular facet* of $zone(\sigma; \Gamma)$ along γ, that is, a facet $g \subseteq \gamma$ whose two incident cells belong to the zone. Thus the induction proceeds not by decreasing the dimension of the arrangement (as was done in the proof of Theorem 5.1), but by reapplying the same machinery to bound the number of vertices of popular facets of the original $zone(\sigma; \Gamma)$. This in turn requires similar bounds on the number of vertices of lower-dimensional popular faces. We refer the reader to Aronov *et al.* [49] for more details.

In general, the zone of a surface in an arrangement of n surfaces in $\mathbb{R}^d$ can be transformed to a single cell in another arrangement of $O(n)$ surface patches in $\mathbb{R}^d$. For example, Let

Γ be a set of n $(d-1)$-simplices in $\mathbb{R}^d$, and let σ be a hyperplane. We split each $\gamma \in \Gamma$ into two polyhedra at the intersection of Δ and σ (if the intersection is nonempty), push these two polyhedra slightly away from each other, and, if necessary, retriangulate each polyhedron into a constant number of simplices. In this manner, we obtain a collection Γ' of $O(n)$ simplices, and all cells of the zone of σ in $\mathcal{A}(\Gamma)$ now fuse into a single cell of $\mathcal{A}(\Gamma')$. Moreover, by the general position assumption, the complexity of the zone of σ in Γ is easily seen to be dominated by the complexity of the new single cell of $\mathcal{A}(\Gamma')$. (The same technique has been used earlier in [140], to obtain a near-linear bound on the complexity of the zone of an arc in a 2-dimensional arrangement of arcs.) Hence, the following theorem is an easy consequence of the result by Aronov and Sharir [52].

Theorem 5.4 *The complexity of the zone of a hyperplane in an arrangement of n $(d-1)$-simplices in $\mathbb{R}^d$ is $O(n^{d-1}\log n)$.*

Using a similar argument one can prove the following.

Theorem 5.5 (Halperin and Sharir [203]) *Let Γ be a collection of n surface patches in $\mathbb{R}^3$, satisfying assumptions (A1) and (A2). The combinatorial complexity of the zone in $\mathcal{A}(\Gamma)$ of an algebraic surface σ of some fixed degree is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε, on the maximum degree of the given surfaces and their boundaries, and on the degree of σ.*

Once the bound on the complexity of a single cell in an arrangement of general algebraic surfaces is extended to higher dimensions, it should immediately yield, using the same machinery, to a similar bound for the zone of a surface in such an arrangement.

6 Levels

The *level* of a point $p \in \mathbb{R}^d$ in an arrangement $\mathcal{A}(\Gamma)$ of a set Γ of monotone surfaces satisfying (A1)–(A2) is the number of surfaces of Γ lying vertically below p. For $0 \le k < n$, we define *k-level* (resp. *$\le k$-level*), denoted by $\mathcal{A}_k(\Gamma)$ (resp. $\mathcal{A}_{\le k}(\Gamma)$), to be the closure of all points on the surfaces of Γ whose level is k (resp. at most k). A face of $\mathcal{A}_k(\Gamma)$ or $\mathcal{A}_{\le k}(\Gamma)$ is a maximal connected portion of a face of $\mathcal{A}(\Gamma)$ consisting of points having a fixed subset of surfaces lying below them. For fully defined functions, any such face coincides with a face of $\mathcal{A}(\Gamma)$. Note that $\mathcal{A}_0(\Gamma)$ is the same as $L(\Gamma)$. If the surfaces in Γ are graphs of totally defined functions, then the level of all points on a face of $\mathcal{A}(\Gamma)$ is the same and $\mathcal{A}_k(\Gamma)$ is a connected monotone surface; otherwise $\mathcal{A}_k(\Gamma)$ may have discontinuities. See Figure 7 for an example of levels in arrangements of lines and segments.

Levels in hyperplane arrangements in $\mathbb{R}^d$ are closely related to *k-sets* of point sets in $\mathbb{R}^d$. Let S be a set of n points in $\mathbb{R}^d$, and let S^* be the set of hyperplanes dual to S. A

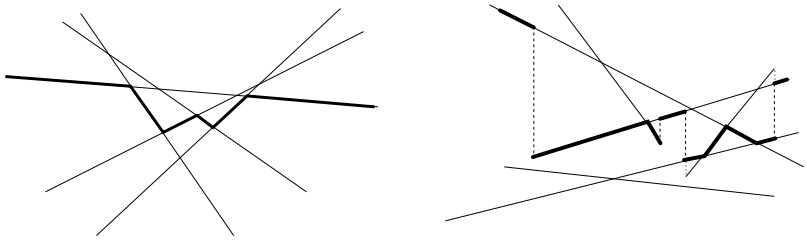

Figure 7: The 2-level in (i) an arrangement of lines, and (ii) in an arrangement of segments.

subset $A \subset S$ is called a *k-set* if $|A| = k$ and A can be strictly separated from $S \setminus A$ by a hyperplane h. The level of point h^*, dual to h, in $\mathcal{A}(S^*)$ is k or $n-k$. The *k-set problem* is to bound the maximum number of k-sets of S (in terms of k and n). It is easy to see that the maximum number of k-sets in a set of n points in $\mathbb{R}^d$ is bounded by the maximum number of facets in the k-level and the $(n-k)$-level in an arrangement of n hyperplanes in $\mathbb{R}^d$.

Let $\psi_k(\Gamma)$ (resp. $\psi_{\leq k}(\Gamma)$) be the total number of faces in $\mathcal{A}_k(\Gamma)$ (resp. $\mathcal{A}_{\leq k}(\Gamma)$). Let $\mathbf{G}$ be a (possibly infinite) family of surfaces in $\mathbb{R}^d$ satisfying assumptions (A1) and (A2). We define $\psi_k(n, d, \mathbf{G}) = \max \psi_k(\Gamma)$ and $\psi_{\leq k}(n, d, \mathbf{G}) = \max \psi_{\leq k}(\Gamma)$, where the maximum in both cases is taken over all subsets $\Gamma \subseteq \mathbf{G}$ of size n. If $\mathbf{G}$ is not important or follows from the context, we will omit the argument $\mathbf{G}$.

The following theorem follows from a result by Clarkson and Shor [111].

Theorem 6.1 (Clarkson and Shor [111]) *Let $\mathbf{G}$ be an infinite family of surfaces satisfying assumptions (A1)–(A3). Then for any $0 \leq k < n - d$,*

$$\psi_{\leq k}(n, d, \mathbf{G}) = O\left((k+1)^d \kappa\left(\frac{n}{k+1}, d, \mathbf{G}\right)\right),$$

where $\kappa(n, d, \mathbf{G})$ is the maximum complexity of the lower envelope of n surfaces in $\mathbf{G}$.

Proof: Let $\Gamma \subset \mathbf{G}$ be a set of n surface patches satisfying assumptions (A1)–(A5). For a subset $X \subseteq \Gamma$ and an integer $0 \leq k \leq |X| - d$, let $V_k(X)$ denote the set of vertices at level k in $\mathcal{A}(\Gamma)$. As is easily seen, $\psi_{\leq k}(\Gamma)$ is proportional to $\sum_{j=0}^{k} |V_j(\Gamma)|$, which we thus proceed to bound. We bound below only the number of vertices in the first k levels that lie in the interior of d surface patches; the other types of vertices are easier to analyze, and the same bound applies to them as well. We choose a random subset $R \subseteq \Gamma$ of size $r = \lfloor n/(k+1) \rfloor$ and bound the expected number of vertices in $V_0(R)$. A vertex $v \in V_j(\Gamma)$ is in $V_0(R)$ if and only if the d surfaces defining v are in R and none of the j surfaces of Γ lying below v are chosen in R, so the probability that $v \in V_0(R)$ is $\binom{n-j-d}{r-d} / \binom{n}{r}$. Hence, easy manipulation of

binomial coefficients implies that

$$\begin{aligned}\mathbf{E}\big[|V_0(R)|\big] &= \sum_{j=0}^{n-d} |V_j(\Gamma)| \frac{\binom{n-j-d}{r-d}}{\binom{n}{r}} \\ &\geq \sum_{j=0}^{k} |V_j(\Gamma)| \frac{\binom{n-j-d}{r-d}}{\binom{n}{r}} \\ &= \Omega\left(\frac{1}{(k+1)^d}\right) \sum_{j=0}^{k} |V_j(\Gamma)|.\end{aligned}$$

Thus

$$\sum_{j=0}^{k} |V_j(\Gamma)| \leq c(k+1)^d \mathbf{E}\big[|V_0(R)|\big], \tag{6.1}$$

for some constant c. Since every vertex in $V_0(R)$ lies on the lower envelope of R, the assertion now follows from the definition of κ. □

Corollary 6.2 *(i)* $\psi_{\leq k}(n,d) = O((k+1)^{1-\varepsilon} n^{d-1+\varepsilon})$.

(ii) Let $\mathbf{H}$ *be the set of all hyperplanes in* $\mathbb{R}^d$. *Then* $\psi_{\leq k}(n,d,\mathbf{H}) = \Theta(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil})$.

Proof: Part (i) follows from Theorems 3.1 and 6.1. Part (ii) follows from the fact that $\kappa(n,d,\mathbf{H}) = \Theta(n^{\lfloor d/2 \rfloor})$. □

If Γ is a set of n lines in the plane, then there is even a tighter upper bound of $kn+1$ on $\psi_{\leq k}(\Gamma)$ for $k \leq n/2$ [36, 277]; see also [176].

In contrast to these bounds on the complexity of $\leq k$-levels, which are tight or almost tight in the worst case, much less is known about the complexity of a single k-level, even for the simplest case of arrangements of lines in the plane. For example, Corollary 6.2(ii), for $d = 2$, implies that the complexity of an average level in an arrangement of lines in the plane is linear, but no upper bound that is even close is known. For a set Γ of n lines in the plane, Lovász [236] proved that $\psi_{\lfloor n/2 \rfloor}(\Gamma) = O(n^{3/2})$.[4] Erdős *et al.* [163] extended his argument to prove that $\psi_k(\Gamma) = O(n\sqrt{k+1})$. Since the original proof many different proofs have been proposed for obtaining the same bound on $\psi_k(\Gamma)$ [8, 193]. Goodman and Pollack [177] proved a similar bound on the maximum complexity of the k-level in an arrangement of pseudo-lines. Erdős *et al.*'s bound was slightly improved by Pach *et al.* [276] to $o(n\sqrt{k+1})$, using a rather complicated argument. Erdős *et al.* [163] constructed, for any n and $0 \leq k < n$, a set Γ of n lines so that $\psi_k(\Gamma) = \Omega(n \log(k+1))$; see Edelsbrunner and

[4] According to L. Lovász [236], the $(n/2)$-set problem was originally posed by A. Simmons, and E. Strauss had constructed a set of points in the plane in which the number of $(n/2)$-sets was $\Omega(n \log n)$.

Welzl for another construction that gives the same lower bound [153]. Klawe *et al.* [225] constructed a set Γ of n pseudo-lines so that $\psi_{n/2}(\Gamma)$ has $n2^{\Omega(\sqrt{\log n})}$ vertices.

A major breakthrough in this direction was recently made by Dey who obtained the following improvement.

Theorem 6.3 (Dey [124]) *Let Γ be a set of n lines in the plane. Then for any $0 \leq k < n$, $\psi_k(\Gamma) = O(n(k+1)^{1/3})$.*

Dey's proof is quite simple and elegant. It uses the following result on geometric graphs, which was independently proved by Ajtai *et al.* [30] and by Leighton [230].[5]

Lemma 6.4 *Let G be a geometric graph with n vertices and $m \geq 4n$ edges. Then there are $\Omega(m^3/n^2)$ pairs of edges in G whose relative interiors cross.*

Proof of Theorem 6.3: For simplicity we assume that n is even and prove the bound for $k = n/2$. We argue in the dual plane, where we have a set S of n points in general position and we wish to establish the asserted bound for the number of *halving segments* of S, where a halving segment is a straight segment connecting a pair of points $u, v \in S$ so that the line passing through u and v has exactly $(n/2) - 1$ points of S below it. Let H denote the set of halving segments.

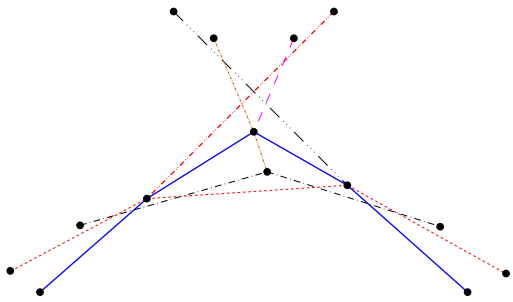

Figure 8: A set of 14 points with 14 halving segments, split into 7 convex x-monotone chains

The segments in H are decomposed into $n/2$ convex x-monotone chains as follows. Let uv be an edge of H, with u lying to the right of v. We rotate the line that passes through u and v clockwise about v and stop as soon as the line overlaps another halving segment vw incident to v. It is easy to check that w lies to the right of v and that uvw is a right turn. We now rotate about w, and continue in this manner until our line becomes vertical. We apply the same procedure 'backwards' by turning the line uv counterclockwise around u and keep iterating until the line becomes vertical. The halving segments that we have

[5] A *geometric graph* $G = (V, E)$ is a graph drawn in the plane so that its vertices are points and its edges are straight segments connecting pairs of these points. A geometric graph need not be planar.

encountered during the whole process constitute one convex polygonal chain. By applying this procedure repeatedly, we obtain the desired decomposition of the entire H into convex x-monotone polygonal chains. Using the properties of halving segments proved by Lovász, we can conclude that the segments are partitioned into $n/2$ convex chains. (These convex chains are in a certain sense dual to the concave chains in the dual line arrangement that were defined by Agarwal *et al.* [8]; see also [193].)

The number of crossing points between two convex chains is bounded by the number of upper common tangents between the same two chains. Any line passing through two points of S is an upper common tangent of at most one pair of chains. Thus there are $O(n^2)$ crossings between the segments in H. By Lemma 6.4, any graph with n vertices and crossing number $O(n^2)$ has at most $O(n^{4/3})$ edges, so S has at most $O(n^{4/3})$ halving segments. A similar, slightly more detailed, argument proves the bound for arbitrary values of k. □

Tamaki and Tokuyama generalized Dey's proof to prove a similar bound on the complexity of the k-level in arrangements of pseudo-lines [315]. Combining the ideas from an old result of Welzl [325] with Dey's proof technique, one can obtain the following generalization. See also [41] for some other generalizations of Dey's result.

Corollary 6.5 *Let Γ be a set of n lines in the plane. Then for any $0 \leq k < n$, $0 < j < n-k$, we have*

$$\sum_{t=k}^{k+j} \psi_t(\Gamma) = O(n(k+1)^{1/3} j^{2/3}).$$

Bárány and Steiger proved a linear upper bound on the expected number of k-sets in a random planar point set [63]; the points are chosen uniformly from a convex region. Edelsbrunner *et al.* [152] proved that if S is a set of points in the plane so that the ratio of the maximum and the minimum distance in S is at most $c\sqrt{n}$ (a so-called *dense* set), then the number of k-sets in S is $O(c\sqrt{n}\psi_k(c\sqrt{n}))$. Applying Dey's result, the number of k-sets in a dense point set is $O(n^{7/6})$. Recently Alt *et al.* [37] have proved that if the points in S lie on a constant number of pairwise disjoint convex curves, then the number of k-sets in S is $O(n)$.

The following question is related to the complexity of levels in arrangements of lines: Let Γ be a set of n lines in the plane. Let Π be a x-monotone path whose vertices are the vertices of $\mathcal{A}(\Gamma)$ and whose edges are contained in the lines of Γ. What is the maximum number of vertices in Π? Matoušek [239] proved that there exists a set Γ of n lines in the plane that contains a x-monotone path with $\Omega(n^{5/3})$ vertices. No subquadratic upper bound is known for this problem.

Agarwal *et al.* [8] proved a nontrivial upper bound on the complexity of the k-level in an arrangement of segments. Combining their argument with that of Dey, one can prove that the maximum complexity of the k-level in a planar arrangement of n segments is

$O(n(k+1)^{1/3}\alpha(n/(k+1)))$. Very little is known on the complexity of a single level in an arrangement of n arcs in the plane. Recently, Tamaki and Tokuyama [316] proved that the complexity of any level in an arrangement of parabolas, each with a vertical axis, is $O(n^{23/12})$. (Their bound actually applies to *pseudo-parabolas*, i.e., graphs of continuous, totally defined, univariate functions, each pair of which intersect at most twice.)

Open Problem 3 *(i) What is the maximum complexity of a level in an arrangement of n lines in the plane?*

(i) What is the maximum complexity of a level in an arrangement of n x-monotone Jordan arcs, each pair of which intersect in at most s points, for some constant $s > 1$?

Bárány *et al.* [62] proved an $O(n^{3-\gamma})$ bound on the complexity of the k-level in arrangements of n planes in $\mathbb{R}^3$, for any k, for some absolute constant $\gamma > 0$. The bound was improved by Chazelle *et al.* [45] and Eppstein [161] to $O(n^{8/3}\text{polylog}n)$, and then by Dey and Edelsbrunner [125] to $O(n^{8/3})$. The best bound known, due to Agarwal *et al.* [8], is $O(n(k+1)^{5/3})$. They also proved a bound on the complexity of the k-level for arrangements of triangles in $\mathbb{R}^3$. A nontrivial bound on the complexity of the k-level in an arrangement of n hyperplanes in $d > 3$ dimensions, of the form $O(n^{d-\varepsilon_d})$, for some constant ε_d that decreases exponentially with d, was obtained in [35, 322]. This has later been slightly improved to $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - \varepsilon_d})$ in [8]. Table 1 summarizes the known upper bounds on k-levels.

Objects	Bound	Source
Lines in $\mathbb{R}^2$	$O(n(k+1)^{1/3})$	[124]
Segments in $\mathbb{R}^2$	$O(n(k+1)^{1/3}\alpha(n/(k+1)))$	[8, 124]
Planes in $\mathbb{R}^3$	$O(n(k+1)^{5/3})$	[8]
Triangles in $\mathbb{R}^3$	$O(n^2(k+1)^{5/6}\alpha(n/(k+1)))$	[8]
Hyperplanes in $\mathbb{R}^d$	$O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - \varepsilon_d})$	[322]
Parabolas in $\mathbb{R}^2$ (Vertical axis)	$O(n^{23/12})$	[316]

Table 1: Upper bounds on k-levels.

7 Many Cells and Incidences

In the previous two sections we bounded the complexity of families of d-dimensional cells in $\mathcal{A}(\Gamma)$ that satisfied certain conditions (e.g., cells intersected by a surface, the cells of level at most k). We can ask a more general question: *What is the complexity of any m distinct cells in $\mathcal{A}(\Gamma)$?* A single cell in an arrangement of lines in the plane can have n edges, but can the total complexity of m cells in an arrangement of lines be $\Omega(mn)$? This is certainly false for $m = \Omega(n^2)$.

We can also formulate the above problem as follows: Let P be a set of m points and Γ a set of n surfaces in $\mathbb{R}^d$ satisfying assumptions (A1) and (A2). Define $\mathcal{C}(P,\Gamma)$ to be the set of cells in $\mathcal{A}(\Gamma)$ that contain at least one point of P. Define $\mu(P,\Gamma) = \sum_{C\in\mathcal{C}(P,\Gamma)} |C|$ and $\mu(m,n,\mathbf{G}) = \max \mu(P,\Gamma)$, where the maximum is taken over all sets P of m points and over all sets Γ of n surfaces in a given class $\mathbf{G}$.

Let $\mathbf{L}$ be the set of all lines in the plane. Canham [84] proved that $\mu(m,n,\mathbf{L}) = O(m^2+n)$, from which it easily follows that $\mu(m,n,\mathbf{L}) = O(m\sqrt{n}+n)$. Although this bound is optimal for $m \le \sqrt{n}$, it is weak for larger values of m. Clarkson *et al.* [108] proved that $\mu(m,n,\mathbf{L}) = \Theta(m^{2/3}n^{2/3}+n)$. Their technique, based on random sampling, is general and constructive. It has led to several important combinatorial and algorithmic results on arrangements [108, 189, 190]. For example, following a similar, but considerably more involved, approach, Aronov *et al.* [46] proved that $\mu(m,n,\mathbf{E}) = O(m^{2/3}n^{2/3}+m\log n+n\alpha(n))$, where $\mathbf{E}$ is the set of all line segments in the plane. Hershberger and Snoeyink [213] proved an $O(m^{2/3}n^{2/3}+n)$ upper bound on the complexity of m distinct cells in the arrangements of n segments in the plane where the segments satisfy certain additional conditions.

Although Clarkson *et al.* [108] proved nontrivial bounds on the complexity of m distinct cells in arrangements of circles (see Table 2 below), no tight bound is known.

Open Problem 4 *What is the maximum complexity of m distinct cells in an arrangement of n circles in the plane?*

Objects	Complexity	Source
Lines in $\mathbb{R}^2$	$\Theta(m^{2/3}n^{2/3}+n)$	[108]
Segments in $\mathbb{R}^2$	$O(m^{2/3}n^{2/3}+n\alpha(n)+n\log m)$	[46]
Unit circles in $\mathbb{R}^2$	$\Theta(m^{2/3}n^{2/3}+n)$	[108]
Circles in $\mathbb{R}^2$	$O(m^{3/5}n^{4/5}+n)$	[108]
Arcs in $\mathbb{R}^2$	$O(m\sqrt{\lambda_q(n)})$	[140]
Planes in $\mathbb{R}^3$	$\Theta(m^{2/3}n)$	[7]
Unit Spheres in $\mathbb{R}^3$	$O(m^{3/4}n^{3/4}+n)$	[108]
Hyperplanes in $\mathbb{R}^d$, $d\ge 4$	$O(m^{1/2}n^{d/2}\log^\gamma n)$ $\gamma=(\lfloor d/2\rfloor-1)/2$	[48]

Table 2: Complexity of many cells.

Complexity of many cells in hyperplane arrangements in higher dimensions was first studied by Edelsbrunner and Haussler [145]. Let $\mathbf{H}$ be the set of all hyperplanes in $\mathbb{R}^d$. They proved that the maximum number of $(d-1)$-dimensional faces in m distinct cells in an arrangement of n hyperplanes in $\mathbb{R}^d$ is $O(m^{1/2}n^{d/2}+n^{d-1})$. Refining an argument by Edelsbrunner *et al.* [143], Agarwal and Aronov [7] improved this bound to $O(m^{2/3}n^{d/3}+n^{d-1})$. By a result of Edelsbrunner and Haussler [145], this bound is tight in the worst case. Aronov *et al.* [48] proved that $\mu(m,n,\mathbf{H}) = O(m^{1/2}n^{d/2}\log^\gamma n)$, where $\gamma = (\lfloor d/2\rfloor-1)/2$.

They also proved several lower bounds on $\mu(m, n, \mathbf{H})$: For odd values of d and $m \leq n$, $\mu(m, n, \mathbf{H}) = \Theta(mn^{\lfloor d/2 \rfloor})$; for m of the form $\Theta(n^{d-2k})$ where $0 \leq k \leq \lfloor d/2 \rfloor$ is an integer, $\mu(m, n, \mathbf{H}) = \Omega(m^{1/2}n^{\lfloor d/2 \rfloor})$; and for arbitrary values of m, $\mu(m, n, \mathbf{H}) = \Omega(m^{1/2}n^{d/2-1/4})$. Agarwal [4], Guibas *et al.* [186], and Halperin and Sharir [201] obtained bounds on "special" subsets of cells in hyperplane arrangements.

A problem closely related to, but somewhat simpler than, the many-cells problem is the *incidence problem*. Here is a simple instance of this problem: Let Γ be a set of n lines and P a set of m points in the plane. Define $\mathcal{I}(P, \Gamma) = \sum_{\ell \in \Gamma} |P \cap \ell|$; set $\mathcal{I}(m, n) = \max \mathcal{I}(P, \Gamma)$, where the maximum is taken over all sets P of m distinct points and over all sets Γ of n distinct lines in the plane. Of course, this problem is interesting only when the lines in Γ are in highly degenerate position. If $n = m^2 + m + 1$, then a finite projective plane of order m has n points and n lines and each line contains $m + 1 = \Omega(n^{1/2})$ points, so the number of incidences between n points and n lines is $\Omega(n^{3/2})$. Szémeredi and Trotter [311] proved that such a construction is infeasible in $\mathbb{R}^2$. In a subsequent paper, Szémeredi and Trotter [312] proved that $\mathcal{I}(m, n) = O(m^{2/3}n^{2/3} + m + n)$. Their proof is, however, quite intricate, and an astronomic constant is hidden in the big-O notation. Their bound is asymptotically tight in the worst case, as shown in [155]. A considerably simpler proof, with a small constant of proportionality in the bound, was given by Clarkson *et al.* [108], based on the random-sampling technique. In fact, the bound on many cells in arrangements of lines immediately yields a similar bound on $\mathcal{I}(m, n)$ [108], but the proof can be somewhat simplified for the incidence problem. Here we present an even more elegant and simpler proof, due to Székely [310], for the bound on $\mathcal{I}(m, n)$ using Lemma 6.4:

Theorem 7.1 (Szémeredi and Trotter [312]) *Let Γ be a set of n lines and P a set of m points in the plane. Then*

$$\mathcal{I}(P, \Gamma) = O(m^{2/3}n^{2/3} + m + n).$$

Proof: We construct a geometric graph $G = (V, E)$ whose vertices are the points of P. We connect two vertices p, q by an edge if the points p and q are consecutive along a line in Γ. Each edge of G is a portion of a line of Γ, and no two edges overlap. Therefore at most $\binom{n}{2}$ pairs of edges cross each other. Note that $\mathcal{I}(P, \Gamma) \leq |E| + n$.

If $|E| \leq 4m$, there is nothing to prove. Otherwise, by Lemma 6.4,

$$\binom{n}{2} \geq \frac{|E|^3}{c|V|^2} \geq \frac{1}{cm^2}(\mathcal{I}(P, \Gamma) - n)^3,$$

which implies that $\mathcal{I}(P, \Gamma) = O(m^{2/3}n^{2/3} + n)$. □

Valtr [319] has studied the incidence problem and its generalization for dense point sets, where the ratio of the maximum and the minimum distances in P is at most $O(\sqrt{n})$.

The incidence problem has been studied for other curves as well. Of particular interest is the number of incidences between points and unit circles in the plane [308, 108] because of its close relationship with the following major open problem in combinatorial geometry, which was originally introduced by Erdős in 1946 [162]: *Let S be a set of n points in the plane. How many pairs of points in S are at distance 1?* Spencer *et al.* [308] had proved, by modifying the proof of Szémeredi and Trotter [312], that the number of incidences between m points and n unit circles is $O(m^{2/3}n^{2/3} + m + n)$. The proofs by Clarkson *et al.* [108] and by Székely [310] have been extended to this case. The incidence bound implies that the number of *unit distances* between the points of S is $O(n^{4/3})$. However, the best-known lower bound is only $n^{1+\Omega((\log\log n)/\log n)}$ [162] (see also [272]).

Open Problem 5 *How many pairs of points in a given planar set of points are at distance 1?*

Füredi [174] showed that if points in S are in convex position, then the number of pairs at distance 1 is $O(n \log n)$; the best-known lower bound is $7n - 12$ by Edelsbrunner and Hajnal [144]. The best-known upper bound on the unit distances in $\mathbb{R}^3$ is $O(n^{3/2})$ [108]. Let S be a set of n points in $\mathbb{R}^3$ so that no four points of P lie on a circle, then the number of pairs of points in S at unit distance is $O(n^{10/7})$ [189].

We can state the incidence problem in higher dimensions. If we do not make any additional assumptions on points and surfaces, the maximum number of incidences between m points and n planes is obviously mn: take a set of n planes passing through a common line and place m points on this line. Agarwal and Aronov [7] proved that if Γ is a set of n planes and P is a set of m points in $\mathbb{R}^3$ so that no three points in P are collinear, then $\mathcal{I}(P, \Gamma) = O(m^{3/5}n^{4/5} + m + n)$. Edelsbrunner and Sharir [150] showed that if Γ is a set of n unit spheres in $\mathbb{R}^3$ and P is a set of m points so that none of the points in P lies in the interior of any sphere, then $\mathcal{I}(P, \Gamma) = O(m^{2/3}n^{2/3} + m + n)$. See [189, 275] for other results on incidences in higher dimensions.

8 Generalized Voronoi Diagrams

An interesting application of the new bounds on the complexity of lower envelopes is to generalized Voronoi diagrams in higher dimensions. Let S be a set of n pairwise-disjoint convex objects in $\mathbb{R}^d$, each of constant description complexity, and let ρ be some metric. The *Voronoi diagram* $\mathrm{Vor}_\rho(S)$ of S under the metric ρ (sometimes also simply denoted as $\mathrm{Vor}(S)$) is a partition of $\mathbb{R}^d$ into maximal connected cells of various dimensions, where each cell C has the following property. There is a subset $S_C \subseteq S$ so that for any $\mathbf{x} \in C$

$$S_C = \{s \in S \ : \ \rho(\mathbf{x}, s) = \min_{s' \in S} \rho(\mathbf{x}, s')\}.$$

Let γ_i be the graph of the function $x_{d+1} = \rho(\mathbf{x}, s_i)$. Set $\Gamma = \{\gamma_i \mid 1 \leq i \leq n\}$. Edelsbrunner and Seidel [148] observed that $\mathrm{Vor}_\rho(S)$ is the minimization diagram of Γ.

In the classical case, in which ρ is the Euclidean metric and the objects in S are singletons (points), the graphs of these distance functions can be replaced by a collection of n hyperplanes in $\mathbb{R}^{d+1}$, using the linearization technique, without affecting the minimization diagram. Hence the maximum possible complexity of $\mathrm{Vor}(S)$ is $O(n^{\lceil d/2 \rceil})$, which actually can be achieved (see, e.g., [226, 297]). In more general settings, though, this reduction is not possible. Nevertheless, the bounds on the complexity of lower envelopes imply that, under reasonable assumption on ρ and on the objects in S, the complexity of the diagram is $O(n^{d+\varepsilon})$, for any $\varepsilon > 0$. While this bound is nontrivial, it is conjectured to be too weak. For example, this bound is near-quadratic for planar Voronoi diagrams, but the complexity of almost every planar Voronoi diagram is only $O(n)$, although there are certain distance functions for which the corresponding planar Voronoi diagram can have quadratic complexity [57].

In three dimensions, the above-mentioned bound for point sites and Euclidean metric is $\Theta(n^2)$. It has been a long-standing open problem to determine whether a similar quadratic or near-quadratic bound holds in 3-space for more general objects and metrics (here the new results on lower envelopes only give an upper bound of $O(n^{3+\varepsilon})$). The problem stated above calls for improving this bound by roughly another factor of n. Since we are aiming for a bound that is two orders of magnitude better than the complexity of $\mathcal{A}(\Gamma)$, it appears to be a considerably more difficult problem than that of lower envelopes. The only hope of making progress here is to exploit the special structure of the distance functions $\rho(x, s)$.

Fortunately, some progress on this problem was made recently. It was shown by Chew *et al.* [107] that the complexity of the Voronoi diagram is $O(n^2 \alpha(n) \log n)$ for the case where the objects of S are lines in $\mathbb{R}^3$ and the metric ρ by a *convex distance function* induced by a convex polytope with a constant number of facets (see [107] for more details). Note that such a distance function is not necessarily a metric, because it will fail to be symmetric if the defining polytope is not centrally symmetric. The L_1 and L_∞ metrics are special cases of such distance functions. The best-known lower bound for the complexity of the diagram in this special case is $\Omega(n^2 \alpha(n))$. Dwyer [130] has shown that the expected complexity of the Voronoi diagram of a set of n random lines in $\mathbb{R}^3$ is $O(n^{3/2})$. In another recent paper [75], it is shown that the maximum complexity of the L_1-Voronoi diagram of a set of n points in $\mathbb{R}^3$ is $\Theta(n^2)$. Finally, it is shown in [313] that the complexity of the three-dimensional Voronoi diagram of point sites under a general polyhedral convex distance function (induced by a polytope with $O(1)$ facets) is $O(n^2 \log n)$.

Open Problem 6 *(i) Is the complexity of the Voronoi diagram of a set S of n lines under the Euclidean metric in $\mathbb{R}^3$ close to n^2?*

(ii) Is the complexity of the Voronoi diagram of a set S of pairwise disjoint convex polyhedra in $\mathbb{R}^3$, with a total of n vertices, close to n^2 under the polyhedral convex distance functions?

An interesting special case of these problems involves *dynamic Voronoi diagrams* for

moving points in the plane. Let S be a set of n points in the plane, each moving along some line at some fixed velocity. The goal is to bound the number of combinatorial changes of the Euclidean $\mathrm{Vor}(S)$ over time. This dynamic Voronoi diagram can easily be transformed into a three-dimensional Voronoi diagram, by adding the time t as a third coordinate. The points become lines in 3-space, and the metric is a distance function induced by a horizontal disk (that is, the distance from a point $p(x_0, y_0, t_0)$ to a line ℓ is the Euclidean distance from p to the point of intersection of ℓ with the horizontal plane $t = t_0$). Here too the open problem is to derive a near-quadratic bound on the complexity of the diagram. Cubic or near-cubic bounds are known for this problem, even under more general settings [170, 188, 303], but subcubic bounds are known only in some very special cases [106].

Next, consider the problem of bounding the complexity of generalized Voronoi diagrams in higher dimensions. As mentioned above, when the objects in S are n points in $\mathbb{R}^d$ and the metric is Euclidean, the complexity of $\mathrm{Vor}(S)$ is $O(n^{\lceil d/2 \rceil})$. As d increases, this becomes significantly smaller than the naive $O(n^{d+1})$ bound or the improved bound, $O(n^{d+\varepsilon})$, obtained by viewing the Voronoi diagram as a lower envelope in $\mathbb{R}^{d+1}$. The same bound of $O(n^{\lceil d/2 \rceil})$ has recently been obtained in [75] for the complexity of the L_∞-diagram of n points in d-space (it was also shown that this bound is tight in the worst case). It is thus tempting to conjecture that the maximum complexity of generalized Voronoi diagrams in higher dimensions is close to this bound. Unfortunately, this was recently shown by Aronov to be false [43], by presenting a lower bound of $\Omega(n^{d-1})$. The sites used in this construction are convex polytopes, and the distance is either Euclidean or a polyhedral convex distance function. For $d = 3$, this lower bound does not contradict the conjecture made above, that the complexity of generalized Voronoi diagrams should be at most near-quadratic in this case. Also, in higher dimensions, the conjecture mentioned above is still not refuted when the sites are singleton points. Finally, for the general case, the construction by Aronov still leaves a gap of roughly a factor of n between the known upper and lower bounds.

9 Union of Geometric Objects

Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a set of n connected d-dimensional sets in $\mathbb{R}^d$. In this section, we want to study the complexity of $K = \bigcup_{i=1}^n K_i$. Most of the work to date on this problem has been in two or three dimensions.

Union of planar objects. Let us assume that each K_i is a *Jordan region*, bounded by a closed Jordan curve γ_i. Kedem *et al.* [220] have proved that if any two boundaries γ_i intersect in at most two points, then ∂K contains at most $6n - 12$ intersection points (provided $n \geq 3$), and that this bound is tight in the worst case. An immediate corollary of their result is that the number of intersection points on the boundary of the union of a collection of homothets of some fixed convex set is linear, because the boundaries of any two such homothetic copies in general position can intersect in at most two points. The bound

also holds when the homothets are not in general position. On the other hand, if pairs of boundaries may intersect in four or more points, then ∂K may contain $\Omega(n^2)$ intersection points in the worst case; see Figure 9.

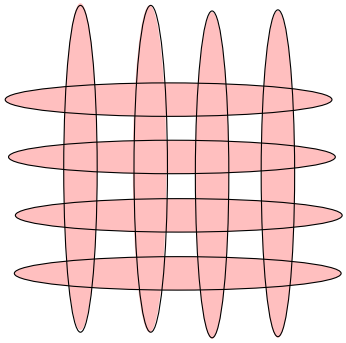

Figure 9: Union of Jordan regions.

This raises the question of what happens if any two boundaries intersect in at most three points. Notice that in general this question is meaningless, since any two closed curves must intersect in an even number of points (assuming nondegenerate configurations). To make the problem interesting, let Γ be a collection of n Jordan arcs, such that both endpoints of each arc $\gamma_i \in \Gamma$ lie on the x-axis, and such that K_i is the region between γ_i and the x-axis. Edelsbrunner *et al.* [139] have shown that the maximum combinatorial complexity of the union K is $\Theta(n\alpha(n))$. The upper bound requires a rather sophisticated analysis of the topological structure of K, and the lower bound follows from the construction by Wiernik and Sharir for lower envelopes of segments [327].

Next, consider the case when each K_i is a triangle in the plane. If the triangles are arbitrary, then a simple modification of the configuration shown in Figure 9 shows that K may have quadratic complexity in the worst case. But in this example the triangles have to be "thin," that is, some of their angles are very small. Matoušek *et al.* [248] have shown that if the given triangles are all *fat*, meaning that each of their angles is at least some fixed constant θ_0, then their union K has only a linear number of *holes* (i.e., connected components of K^c), and that the combinatorial complexity of K is $O(n \log\log n)$; the constants of proportionality in these bounds depend on θ_0. Alt *et al.* [38] proved that the complexity of the union of n fat wedges is $O(n)$. See [38, 320, 156] for other results on the union of fat objects. M. Bern asked the following related question.

Open Problem 7 *Let $\Delta = \{\Delta_1, \ldots, \Delta_n\}$ be a set of n triangles in the plane. Let a_i be the aspect ratio of the smallest rectangle enclosing Δ_i. Suppose $\sum_{i=1}^n a_i = O(n)$. What is the complexity of $\bigcup_{i=1}^n \Delta_i$?*

Recently, Efrat and Sharir [158] considered the case in which $\mathcal{K}$ is a collection of n *fat* convex regions in the plane, each pair of whose boundaries intersect in at most some

constant number s of points. Here fatness means that there exists a constant α such that for each object of S the ratio between the radii of its smallest enclosing disk and its largest inscribed disk is at most α. They showed that the complexity of the union K is $O(n^{1+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε, s, and α. Their proof requires as an initial but important substep an analysis of the number of *regular vertices* of the union: these are vertices of the union that are incident to two boundaries that intersect exactly twice. In fact, the analysis by Efrat and Sharir can only handle directly the irregular vertices of the union. Nevertheless, motivated by this problem, Pach and Sharir [274] have shown that, for an arbitrary collection of n convex regions, each pair of whose boundaries cross in a constant number of points, one has $R \leq 2I + 6n - 12$, where R (resp. I) is the number of regular (resp. irregular) vertices on the boundary of the union. This result has been used in [158] to obtain their near-linear bound. Nevertheless, regular vertices are interesting in their own right, and some additional results concerning them have recently been obtained by Aronov *et al.* [47]. First, if there are only regular vertices (i.e., every pair of boundaries intersect at most twice), then the inequality obtained by [274] implies that the complexity of the union in this case is at most $6n - 12$, so the result by Pach and Sharir extends the older results of [220]. In general, though, I can be quadratic, so the above inequality only yields a quadratic upper bound on the number of regular vertices of the union. However, it was shown in [47] that in many cases R is subquadratic. This is the case when the given regions are such that every pair of boundaries cross at most a constant number of times. If in addition all the regions are convex, the upper bound is close to $O(n^{3/2})$.

Aronov and Sharir [53] proved that the complexity of the union of n convex polygons in $\mathbb{R}^2$ with a total of s vertices is $O(n^2 + s\alpha(n))$.

Union in three and higher dimensions. Little is known about the complexity of the union in higher dimensions. It was recently shown in [75] that the maximum complexity of the union of n axis-parallel hypercubes in $\mathbb{R}^d$ is $\Theta(n^{\lceil d/2 \rceil})$, and this improves to $\Theta(n^{\lfloor d/2 \rfloor})$ if all the hypercubes have the same size. However, the following problem remains open.

Open Problem 8 *What is the complexity of the union of n congruent cubes in $\mathbb{R}^3$?*

Aronov and Sharir [51] proved that the complexity of the union of n convex polyhedra in $\mathbb{R}^3$ with a total of s faces is $O(n^3 + sn\log^2 n)$. The bound was improved by Aronov *et al.* [55] to $O(n^3 + sn\log s)$.

Unions of objects also arise as subproblems in the study of generalized Voronoi diagrams, as follows. Let S and ρ be as in the previous section (say, for the 3-dimensional case). Let K denote the region consisting of all points $x \in \mathbb{R}^3$ whose smallest distance from a site in S is at most r, for some fixed parameter $r > 0$. Then $K = \bigcup_{s \in S} B(s, r)$, where $B(s, r) = \{x \in \mathbb{R}^3 \mid \rho(x, s) \leq r\}$. We thus face the problem of bounding the combinatorial complexity of the union of n objects in $\mathbb{R}^3$ (of some special type). For example, if S is a set of

lines and ρ is the Euclidean distance, the objects are n congruent infinite cylinders in $\mathbb{R}^3$. In general, if the metric ρ is a distance function induced by some convex body P, the resulting objects are the *Minkowski sums* $s \oplus (-rP)$, for $s \in S$, where $A \oplus B = \{x+y \mid x \in A,\ y \in B\}$. Of course, this problem can also be stated in any higher dimension.

Since it has been conjectured that the complexity of the whole Voronoi diagram in $\mathbb{R}^3$ should be near-quadratic , the same conjecture should apply to the (simpler) structure K (whose boundary can be regarded as a *level curve* of the diagram at *height* r; it does indeed correspond to the cross-section at height r of the lower envelope in $\mathbb{R}^4$ that represents the diagram). Recently, this conjecture was confirmed by Aronov and Sharir in [54], in the special case where both P and the objects of S are convex polyhedra. They specialized their analysis of the union of convex polytopes to obtain an improved bound in the special case in which the polyhedra in question are Minkowski sums of the form $R_i \oplus P$, where the R_i's are n pairwise-disjoint convex polyhedra, P is a convex polyhedron, and the total number of faces of these Minkowski sums is s. The improved bounds are $O(ns \log n)$ and $\Omega(ns\alpha(n))$. They are indeed near-quadratic, as conjectured.

Recently, Agarwal and Sharir [26] showed that if S is a set of n lines and P is a sphere in $\mathbb{R}^3$, i.e., $\mathcal{K}$ is a set of n congruent cylinders, then the complexity of K is $O(n^{8/3+\varepsilon})$, for any $\varepsilon > 0$. Their proof works even if S is a set of segments in $\mathbb{R}^3$.

Open Problem 9 *Let Δ be a set of pairwise disjoint triangles in $\mathbb{R}^3$ and let B be a unit-radius ball. What is the complexity of the Minkowski sum of Δ and B?*

10 Decomposition of Arrangements

Many applications call for decomposing each cell of the arrangement into constant size cells; see Sections 12 and 13 for a sample of such applications. In this section we describe a few general schemes that have been proposed for decomposition of arrangements.

10.1 Triangulating hyperplane arrangements

Each k-dimensional cell in an arrangement of hyperplanes is a convex polyhedron, so we can triangulate it into k-simplices. If the cell is unbounded, some of the simplices in the triangulation will be unbounded. A commonly used scheme to triangulate a convex polytope $\mathcal{P}$ is the so-called *bottom-vertex triangulation*, denoted $\mathcal{P}^{\nabla}$. It recursively triangulates every face of $\mathcal{P}$ as follows. An edge is a one-dimensional simplex, so there is nothing to do. Suppose we have triangulated all j-dimensional cells of $\mathcal{P}$ for $j < k$. We now triangulate a k-dimensional cell C as follows. Let v be the vertex of C with the minimum x_d-coordinate. For each $(k-1)$-dimensional simplex Δ lying on the boundary of C but not containing v (Δ was constructed while triangulating a $(k-1)$-dimensional cell incident to C), we extend Δ to a k-dimensional simplex by taking the convex hull of Δ and v; see Figure 10(i).

(Unbounded cells require some care in this definition; see [110]). The number of simplices in $\mathcal{P}^\nabla$ is proportional to the number of vertices in $\mathcal{P}$.

If we want to triangulate the entire arrangement or more than one of its cells, we compute the bottom-vertex triangulation f^∇ for each face f in the increasing order of their dimension. Let $\mathcal{A}^\nabla(\Gamma)$ denote the bottom-vertex triangulation of $\mathcal{A}(\Gamma)$. A useful property of $A^\nabla(\Gamma)$ is that each simplex $\Delta \in \mathcal{A}^\nabla(\Gamma)$ is defined by a set $D(\Delta)$ of at most $d(d+3)/2$ hyperplanes of Γ, in the sense that $\Delta \in \mathcal{A}^\nabla(D(\Delta))$. Moreover, if $\mathcal{K}(\Delta) \subseteq \Gamma$ is the subset of hyperplanes intersecting Δ, then $\Delta \in \mathcal{A}^\nabla(R)$, for a subset $R \subseteq \Gamma$, if and only if $D(\Delta) \subseteq R$ and $\mathcal{K}(\Delta) \cap R = \emptyset$. A disadvantage of bottom-vertex triangulation is that some vertices may have large degree. Methods for obtaining low-degree triangulations have been proposed in two and three dimensions [128].

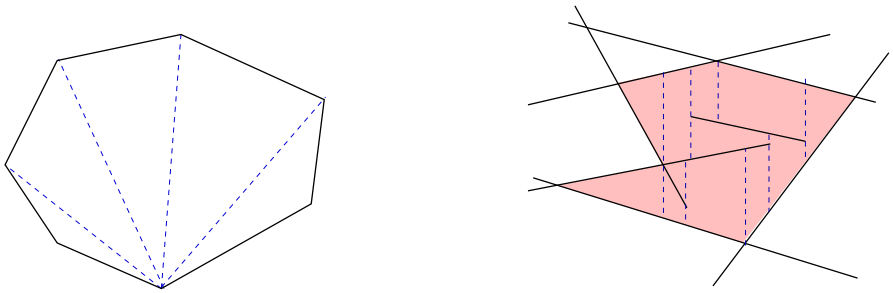

Figure 10: (i) Bottom vertex triangulation of a convex polygon; (ii) vertical decomposition of a cell in an arrangement of segments.

10.2 Vertical decomposition

Unfortunately, the bottom-vertex triangulation scheme does not work for arrangements of surfaces. Collins [114] described a general decomposition scheme, called *cylindrical algebraic decomposition*, that decomposes $\mathcal{A}(\Gamma)$ into $(bn)^{2^{O(d)}}$ cells, each semialgebraic of constant description complexity (however, the maximum algebraic degree involved in defining a cell grows exponentially with d) and homeomorphic to a ball of the appropriate dimension. Moreover, his algorithm produces a cell complex, i.e., closures of any two cells are either disjoint or their intersection is the closure of another lower-dimensional cell of the decomposition. This bound is quite far from the known trivial lower bound of $\Omega(n^d)$, which is a lower bound on the size of the arrangement. A significantly better scheme for decomposing arrangements of general surfaces is their *vertical decomposition*. Although vertical decompositions of polygons in the plane have been in use for a long time, it was extended to higher dimensions only in the late 1980s. We describe this method briefly.

Let C be a d-dimensional cell in $\mathcal{A}(\Gamma)$. The vertical decomposition, $C^{||}$, is computed as follows. We erect a vertical 'wall' up and down (in the x_d-direction) within C from each $(d-2)$-dimensional face of C and from points of vertical tangencies (i.e., the points at

which the tangent planes are parallel to the x_d-direction), and extend these walls until they hit another surface (or, failing this, all the way to $\pm\infty$). This results in a decomposition of C into subcells so that each subcell has a unique top facet and a unique bottom facet, and every vertical line cuts the subcell in a connected (possibly empty) interval. We next project each resulting subcell τ on the hyperplane $x_d = 0$. Let C_τ be the projected cell. We apply recursively the same technique to C_τ and compute its vertical decomposition $C_\tau^{||}$. (We continue the recursion in this manner until we reach $d = 1$.) We then "lift" $C_\tau^{||}$ back into $\mathbb{R}^d$, by extending each subcell $c \in C_\tau^{||}$ into the vertical cylinder $c \times \mathbb{R}$, and by clipping the cylinder within τ. Using a standard argument, it can be shown that each cell of $C^{||}$ is semialgebraic set of constant description complexity. In fact, they have the same structure as the Collins cells, but the number of subcells in $C^{||}$ is much smaller than that in the Collins decomposition of C. Applying the above step to each cell of $\mathcal{A}(\Gamma)$, we obtain the vertical decomposition $\mathcal{A}^{||}(\Gamma)$ of $\mathcal{A}(\Gamma)$. Note $\mathcal{A}^{||}(\Gamma)$ is not a cell complex.

It is easily seen that the complexity of the vertical decomposition of a cell in the plane is proportional to the number of edges in the cell. However, this is no longer the case in higher dimensions: Already for the case of a convex polytope with n facets in $\mathbb{R}^3$, the vertical decomposition may have complexity $\Omega(n^2)$.

Theorem 10.1 (Chazelle *et al.* [95, 96]) *The number of cells in the vertical decomposition $\mathcal{A}^{||}(\Gamma)$ of the arrangement $\mathcal{A}(\Gamma)$, for a set Γ of n surface patches in $\mathbb{R}^d$ satisfying (A1)–(A2), is $O(n^{2d-4}\lambda_q(n))$.*

The only known lower bound on the size of $\mathcal{A}^{||}(\Gamma)$ is the trivial $\Omega(n^d)$, so there is a considerable gap here, for $d > 3$; for $d = 3$ the two bounds nearly coincide. Improving the upper bound appears to be very challenging. This problem has been open since 1989; it seems difficult enough to preempt, at the present state of knowledge, any specific conjecture on the true maximum complexity of the vertical decomposition of arrangements in $d > 3$ dimensions.

Open Problem 10 *What is the complexity of the vertical decomposition of the arrangement of n surfaces in $\mathbb{R}^4$ satisfying assumptions (A1)–(A2)?*

The bound stated above applies to the vertical decomposition of an entire arrangement of surfaces. In many applications, however, one is interested in the vertical decomposition of only a portion of the arrangement, e.g., a single cell, the lower envelope, the zone of some surface, a specific collection of cells of the arrangement, etc. Since, in general, the complexity of such a portion is known (or conjectured) to be smaller than the complexity of the entire arrangement, one would like to conjecture that a similar phenomenon applies to vertical decompositions. Recently, it was shown by Schwarzkopf and Sharir [295] that the complexity of the vertical decomposition of a single cell in an arrangement of n surface patches in $\mathbb{R}^3$, as above, is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. A similar near-quadratic bound has been

obtained by Agarwal *et al.* [9] for the vertical decomposition of the region enclosed between the envelope and the upper envelope of two sets of bivariate surface patches. Another recent result by Agarwal *et al.* [14] gives a bound on the complexity of the vertical decomposition of $\mathcal{A}_{\leq k}(\Gamma)$ for a set Γ of surfaces in $\mathbb{R}^3$, which is only slightly larger that the worst-case complexity of $\mathcal{A}_{\leq k}(\Gamma)$.

Open Problem 11 *What is the complexity of the vertical decomposition of the minimization diagram of n surfaces in $\mathbb{R}^4$ satisfying assumptions (A1)–(A2)?*

Agarwal and Sharir [25] proved a near-cubic upper bound on the complexity of the vertical decomposition in the special case when the surfaces are graphs of trivariate polynomials and the intersection surface of any pair of surfaces is xy-monotone. In fact, their bound holds for a more general setting; see the original paper for details.

An interesting special case of vertical decomposition is that of hyperplanes. For such arrangements, the vertical decomposition is a too cumbersome construct, because, as described above, one can use the bottom-vertex triangulation (or any other triangulation) to decompose the arrangement into $\Theta(n^d)$ simplices. Still, it is probably a useful exercise to understand the complexity of the vertical decomposition of an arrangement of n hyperplanes in $\mathbb{R}^d$. A recent result by Guibas *et al.* [187] gives an almost tight bound of $O(n^4 \log n)$ for this quantity in $\mathbb{R}^4$, but nothing significantly better than the general bound is known for $d > 4$. Another interesting special case is that of triangles in 3-space. This has been studied by [120, 314], where almost tight bounds were obtained for the case of a single cell ($O(n^2 \log^2 n)$), and for the entire arrangement ($O(n^2 \alpha(n) \log n + K)$, where K is the complexity of the undecomposed arrangement). The first bound is slightly better than the general bound of [295] mentioned above. Tagansky [314] also derives sharp complexity bounds for the vertical decomposition of many cells in an arrangement of simplices, including the case of all nonconvex cells.

Objects	Bound	Source
Surfaces in $\mathbb{R}^d$, $d \geq 3$	$O(n^{2d-4}\lambda_q(n))$	[95, 304]
Triangles in $\mathbb{R}^3$	$O(n^2 \alpha(n) \log n + K)$	[120, 314]
Surfaces in $\mathbb{R}^3$, single cell	$O(n^{2+\varepsilon})$	[295]
Triangles in $\mathbb{R}^3$, zone w.r.t. an algebraic surface	$\Theta(n^2 \log^2 n)$	[314]
Surfaces in $\mathbb{R}^3$, ($\leq k$)-level	$O(n^{2+\varepsilon}k)$	[14]
Hyperplanes in $\mathbb{R}^4$	$O(n^4 \log n)$	[187]

Table 3: Combinatorial bounds on the maximum complexity of the vertical decomposition of n surfaces. In the second row, K is the combinatorial complexity of the arrangement.

10.3 Other decomposition schemes

Linearization, defined in Section 3, can be used to decompose the cells of the arrangement $\mathcal{A}(\Gamma)$ into cells of constant description complexity as follows. Suppose Γ admits a linearization of dimension k, i.e., there is a transformation $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ that maps each point $\mathbf{x} \in \mathbb{R}^d$ to a point $\varphi(\mathbf{x}) \in \mathbb{R}^k$, each surface $\gamma_i \in \Gamma$ to a hyperplane $h_i \subset \mathbb{R}^k$, and $\mathbb{R}^d$ to a d-dimensional surface $\Sigma \subseteq \mathbb{R}^k$. Let $H = \{h_i \mid 1 \leq i \leq n\}$. We compute the bottom-vertex triangulation $\mathcal{A}^\nabla(H)$ of $\mathcal{A}(H)$. For each simplex $\Delta \in \mathcal{A}^\nabla(H)$, let $\overline{\Delta} = \Delta \cap \Sigma$, and let $\Delta^* = \varphi^{-1}(\overline{\Delta})$ be the back projection of $\overline{\Delta}$ onto $\mathbb{R}^d$; Δ^* is a semialgebraic cell of constant description complexity. Set $\Xi = \{\Delta^* \mid \Delta \in \mathcal{A}^\nabla(H)\}$. Ξ is a decomposition of $\mathcal{A}(\Gamma)$ into cells of constant description complexity. If a simplex $\Delta \in \mathcal{A}^\nabla(H)$ intersects Σ, then Δ lies in the triangulation of a cell in $zone(\Sigma; H)$. Therefore, by Theorem 5.3, $|\Xi| = O(n^{\lfloor (d+k)/2 \rfloor} \log^\gamma n)$, where $\gamma = (d+k) \pmod 2$. Hence, we can conclude the following.

Theorem 10.2 *Let Γ be a set of hypersurfaces in $\mathbb{R}^d$ of degree at most b. If Γ admits a linearization of dimension k, then $\mathcal{A}(\Gamma)$ can be decomposed into $O(n^{\lfloor (d+k)/2 \rfloor} \log^\gamma n)$ cells of constant description complexity, where $\gamma = d + k \pmod 2$.*

As shown in Section 3, spheres in $\mathbb{R}^d$ admit a linearization of dimension $d+1$, therefore, the arrangement of n spheres in $\mathbb{R}^d$ can be decomposed into $O(n^d \log n)$ cells of constant description complexity.

Aronov and Sharir [50] proposed another scheme for decomposing arrangements of triangles in $\mathbb{R}^3$ by combining vertical decomposition and triangulation. They first decompose each three-dimensional cell of the arrangement into convex polyhedron, using an incremental procedure, and then they compute a bottom-vertex triangulation of each polyhedron. Other specialized decomposition schemes in $\mathbb{R}^3$ have been proposed in [205, 249].

10.4 Cuttings

All the decomposition schemes described in this section decompose $\mathbb{R}^d$ into cells of constant description complexity, so that each cell lies entirely in a single face of $\mathcal{A}(\Gamma)$. In many applications, however, it suffices to decompose $\mathbb{R}^d$ into cells of constant description complexity so that each cell intersects only a few surfaces of Γ. Such a decomposition lies at the heart of divide-and-conquer algorithms for numerous geometric problems.

Let Γ be a set of n surfaces in $\mathbb{R}^d$ satisfying assumptions (A1)–(A2). For a parameter $r \leq n$, a family $\Xi = \{\Delta_1, \ldots, \Delta_s\}$ of cells of constant description complexity with pairwise disjoint interiors is called a $(1/r)$-*cutting* of $\mathcal{A}(\Gamma)$ if the interior of each cell in Ξ is crossed by at most n/r surfaces of Γ and Ξ covers $\mathbb{R}^d$. If Γ is a set of hyperplanes, then Ξ is typically a set of simplices. Cuttings have led to efficient algorithms for a wide range of geometric problems and to improved bounds for several combinatorial problems. For example, the

proof by Clarkson *et al.* [108] on the complexity of m distinct cells in arrangements of lines uses cuttings; see the survey papers [3, 243] for a sample of applications of cuttings.

Clarkson [109] proved that a $(1/r)$-cutting of size $O(r^d \log^d r)$ exists for a set of hyperplanes in $\mathbb{R}^d$. The bound was improved by Chazelle and Friedman [100] to $O(r^d)$; see also [1, 237, 241]. An easy counting argument shows that this bound is optimal for any nondegenerate arrangement. There has been considerable work on computing optimal $(1/r)$-cuttings efficiently [1, 93, 207, 237, 241]. Chazelle [93] showed that a $(1/r)$-cutting for a set of n hyperplanes in $\mathbb{R}^d$ can be computed in time $O(nr^{d-1})$.

Using Haussler and Welzl's result on ε-nets [209], one can show that if, for any subset $R \subseteq \Gamma$, there exists a canonical decomposition of $\mathcal{A}(R)$ into at most $g(|R|)$ cells of constant description complexity, then there exists a $(1/r)$-cutting of $\mathcal{A}(\Gamma)$ of size $O(g(r \log r))$. By the result of Chazelle *et al.* [95] on the vertical decomposition of $\mathcal{A}(\Gamma)$, there exists a $(1/r)$-cutting of size $O((r \log r)^{2d-3+\varepsilon})$ of $\mathcal{A}(\Gamma)$. On the other hand, if Γ admits a linearization of dimension k, then there exists a $(1/r)$-cutting of size $O((r \log r)^{\lfloor (d+k)/2 \rfloor} \log r)$.

11 Representation of Arrangements

Before we begin to present algorithms for computing arrangements and their substructures, we need to describe how we represent arrangements and their substructures. Planar arrangements of lines can be represented using any standard data structure for representing planar graphs such as quad-edge or winged-edge data structures [192, 324]. However, representation of arrangements in higher dimensions is challenging because the topology of cells may be rather complex. Exactly how an arrangement is represented largely depends on the specific application for which we need to compute it. For example, representations may range from simply computing a representative point within each cell, or the vertices of the arrangement, to storing various spatial relationships between cells. We first review representations of hyperplane arrangements and then discuss surface arrangements.

Hyperplane arrangements. A simple way to represent a hyperplane arrangement $\mathcal{A}(\Gamma)$ is by storing its 1-*skeleton* [131]. That is, we construct a graph (V, E) whose nodes are the vertices of the arrangement. There is an edge between two nodes v_i, v_j if they are endpoints of an edge of the arrangement. Using the 1-skeleton of $\mathcal{A}(\Gamma)$, we can traverse the entire arrangement in a systematic way. The incidence relationship of various cells in $\mathcal{A}(\Gamma)$ can be represented using a data structure called *incidence graph*. A k-dimensional cell C is called a *subcell* of a $(k+1)$-dimensional cell C' if C lies on the boundary of C'; C' is called the *supercell* of C. We assume that the empty set is a (-1)-dimensional cell of $\mathcal{A}(\Gamma)$, which is a subcell of all vertices of $\mathcal{A}(\Gamma)$; and $\mathbb{R}^d$ is a $(d+1)$-dimensional cell, which is the supercell of all d-dimensional cells of $\mathcal{A}(\Gamma)$. The incidence graph of $\mathcal{A}(\Gamma)$ has a node for each cell of $\mathcal{A}(\Gamma)$, including the (-1)-dimensional and $(d+1)$-dimensional cells. There is a (directed) arc from a node C to another node C' if C is a subcell of C'; see Figure 11. Note that the

incidence graph forms a lattice. Many algorithms for computing the arrangement construct the incidence graph of the arrangement.

A disadvantage of 1-skeletons and incidence graphs is that they do not encode ordering information of cells. For examples, in planar arrangements of lines or segments, there is a natural ordering of edges incident to a vertex or of the edges incident to a two-dimensional face. The quad-edge data structure encodes this information for planar arrangements. Dobkin and Laszlo [129] extended the quad-edge data structure to $\mathbb{R}^3$, which was later extended to higher dimensions [78, 233, 234]. Dobkin *et al.* [126] described an algorithm for representing a simple polygon as a short Boolean formula, which can be used to store faces of segment arrangements to answer various queries efficiently.

Surface arrangements. Representing arrangements of surface patches is considerably more challenging than representing hyperplane arrangements because of the complex topology that cells in such an arrangement can have. A very simple representation of $\mathcal{A}(\Gamma)$ is to store a representative point from each cell of $\mathcal{A}(\Gamma)$ or to store the vertices of $\mathcal{A}(\Gamma)$. An even coarser representation of arrangements of graphs of polynomials is to store all realizable sign sequences. It turns out that this simple representation is sufficient for some applications [34, 72]. The notion of 1-skeleton can be generalized to arrangements of surfaces. However, all the connectivity information cannot be encoded by simply storing vertices and edges of the arrangement. Instead we need a finer one-dimensional structure, known as the *roadmap*. Road maps were originally introduced by Canny [85, 87] to represent a semialgebraic set. We can extend the notion of roadmaps to entire arrangements. Roughly speaking, a roadmap $\mathcal{R}(\Gamma)$ of $\mathcal{A}(\Gamma)$ is a one-dimensional semialgebraic set that satisfies the following two conditions.

(R1) For every cell C in $\mathcal{A}(\Gamma)$, $C \cap \mathcal{R}(\Gamma)$ is nonempty and connected.

(R2) Let C_w be the cross-section of a cell $C \in \mathcal{A}(\Gamma)$ at the hyperplane $x_1 = w$. For any $w \in \mathbb{R}$ and for cell $C \in \mathcal{A}(\Gamma)$, $C_w \neq \emptyset$ implies that every connected component of C_w intersects $\mathcal{R}(\Gamma)$.

Intuitively, a roadmap adds new *arcs* so that all the cells are connected and one can traverse the entire arrangement. same connected component of S. We can also define a roadmap of a substructure of the arrangement. See [66, 85] for details on roadmaps.

A roadmap does not represent "ordering" of cells in the arrangement or adjacency relationship among various cells. If we want to encode the adjacency relationship among higher dimensional cells of $\mathcal{A}(\Gamma)$, we can compute the vertical decomposition or the cylindrical algebraic decomposition of $\mathcal{A}(\Gamma)$ and compute the adjacency relationship of cells in the decomposition [42, 293]. Brisson [78] describes the *cell-tuple* data structure that encodes topological structures, ordering among cells, the boundaries of cells, and other information for cells of surface arrangements.

Many query-type applications (e.g., point location, ray shooting) call for preprocessing $\mathcal{A}(\Gamma)$ into a data structure so that various queries can be answered efficiently. In these cases, instead of storing various cells of an arrangement explicitly, we can store the arrangement implicitly, e.g., using cuttings. Chazelle *et al.* [96] have described how to preprocess arrangements of surfaces for point-location queries; Agarwal *et al.* [9] have described data structures for storing lower envelopes in $\mathbb{R}^4$ for point-location queries.

12 Computing Arrangements

We now review algorithms to compute the arrangement $\mathcal{A}(\Gamma)$ of a set Γ of n surface patches satisfying assumptions (A1)–(A2). As in Chapter DS-??, we need to assume here an appropriate model of computation in which various primitive operations on a constant number of surfaces can be performed in constant time. We will assume an infinite-precision real arithmetic model in which the roots of any polynomial of constant degree can be computed exactly in constant time.

Constructing arrangements of hyperplanes and simplices. Edelsbrunner *et al.* [147] describe an incremental algorithm that computes in time $O(n^d)$ the incidence graph of $\mathcal{A}(\Gamma)$, for a set Γ of n hyperplanes in $\mathbb{R}^d$. Roughly speaking, their algorithm adds the hyperplanes of Γ one by one and maintains the incidence graph of the arrangement of the hyperplanes added so far. Let Γ_i be the set of hyperplanes added in the first i stages, and let γ_{i+1} be the next hyperplane to be added. In the $(i+1)$st stage, the algorithm traces γ_{i+1} through $\mathcal{A}(\Gamma_i)$. If a k-face f of $\mathcal{A}(\Gamma_i)$ does not intersect γ_i, then f remains a face of $\mathcal{A}(\Gamma_{i+1})$. If f intersects γ_{i+1}, then $f \in \mathit{zone}(\gamma_{i+1}; \Gamma_i)$ and f is split into two k-faces f^+, f^-, lying in the two open halfspaces bounded by γ_{i+1}, and a $(k-1)$-face $f' = f \cap \gamma_{i+1}$. The algorithm therefore checks the faces of $\mathit{zone}(\gamma_{i+1}; \Gamma_i)$ whether they intersect γ_{i+1}. For each such intersecting face, it adds corresponding nodes in the incidence graph and updates the edges of the incidence graph. The $(i+1)$st stage can be completed in time proportional to the complexity of $\mathit{zone}(\gamma_{i+1}; \Gamma_i)$, which is $O(i^{d-1})$; see [132, 147]. Hence, the overall running time of the algorithm is $O(n^d)$.

A drawback of the algorithm just described is that it requires $O(n^d)$ "working" storage because it has to maintain the entire arrangement constructed so far in order to determine which of the cells intersect the new hyperplane. An interesting question is whether $\mathcal{A}(\Gamma)$ can be computed using only $O(n)$ working storage. Edelsbrunner and Guibas [138] proposed the *topological sweep* algorithm that can construct the arrangement of n lines in $O(n^2)$ time using $O(n)$ working storage. Their algorithm, which is a generalization of the sweep-line algorithm of Bentley and Ottmann [68], sweeps the plane by a pseudo-line. The algorithm by Edelsbrunner and Guibas can be extended to enumerate all vertices in an arrangement of n hyperplanes in $\mathbb{R}^d$ in $O(n^d)$ time using $O(n)$ space. See [40, 56, 151] for other topological-sweep algorithms. Avis and Fukuda [59] developed an algorithm that can enumerate in

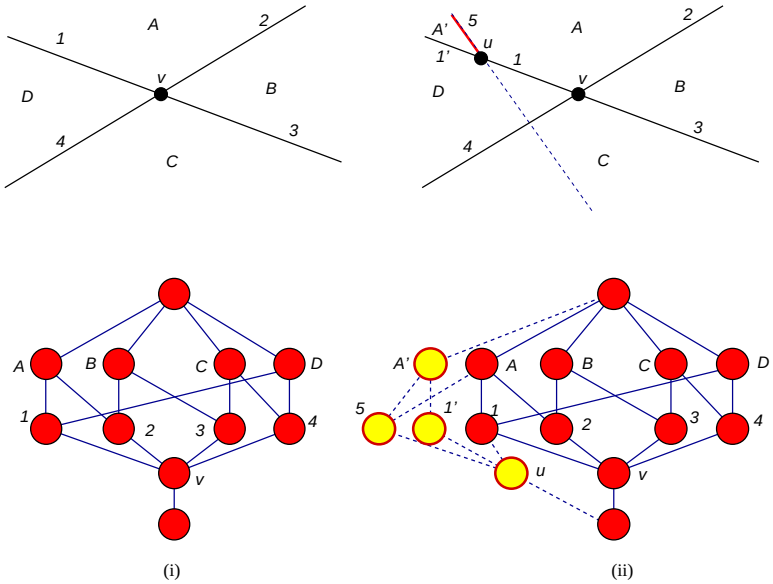

Figure 11: (i) Incidence graph of the arrangement of 2 lines. (ii) Adding a new line; incremental changes in the incidence graph as the vertex v, the edge 5 and the face A' are added.

$O(n^2k)$ time, using $O(n)$ space, all k vertices of the arrangement of a set Γ of n hyperplanes in $\mathbb{R}^d$ in which every vertex is incident to d hyperplanes. Their algorithm is useful when there are many parallel hyperplanes in Γ. See also [60, 172] for some related results.

Using the random-sampling technique, Clarkson and Shor [111] developed an $O(n \log n + k)$ expected time algorithm for constructing the arrangement of a set Γ of n line segments in the plane; here k is the number of vertices in $\mathcal{A}(\Gamma)$; see also [262, 263]. Chazelle and Edelsbrunner [94] developed a deterministic algorithm that can construct $\mathcal{A}(\Gamma)$ in time $O(n \log n + k)$, using $O(n+k)$ storage. The space complexity was improved to $O(n)$, without affecting the asymptotic running time, by Balaban [61]. If Γ is a set of n triangles in $\mathbb{R}^3$, $\mathcal{A}^{||}(\Gamma)$ can be constructed in $O(n^2 \log n + k)$ expected time using a randomized incremental algorithm [95, 304].

Chazelle and Friedman [101] describe an algorithm that can preprocess a set Γ of n hyperplanes into a data structure of size $O(n^d / \log^d n)$ so that a point-location query can be answered in $O(\log n)$ time. Their algorithm was later simplified by Matoušek [245] and Chazelle [93]. Mulmuley and Sen [265] developed a randomized dynamic data structure of size $O(n^d)$ for point location in arrangements of hyperplanes that can answer a point-location query in $O(\log n)$ expected time and can insert or delete a hyperplane in $O(n^{d-1} \log n)$ expected time. [com]Check the bound.[ment] Hagerup *et al.* [195] describe a ← randomized parallel algorithm for constructing the arrangement of hyperplanes under the CRCW model. Their algorithm runs in $O(\log n)$ time using $O(n^d / \log n)$ expected number

of processors. A deterministic algorithm under the CREW model with the same worst-case performance was proposed by Goodrich [179].

There has been some work on constructing arrangements of lines and segments using floating-point (finite precision) arithmetic. Milenkovic [257] developed a general technique called *double-precision geometry* that can be applied to compute arrangements of lines and segments in the plane. For example, if the coefficients of each line in a set Γ of n lines are represented using at most b bits, then his technique can compute $\mathcal{A}(\Gamma)$ in $O(n^3 \log n)$ time using at most $b + 20$ bits of precision. A careful implementation of the algorithm by Edelsbrunner *et al.* requires $3b$ bits of precision. Because of finite-precision arithmetic, Milenkovic's technique computes the coordinates of vertices approximately, and therefore produces a planar geometric graph, which is an arrangement of pseudolines. If the approximate arithmetic used by his algorithm makes relative error ε, then the maximum error in the coordinates of vertices of $\mathcal{A}(\Gamma)$ computed by his algorithm is $O(\sqrt{\varepsilon})$. Fortune and Milenkovic [168] showed that the sweep-line and incremental algorithms can be implemented so that the maximum error in the coordinates of vertices is at most $O(n\varepsilon)$. For all practical purposes this approach is better than the one described in [257]. See [182, 185, 256, 258] for a few additional results on constructing arrangements using floating-point arithmetic.

Constructing arrangements of surfaces. The algorithm by Edelsbrunner *et al.* [147] for computing hyperplane arrangements can be extended to computing the vertical decomposition $\mathcal{A}^{||}(\Gamma)$ for a set Γ of n arcs in the plane. In the $(i+1)$st step, the algorithm traces γ_{i+1} through $zone(\gamma_{i+1}; \Gamma_i)$ and updates the trapezoids of $\mathcal{A}^{||}(\Gamma_i)$ that intersect γ_{i+1}. The running time of the $(i+1)$st stage is $O(\lambda_{s+2}(i))$, where s is the maximum number of intersection points between a pair of arcs in Γ. Hence, the overall running time of the algorithm is $O(n\lambda_{s+2}(n))$. Suppose Γ is a set of arcs in the plane in general position. If the arcs in Γ are added in a random order and a "history dag," as described in Chapter DS-??, is used to efficiently find the trapezoids of $\mathcal{A}^{||}(\Gamma_i)$ that intersect γ_{i+1}, the expected running time of the algorithm can be improved to $O(n \log n + k)$, where k is the number of vertices in $\mathcal{A}(\Gamma)$.

Very little is known about computing the arrangement of a set Γ of surfaces in higher dimensions. Chazelle *et al.* [95] have shown that $\mathcal{A}^{||}(\Gamma)$ can be computed in randomized expected time $O(n^{2d-3+\varepsilon})$, using the random-sampling technique. Their algorithm can be made deterministic without increasing its asymptotic running time, but the deterministic algorithm is considerably more complex.

There has been some work for computing arrangements under the more realistic model of precise rational arithmetic model used in computational real algebraic geometry [71]. Canny [88] had described an $(nb)^{O(d)}$-time algorithm for computing a sample point from each cell of the arrangement of a set of n hypersurfaces in $\mathbb{R}^d$, each of degree at most b. The running time was improved by Basu *et al.* [67] to $n^{d+1}b^{O(d)}$. Basu *et al.* [66] described an $n^{d+1}b^{O(d^2)}$-time algorithm for computing the roadmap of a semialgebraic set defined by n polynomials, each of degree at most b. Although their goal is to develop the road map of

a semialgebraic set, their algorithm first constructs the road map of the entire arrangement of the surfaces defining the semialgebraic set and then outputs the appropriate portion of the map.

13 Computing Substructures in Arrangements

13.1 Lower envelopes

Let Γ be a set of surface patches satisfying assumptions (A1)–(A3). We want to compute the minimization diagram $\mathcal{M}(\Gamma)$ of Γ. We described in Chapter DS-?? algorithms for computing the minimization diagram of a set of arcs in the plane. In this chapter we will focus on minimization diagrams of sets of surface patches in higher dimensions. There are again several choices, depending on the application, as to what exactly we want to compute. The simplest choice is to compute the vertices or the 1-skeleton of $\mathcal{M}(\Gamma)$. A more difficult task is to compute all the faces of $\mathcal{M}(\Gamma)$ and represent them using any of the mechanisms described in the previous section. Another challenging task, which is required in many applications, is to store Γ into a data structure so that $L_\Gamma(\mathbf{x})$, for any point $\mathbf{x} \in \mathbb{R}^{d-1}$, can be computed efficiently.

For collections Γ of surface patches in $\mathbb{R}^3$, the minimization diagram $\mathcal{M}(\Gamma)$ is a planar subdivision. In this case, the latter two tasks are not significantly harder than the first one, because we can preprocess $\mathcal{M}(\Gamma)$ using any optimal planar point-location algorithm [122]. Several algorithms have been developed for computing the minimization diagram of bivariate (partial) surface patches [9, 73, 74, 118, 303]. Some of these techniques use randomized algorithms, and their expected running time is $O(n^{2+\varepsilon})$, which is comparable with the maximum complexity of the minimization diagram of bivariate surface patches. The simplest algorithm is probably the deterministic divide-and-conquer algorithm presented by Agarwal *et al.* [21]. It partitions Γ into two subsets Γ_1, Γ_2 of roughly equal size, and computes recursively the minimization diagrams $\mathcal{M}_1$, $\mathcal{M}_2$ of Γ_1 and Γ_2, respectively. It then computes the overlay $\mathcal{M}^*$ of $\mathcal{M}_1$ and $\mathcal{M}_2$. Over each face f of $\mathcal{M}^*$ there are only (at most) two surface patches that can attain the final envelope (the one attaining $L(\Gamma_1)$ over f and the one attaining $L(\Gamma_2)$ over f), so we compute the minimization diagram of these two surface patches over f, replace f by this refined diagram, and repeat this step for all faces of $\mathcal{M}^*$. We finally merge any two adjacent faces f, f' of the resulting subdivision if the same surface patches attain $L(\Gamma)$ over both f and f'. It is easy to see that the cost of this step is proportional to the number of faces of $\mathcal{M}^*$. By the result of Agarwal *et al.* [21], $\mathcal{M}^*$ has $O(n^{2+\varepsilon})$ faces. This implies that the complexity of the above divide-and-conquer algorithm is $O(n^{2+\varepsilon})$. If Γ is a set of triangles in $\mathbb{R}^3$, the running time of the algorithm is $O(n^2\alpha(n))$ [141]. This divide-and-conquer algorithm can also be used to compute $S(\Gamma, \Gamma')$, the region lying above all surface patches of one collection Γ' and below all surface patches of another collection Γ, in time $O(n^{2+\varepsilon})$, where $n = |\Gamma| + |\Gamma'|$ [21].

A more difficult problem is to devise *output-sensitive* algorithms for computing $\mathcal{M}(\Gamma)$, whose complexity depends on the actual combinatorial complexity of the envelope. A rather complex algorithm is presented by De Berg [117] for the case of triangles in $\mathbb{R}^3$, whose running time is $O(n^{4/3+\varepsilon}+n^{4/5+\varepsilon}k^{4/5})$, where k is the number of vertices in $\mathcal{M}(\Gamma)$. [com Check the bound.ment] If the triangles in Γ are pairwise disjoint, the running time can be improved to $O(n^{1+\varepsilon}+n^{2/3+\varepsilon}k^{2/3})$ [17, 117].

The algorithm by Edelsbrunner *et al.* [141] can be extended to compute in $O(n^{d-1}\alpha(n))$ time all faces of the minimization diagram of $(d-1)$-simplices in $\mathbb{R}^d$ for $d \geq 4$. However, little is known about computing the minization diagram of more general surface patches in $d \geq 4$ dimensions. Let Γ be a set of surface patches in $\mathbb{R}^d$ satisfying assumptions (A1)–(A2). Agarwal *et al.* [9] showed that all vertices, edges and 2-faces of $\mathcal{M}(\Gamma)$ can be computed in randomized expected time $O(n^{d-1+\varepsilon})$. We sketch their algorithm below.

Assume that Γ satisfies assumptions (A1)–(A5). Fix a $(d-2)$-tuple of surface patches, say $\gamma_1, \ldots, \gamma_{d-2}$, and decompose their common intersection $\bigcap_{i=1}^{d-2}\gamma_i$ into smooth, x_1x_2-monotone, connected patches, using a stratification algorithm. Let Π be one such piece. Each surface γ_i, for $i \geq d-1$, intersects Π at a curve ξ_i, which partitions Π into two regions. If we regard each γ_i as the graph of a partially defined $(d-1)$-variate function, then we can define $K_i \subseteq \Pi$ to be the region whose projection on the hyperplane $H: x_d = 0$ consists of points $\mathbf{x}$ at which $\gamma_i(\mathbf{x}) \geq \gamma_1(\mathbf{x}) = \cdots = \gamma_{d-2}(\mathbf{x})$. The intersection $Q = \bigcap_{i \geq d-1} K_i$ is equal to the portion of Π that appears along the lower envelope $L(\Gamma)$. We repeat this procedure for all patches of the intersection $\bigcap_{i=1}^{d-2}\gamma_i$ and for all $(d-2)$-tuples of surface patches. This will give all the vertices, edges and 2-faces of $L(\Gamma)$.

Since Π is x_1x_2-monotone 2-manifold, computing Q is essentially the same as computing the intersection of $n-d+2$ planar regions. Q can thus be computed using an appropriate variant of the randomized incremental approach [118, 98]. It adds $\xi_i = \gamma_i \cap \Pi$ one by one in a random order (ξ may consist of $O(1)$ arcs), and maintains the intersection of the regions K_i for the arcs added so far. Let Q_r denote this intersection after r arcs have been added. We maintain the "vertical decomposition" of Q_r (within Π), and represent Q_r as a collection of *pseudo-trapezoids*. We maintain additional data structures, including a *history dag* and a union-find structure, and proceed exactly as in [118, 98] (See DS-??). We omit here the details.

We define the *weight* of a pseudo-trapezoid τ to be the number of surface patches γ_i, for $i \geq d-1$, whose graphs either cross τ or hide τ completely from the lower envelope (excluding the up to 4 function graphs whose intersections with Π *define* τ). The cost of the above procedure, summed over all $(d-2)$-tuples of Γ, is proportional to the number of pseudo-trapezoids that are created during the execution of the algorithm, plus the sum of their weights, plus an overhead term of $O(n^{d-1})$ needed to prepare the collections of arcs ξ_i over all two-dimensional patches Π. Modifying the analysis in the papers cited above, Agarwal *et al.* prove the following.

Theorem 13.1 (Agarwal *et al.* [9]) *Let Γ be a set of n surface patches in $\mathbb{R}^d$ satisfying*

assumptions (A1)–(A2). The vertices, edges, and 2-faces of $\mathcal{M}(\Gamma)$ can be computed in randomized expected time $O(n^{d-1+\varepsilon})$, for any $\varepsilon > 0$.

For $d = 4$, the above algorithm can be extended to compute the incidence graph (or cell-tuple structure) of $\mathcal{M}(\Gamma)$. Their approach, however, falls short of computing such representations for $d > 4$. Agarwal *et al.* also show that the three-dimensional point-location algorithm by Preparata and Tamassia [284] can be extended to preprocess a set of trivariate surface patches in time $O(n^{3+\varepsilon})$ into a data structure of size $O(n^{3+\varepsilon})$ so that $L_\Gamma(\mathbf{x})$, for any point $\mathbf{x} \in \mathbb{R}^3$, can be computed in $O(\log^2 n)$ time.

Open Problem 12 *Let Γ be a set of n surface patches in $\mathbb{R}^d$, for $d > 4$, satisfying assumptions (A1)–(A3). How fast can Γ be preprocessed, so that $L_\Gamma(\mathbf{x})$, for a query point $\mathbf{x} \in \mathbb{R}^{d-1}$, can be computed efficiently?*

13.2 Single cells

Computing a single cell in an arrangement of n hyperplanes in $\mathbb{R}^d$ is equivalent, by duality, to computing the convex hull of a set of n points in $\mathbb{R}^d$ and is therefore a widely studied problem; see, e.g., [132, 299] for a summary of known results. For $d \geq 4$, an $O(n^{\lfloor d/2 \rfloor})$ expected-time algorithm for this problem was proposed by Clarkson and Shor [111] (see also [298]), which is optimal in the worst case. By derandomizing this algorithm, Chazelle [21] developed an $O(n^{\lfloor d/2 \rfloor})$-time deterministic algorithm. A somewhat simpler algorithm with the same running time was later proposed by Brönnimann *et al.* [80]. This result implies that the Euclidean Voronoi diagram of a set of n points in $\mathbb{R}^d$ can be computed in time $O(n^{\lceil d/2 \rceil})$.

Since the complexity of a cell may vary between $O(1)$ and $O(n^{\lfloor d/2 \rfloor})$, output-sensitive algorithms have been developed for computing a single cell in hyperplane arrangements [103, 224]. For $d \leq 3$, Clarkson and Shor [111] gave randomized algorithms with expected time $O(n \log h)$, where h is the complexity of the cell, provided that the planes are in general position. Simple deterministic algorithms with the same worst-case bound were developed by Chan [90]. Seidel [296] proposed an algorithm whose running time is $O(n^2 + h \log n)$; the first term can be improved to $O(n^{2-2/(\lfloor d/2 \rfloor + 1)} \log^c n)$ [244] or to $O((nh)^{1-1/(\lfloor d/2 \rfloor + 1)} \log^c n)$ [91]. Chan *et al.* [92] described another output-sensitive algorithm whose running time is $O((n + (nf)^{1-1/\lceil d/2 \rceil} + fn^{1-2/\lceil d/2 \rceil}) \log^c n)$. Avis *et al.* [58] described an algorithm that can compute in $O(nf)$ time, using $O(n)$ space, all f vertices of a cell in an arrangement of n hyperplanes in $\mathbb{R}^d$; see also [77, 171]. All these output-sensitive bounds hold only for simple arrangements. Although many of these algorithms can be extended to nonsimple arrangements, the running time increases.

As mentioned in Chapter DS-??, Guibas *et al.* [191] developed an $O(\lambda_{s+2}(n) \log^2 n)$-time algorithm for computing a single face in an arrangement of n arcs, each pair of which intersect in at most s points. Later a randomized algorithm with expected time $O(\lambda_{s+2}(n) \log n)$ was developed by Chazelle *et al.* [98]. Since the complexity of the vertical decomposition of

a single cell in an arrangement of n surface patches in $\mathbb{R}^3$ is $O(n^{2+\varepsilon})$ [295], an application of the random-sampling technique yields an algorithm for computing a single cell in time $O(n^{2+\varepsilon})$ in an arrangement of n surface patches in $\mathbb{R}^3$ [295]. If Γ is a set of triangles, the running time can be improved to $O(n^2 \log^3 n)$ [118]. Halperin [196] developed faster algorithms for computing a single cell in arrangements of "special" classes of bivariate surfaces that arise in motion-planning applications.

13.3 Levels

Constructing the $\leq k$-level. Let Γ be a set of n arcs in the plane, each pair of which intersect in at most s points. $\mathcal{A}_{\leq k}(\Gamma)$ can be computed by a simple divide-and-conquer algorithm as follows [301]. Partition Γ into two subsets Γ_1, Γ_2, each of size at most $\lceil n/2 \rceil$, compute recursively $\mathcal{A}_{\leq k}(\Gamma_1), \mathcal{A}_{\leq k}(\Gamma_2)$, and then use a sweep-line algorithm to compute $\mathcal{A}_{\leq k}(\Gamma)$ from $\mathcal{A}_{\leq k}(\Gamma_1)$ and $\mathcal{A}_{\leq k}(\Gamma_2)$. The time spent in the merge step is proportional to the number of vertices in $\mathcal{A}_{\leq k}(\Gamma_1), \mathcal{A}_{\leq k}(\Gamma_2)$ and the number of intersections points between the edges of two subdivisions, each of which is a vertex of $\mathcal{A}(\Gamma)$ whose level is at most $2k$. Using Theorem 6.1, the total time spent in the merge step is $O(\lambda_{s+2}(n)k \log n)$. Hence, the overall running time of the algorithm is $O(\lambda_{s+2}(n)k \log^2 n)$. If we use a randomized incremental algorithm that adds arcs one by one in a random order and maintains $\mathcal{A}_{\leq k}(\Gamma_i)$, where Γ_i is the set of arcs added so far, the expected running time of the algorithm is $O(\lambda_{s+2}(n)k \log(n/k))$; see, e.g., [264]. Everett *et al.* [164] showed that if Γ is a set of n lines, the expected running time can be improved to $O(n \log n + nk)$. Recently Agarwal *et al.* [13] gave another randomized incremental algorithm that can compute $\mathcal{A}_{\leq k}(\Gamma)$ in expected time $O(\lambda_{s+2}(n)(k + \log n))$.

In higher dimensions, little is known about computing $\mathcal{A}_{\leq k}(\Gamma)$, for collections Γ of surface patches. For $d = 3$, Mulmuley [264] gave a randomized incremental algorithm for computing the $\leq k$-level in an arrangement of n planes whose expected running time is $O(nk^2 \log(n/k))$. The expected running time can be improved to $O(n \log^3 n + nk^2)$ using the algorithm by Agarwal *et al.* [13]. There are, however, several technical difficulties in extending this approach to arrangements of surface patches. Using the random-sampling technique, Agarwal *et al.* [14] developed an $O(n^{2+\varepsilon}k)$ expected-time algorithm for computing $\mathcal{A}_{\leq k}(\Gamma)$, for a collection Γ of n surface patches in $\mathbb{R}^3$. Their algorithm can be derandomized without affecting the asymptotic running time. For $d \geq 4$, Mulmuley's algorithm can compute the $\leq k$-level in arrangements of n hyperplanes in expected time $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$. No efficient algorithm is known for computing the $\leq k$-level in surface arrangements because no nontrivial bound is known for the complexity of the vertical decomposition of $\mathcal{A}_{\leq k}(\Gamma)$ in $d \geq 4$ dimensions.

Constructing a single level. Edelsbrunner and Welzl [154] gave an $O(n \log n + b \log^2 n)$-time algorithm to construct the k-level in an arrangement of n lines in the plane, where b is the number of vertices of the k-level. This bound was slightly improved by Cole *et*

al. [113] to $O(n \log n + b \log^2 k)$. However, these algorithms do not extend to computing the k-level in arrangements of curves. The approach by Agarwal *et al.* [13] can compute the k-level in an arrangement of lines in randomized expected time $O(n \log^2 n + nk^{1/3} \log^{2/3} n)$, and it extends to arrangements of curves and to arrangements of hyperplanes. Agarwal and Matoušek [19] describe an output-sensitive algorithm for computing the k-level in an arrangement of planes. The running time of their algorithm, after a slight improvement by Chan [91], is $O(n \log b + b^{1+\varepsilon})$, where b is the number of vertices of the k-level. Their algorithm can compute the k-level in an arrangement of hyperplanes in $\mathbb{R}^d$ in time $O(n \log b + (nb)^{1-1/(\lfloor d/2 \rfloor + 1)+\varepsilon} + bn^{1-2/(\lfloor d/2 \rfloor + 1)+\varepsilon})$. As in the case of single cells, all the output-sensitive algorithms assume that the hyperplanes are in general position.

13.4 Marked cells

Let Γ be a set of n lines in the plane and S a set of m points in the plane. Edelsbrunner *et al.* [142] presented a randomized algorithm, based on the random-sampling technique, for computing $\mathcal{C}(S, \Gamma)$, the set of cells in $\mathcal{A}(\Gamma)$ that contain at least one point of S, whose expected running time is $O(m^{2/3-\varepsilon} n^{2/3+2\varepsilon} \log n + m \log n + n \log n \log m)$, for any $\varepsilon > 0$. A deterministic algorithm with running time $O(m^{2/3} n^{2/3} \log^c n + n \log^3 n + m \log n)$ was developed by Agarwal [2]. However, both algorithms are rather complicated. A simple randomized divide-and-conquer algorithm, with $O((m\sqrt{n}+n) \log n)$ expected running time, was recently proposed by Agarwal *et al.* [20]. Using random sampling, they improved the expected running time to $O(m^{2/3} n^{2/3} \log^{2/3}(n/\sqrt{m}) + (m+n) \log n)$. If we are interested in computing the incidences between Γ and S, the best-known algorithm is by Matoušek whose expected running time is $O(m^{2/3} n^{2/3} 2^{O(\log^*(m+n))} + (m+n) \log(m+n))$ [246]. His algorithm can be extended to higher dimensions. The number of incidences between m points and n hyperplanes in $\mathbb{R}^d$ can be counted in time $O((mn)^{1-1/(d+1)} 2^{O(\log^*(m+n))} + (m+n) \log(m+n))$ [246].

The above algorithms can be modified to compute marked cells in arrangements of segments in the plane. The best-known randomized algorithm is by Agarwal *et al.* [20] whose running time is $O(m^{2/3} n^{2/3} \log^{2/3}(n/\sqrt{m}) \alpha^{1/3}(n/\sqrt{m}) + (m + n \log n) \log n)$. Little is known about computing marked cells in arrangements of arcs in the plane. Using a randomized incremental algorithm, $\mathcal{C}(S, \Gamma)$ can be computed in expected time $O(\lambda_{s+2}(n) \sqrt{m} \log n)$, where s is the maximum number of intersection points between a pair of arcs in Γ [304]. If Γ is a set of n unit-radius circles and S is a set of m points in the plane, the incidences between Γ and S can be computed using Matoušek's algorithm [246].

Randomized incremental algorithms can be used to construct marked cells in arrangements of hyperplanes in higher dimensions in time close to their worst-case complexity. For example, if Γ is a set of n planes in $\mathbb{R}^3$ and S is a set of m points in $\mathbb{R}^3$, then the incidence graph of cells in $\mathcal{C}(S, \Gamma)$ can be computed in expected time $O(nm^{2/3} \log n)$ [118]. For $d \geq 4$, the expected running time is $O(m^{1/2} n^{d/2} \log^\gamma n)$, where $\gamma = (\lfloor d/2 \rfloor - 1)/2$. De Berg *et al.* [123] describe an efficient point-location algorithm in the zone of a k-flat in an

arrangement of hyperplanes in $\mathbb{R}^d$. Their algorithm can answer a query in $O(\log n)$ time using $O(n^{\lfloor (d+k)/2 \rfloor} \log^{\gamma} n)$ space, where $\gamma = d + k \pmod 2$.

13.5 Union of objects

Let Γ be a set of n semialgebraic simply connected regions in the plane, each of constant description complexity. The union of Γ can be computed in $O(f(n) \log^2 n)$ time by a divide-and-conquer technique, similar to that described in Section 13.3 for computing $\mathcal{A}_{\leq k}(\Gamma)$. Here $f(m)$ is the maximum complexity of the union of a subset of Γ of size m. Alternatively, $\bigcup \Gamma$ can be computed in $O(f(n) \log n)$ expected time using the *lazy* randomized incremental algorithm by De Berg *et al.* [118]. As a consequence, the union of n convex fat objects, each of constant description complexity, can be computed in $O(n^{1+\varepsilon})$ time, for any $\varepsilon > 0$; see Section 9.

Aronov *et al.* [55] modified the approach by Agarwal *et al.* [9] so that the union of n convex polytopes in $\mathbb{R}^3$ with a total of s vertices can be computed in expected time $O(sn \log n \log s + n^3)$. The same approach can be used to compute the union of n congruent cylinders in time $O(n^{8/3+\varepsilon})$. (Again, consult Section 9 for the corresponding bounds on the complexity of the union.)

Many applications call for computing the volume or surface area of $\bigcup \Gamma$ instead of its combinatorial structure. Overmars and Yap [270] showed that the volume of the union of n axis-parallel boxes in $\mathbb{R}^d$ can be computed in $O(n^{d/2} \log n)$ time. Edelsbrunner [134] gave an elegant formula for the volume and the surface area of the union of n balls in $\mathbb{R}^d$, which can be used to compute the volume efficiently.

14 Applications

In this section we present a sample of applications of arrangements. We discuss a few specific problems that can be reduced to bounding the complexity of various substructures of arrangements of surfaces or to computing these substructures. We also mention a few general areas that have motivated several problems involving arrangements and in which arrangements have played an important role.

14.1 Range searching

A typical range searching problem is defined as follows: *Preprocess a set S of n points in $\mathbb{R}^d$, so that all points of S lying in a query region can be reported (or counted) quickly.* A special case of range searching is halfspace range searching, in which the query region is a halfspace. Because of numerous applications, range searching has received much attention during the last twenty years. See [16, 247] for recent surveys on range searching and its

applications.

If we define the dual of a point $p = (a_1, \ldots, a_d)$ to be the hyperplane $p^* : x_d = -a_1 x_1 - \cdots - a_{d-1} x_{d-1} + a_d$, and the dual of a hyperplane $h : x_d = b_1 x_1 + \cdots + b_{d-1} x_{d-1} + b_d$ to be the point $h^* = (b_1, \ldots b_d)$, then p lies above (resp. below, on) h if and only if the hyperplane p^* lies above (resp. below, on) the point h^*. Hence, halfspace range searching has the following equivalent "dual" formulation: Preprocess a set Γ of n hyperplanes in $\mathbb{R}^d$ so that the hyperplanes of H lying below a query point can be reported quickly, or the level of a query point can be computed quickly. Using the point-location data structure for hyperplane arrangements given in [93], the level of a query point can be computed in $O(\log n)$ time using $O(n^d/\log^d n)$ space. This data structure can be modified to report all t hyperplanes lying below a query point in time $O(\log n + t)$. Chazelle *et al.* [102] showed, using results on arrangements, that a two-dimensional halfspace range reporting query can be answered in $O(\log n + t)$ time using $O(n)$ space [102]. In higher dimensions, by constructing $(1/r)$-cuttings for $\mathcal{A}_{\leq k}(\Gamma)$, Matoušek [242] developed a data structure that can answer a halfspace range reporting query in time $O(\log n + t)$ using $O(n^{\lfloor d/2 \rfloor} \log^c n)$ space, for some constant c. He also developed a data structure that can answer a query in time $O(n^{1-1/\lfloor d/2 \rfloor} \log^c n + t)$ using $O(n \log\log n)$ space [242]. See also [6, 104]. Using linearization, a semialgebraic range-searching query, where one wants to report all points of S lying inside a semialgebraic set of constant description complexity, can be answered efficiently using some of the halfspace range-searching data structures [18, 329].

Point-location in hyperplane arrangements can be used for simplex range searching [105], ray shooting [17, 18, 249], and several other geometric searching problems [28].

14.2 Terrain visualization

Let Σ be a *polyhedral terrain* in $\mathbb{R}^3$ with n edges; that is, Σ is the graph of a continuous piecewise-linear bivariate function, so it intersects each vertical line in exactly one point. The *orthographic view* of Σ in direction $b \in \mathbb{S}^2$ is the decomposition of Π, a plane normal to the direction b and placed at infinity, into maximal regions so that the rays emerging in direction b from all points in such a region hit the same face of Σ, or none of them hit Σ. The *perspective view* of Σ from a point $a \in \mathbb{R}^3$ is the decomposition of $\mathbb{S}^2$ into maximal connected regions so that, for each region $R \subseteq \mathbb{S}^2$ and for all points $b \in R$, either the first intersection point of Σ and the ray r emanating from a in direction b lie in the same face of Σ (which depends on R), or none of these rays meet Σ. The orthographic (resp. perspective) *aspect graph* of Σ represents all topologically different orthographic (resp. perspective) views of Σ. For background and a survey of recent research on aspect graphs, see [76]. Here we will show how the complexity bounds for lower envelopes can be used to derive near-optimal bounds on the aspect graphs of polyhedral terrains.

A pair of parallel rays (ρ_1, ρ_2) is called *critical* if for each $i = 1, 2$, the source point of ρ_i lies on an edge a_i of Σ, ρ_i passes through three edges of Σ (including a_i), and ρ_i does not intersect the (open) region lying below Σ. It can be shown that the number of

topologically different orthographic views of Σ is $O(n^5)$ plus the number of critical pairs of parallel rays. Fix a pair a_1, a_2 of edges of Σ. Agarwal and Sharir [23] define, for each pair (a_1, a_2) of edges of Σ, a collection $\mathcal{F}_{a_1,a_2}$ of n trivariate functions, so that every pair (ρ_1, ρ_2) of critical rays, where ρ_i emanates from a point on a_i (for $i = 1, 2$), corresponds to a vertex of $\mathcal{M}(\mathcal{F}_{a_1,a_2})$. They also show that the graphs of the functions in $\mathcal{F}_{a_1,a_2}$ satisfy assumptions (A1)–(A2). Using Theorem 3.1 and summing over all pairs of edges of Σ, we can conclude that the number of critical pairs of rays, and thus the number of topologically different orthographic views of Σ, is $O(n^{5+\varepsilon})$. Using a more careful analysis, Halperin and Sharir [202] proved that the number of different orthographic views is $n^5 2^{O(\sqrt{\log n})}$. De Berg *et al.* [121] have constructed a terrain for which there are $\Omega(n^5)$ topologically different orthographic views. If Σ is an arbitrary polyhedral set with n edges, the maximum possible number of topologically different orthographic views of Σ is $\Theta(n^6)$ [282]. De Berg *et al.* [121] showed that if Σ is a set of k pairwise-disjoint convex polytopes with a total of n vertices, then the number of orthographic views is $O(n^4k^2)$; the best-known lower bound is $\Omega(n^2k^4)$.

Agarwal and Sharir extended their approach to bound the number of perspective views of a terrain. They argue that the number of perspective views of Σ is proportional to the number of triples of rays emanating from a common point, each of which passes through three edges of Σ before intersecting the open region lying below Σ. Following a similar approach to the one sketched above, they reduce the problem to the analysis of lower envelopes of $O(n^3)$ families of 5-variate functions, each family consisting of $O(n)$ functions that satisfy assumptions (A1)–(A2). This leads to an overall bound of $O(n^{8+\varepsilon})$ for the number of topologically different perspective views of Σ. This bound is also known to be almost tight in the worst case, as follows from another lower-bound construction given by De Berg *et al.* [121]. Again, in contrast, If Σ is an arbitrary polyhedral set with n edges, the maximum possible number of topologically different perspective views of Σ is $\Theta(n^9)$ [282].

14.3 Transversals

Let S be a set of n compact convex sets in $\mathbb{R}^d$. A hyperplane h is called a *transversal* of S if h intersects every member of S. Let $\mathcal{T}(S)$ denote the space of all hyperplane transversals of S. We wish to study the structure of $\mathcal{T}(S)$. To facilitate this study, we apply the dual transform described in Section 14.1. Let $h : x_d = a_1x_1 + \cdots + a_{d-1}x_{d-1} + a_d$ be a hyperplane that intersects a set $s \in S$. Translate h up and down until it becomes tangent to s. Denote the resulting upper and lower tangent hyperplanes by

$$x_d = a_1x_1 + \cdots + a_{d-1}x_{d-1} + U_s(a_1, \ldots, a_{d-1})$$

and

$$x_d = a_1x_1 + \cdots + a_{d-1}x_{d-1} + L_s(a_1, \ldots, a_{d-1}),$$

respectively. Then we have

$$L_s(a_1, \ldots, a_{d-1}) \leq a_d \leq U_s(a_1, \ldots, a_{d-1}).$$

Now if h is a transversal of S, we must have

$$\max_{s \in S} L_s(a_1, \ldots, a_{d-1}) \leq a_d \leq \min_{s \in S} U_s(a_1, \ldots, a_{d-1}).$$

In other words, if we define $\Gamma = \{U_s \mid s \in S\}$ and $\Gamma' = \{L_s \mid s \in S\}$, then $\mathcal{T}(S)$ is $\mathcal{S}(\Gamma, \Gamma')$, the region lying above the lower envelope of Γ and below the upper envelope of Γ'. The results of Agarwal *et al.* [21] imply that if each set in S has constant description complexity, then the complexity of $\mathcal{T}(S)$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$ in $\mathbb{R}^3$. The results in [21] concerning the complexity of the vertical decomposition of $S(\Gamma, \Gamma')$ imply that $\mathcal{T}(S)$ can be constructed in $O(n^{2+\varepsilon})$ time. No sharp bounds are known on $\mathcal{T}(S)$ in higher dimensions. However, in four dimensions, using the algorithm by Agarwal *et al.* [9] for point location in the minimization diagram of trivariate functions, we can preprocess S into a data structure of size $O(n^{3+\varepsilon})$ so that we can determine in $O(\log n)$ time whether a hyperplane h is a transversal of S.

The problem can be generalized by considering lower-dimensional transversals. For example, in $\mathbb{R}^3$ we can also consider the space of all line transversals of S (lines that meet every member of S). By mapping lines in $\mathbb{R}^3$ into points in $\mathbb{R}^4$, and by using an appropriate parametrization of the lines, the space of all line transversals of S can be represented as the region in $\mathbb{R}^4$ enclosed between the upper envelope and the lower envelope of two respective collections of surfaces. Pellegrini and Shor [280] showed that if S is a set of triangles in $\mathbb{R}^3$, then the space of line transversals of S has $n^3 2^{O(\sqrt{\log n})}$ complexity. The bound was slightly improved by Agarwal [4] to $O(n^3 \log n)$. He reduced the problem to bounding the complexity of a family of cells in an arrangement of $O(n)$ hyperplanes in $\mathbb{R}^5$. Agarwal *et al.* [10] proved that the complexity of the space of line transversals for a set of n balls in $\mathbb{R}^3$ is $O(n^{3+\varepsilon})$. Their argument works even if S is a set of homothets of a convex region of constant description complexity in $\mathbb{R}^3$.

14.4 Geometric optimization

In the past few years, many problems in geometric optimization have been attacked by techniques that reduce the problem to constructing and searching in various substructures of surface arrangements. Hence, the area of geometric optimization is a natural extension, and a good application area, of the study of arrangements. See [24] for a recent survey on geometric optimization.

One of the basic techniques for geometric optimization is the *parametric searching* technique, originally proposed by Megiddo [253]. This technique reduces the optimization problem to a *decision problem*, where one needs to compare the optimal value to a given parameter. In most cases, the decision problem is easier to solve than the optimization problem. The parametric searching technique proceeds by a parallel simulation of a generic version of the decision procedure with the (unknown) optimum value as an input parameter. In most applications, careful implementation of this technique leads to a solution of the optimization problem whose running time is larger than that of the decision algorithm only by

a polylogarithmic factor. See [24] for a more detailed survey of parametric searching and its applications.

Several alternatives to parametric searching have been developed during the past decade. They use randomization [240, 25, 89], expander graphs [217], and searching in monotone matrices [169]. Like parametric searching, all these techniques are based on the availability of an efficient procedure for the decision problem. When applicable, they lead to algorithms with running times that are similar to, and sometimes slightly better than, those yielded by parametric searching.

These methods have been used to solve a wide range of geometric optimization problems, many of which involve arrangements. We mention a sample of such results.

Slope selection. Given a set S of n points in $\mathbb{R}^2$ and an integer k, find the line with the kth smallest slope among the lines passing through pairs of points of S. If we dualize the points in S to a set Γ of lines in $\mathbb{R}^2$, the problem becomes that of computing the kth leftmost vertex of $\mathcal{A}(\Gamma)$. Cole *et al.* [112] developed a rather sophisticated $O(n \log n)$-time algorithm for this problem, which is based on parametric searching. (Here the decision problem is to count the number of vertices of the arrangement that lie to the left of a given vertical line.) A considerably simpler algorithm, based on $(1/r)$-cuttings, was later proposed by Brönnimann and Chazelle [79]. See also [240, 216].

Distance selection. Given a set S of n points in $\mathbb{R}^2$ and a parameter $k \leq \binom{n}{2}$, find the k-th largest distance among the points of S [12, 217]. The corresponding decision problem reduces to point location in a set of congruent disks in $\mathbb{R}^2$. Specifically, given a set Γ of m congruent disks in the plane, we wish to count efficiently the number of containments between disks of Γ and points of S. This problem can be solved using parametric searching [12], expander graphs [217], or randomization [240]. The best-known deterministic algorithm, given by Katz and Sharir [217], runs in $O(n^{4/3} \log^{3+\varepsilon} n)$ time.

Segment center. Given a set S of n points in $\mathbb{R}^2$ and a line segment e, find a placement of e that minimizes the largest distance from the points of S to e [15, 157]. The decision problem reduces to determining whether given two families Γ and Γ' of bivariate surfaces, $\mathcal{S}(\Gamma, \Gamma')$, the region lying between L_Γ and $U_{\Gamma'}$, is empty. Exploiting the special properties of Γ and Γ', Efrat and Sharir [157] show that the complexity of $\mathcal{S}(\Gamma, \Gamma')$ is $O(n \log n)$. They describe an $O(n^{1+\varepsilon})$-time to determine whether $S(\Gamma, \Gamma')$ is empty, which leads to an $O(n^{1+\varepsilon})$-time algorithm for the segment-center problem.

Extremal polygon placement. Given a convex m-gon P and a closed polygonal environment Q with n vertices, find the largest similar copy of P that is fully contained in Q [305]. Here the decision problem is to determine whether P, with a fixed scaling

factor, can be placed inside Q; this is a variant of the corresponding motion planning problem for P inside Q, and is solved by constructing an appropriate representation of the 3-dimensional free configuration space, as a collection of cells in a corresponding 3-dimensional arrangement of surfaces. The running time of the whole algorithm is only slightly larger than the time needed to solve the fixed-size placement problem. The best running time is $O(mn\lambda_6(mn)\log^3 mn\log^2 n)$ [11]; see also [222, 305]. If Q is a convex n-gon, the largest similar copy of P that can be placed inside Q can be computed in $O(mn^2\log n)$ time [5];

Diameter in 3D. Given a set S of n points in $\mathbb{R}^3$, determine the maximum distance between a pair of points in S. The problem is reduced to determining whether S lies in the intersection of a given set Γ of n congruent balls. A randomized algorithm with $O(n\log n)$ expected time was proposed by Clarkson and Shor [111]. A series of papers [97, 250, 286, 285] describe near-linear-time deterministic algorithms. The best-known deterministic algorithm runs in $O(n\log^2 n)$ time [285].

Width in 3D. Given a set S of n points in $\mathbb{R}^3$, determine the smallest distance between two parallel planes enclosing S between them. This problem has been studied in a series of papers [9, 25, 97], and the currently best known randomized algorithms computes the width in $O(n^{3/2+\varepsilon})$ expected time [25]. The technique used in attacking the decision problems for this and the two following problems reduce them to point location in the region above the lower envelope of a collection of trivariate functions in $\mathbb{R}^4$.

Biggest stick in a simple polygon: Compute the longest line segment that can fit inside a given simple polygon with n edges. The current best solution is $O(n^{3/2+\varepsilon})$ [25] (see also [9, 27]).

Minimum-width annulus: Compute the annulus of smallest width that encloses a given set of n points in the plane. This problem arises in fitting a circle through a set of points in the plane. Again, the current best solution is $O(n^{3/2+\varepsilon})$ [25] (see also [9, 27]).

Geometric matching. Consider the problem where we are given two sets S_1, S_2 of n points in the plane, and we wish to compute a minimum-weight matching in the complete bipartite graph $S_1 \times S_2$, where the weight of an edge (p, q) is the Euclidean distance between p and q. One can also consider the analogous nonbipartite version of the problem, which involves just one set S of $2n$ points, and the complete graph on S. The goal is to explore the underlying geometric structure of these graphs, to obtain faster algorithms than those available for general abstract graphs. Vaidya [318] had shown that both the bipartite and the nonbipartite versions of the problem can be solved in time close to $O(n^{5/2})$. A fairly

sophisticated application of vertical decomposition in three-dimensional arrangements, given in [14], has improved the running time for the bipartite case to $O(n^{2+\varepsilon})$.

Center point. A *center point* of a set S of n points in the plane is a point $\pi \in \mathbb{R}^2$ so that each line ℓ passing through π has the property that at least $\lfloor n/3 \rfloor$ points lie in each halfplane bounded by ℓ. It is well known that such a center point always exists [132]. If we dualize S to a set Γ of n lines in the plane, then π^*, the line dual to π, lies between $\mathcal{A}_{\lfloor n/3 \rfloor}(\Gamma)$ and $\mathcal{A}_{\lceil 2n/3 \rceil}(\Gamma)$. Cole *et al.* [113] described an $O(n \log^3 n)$-time algorithm for computing a center point of S, using parametric searching. The problem of computing the set of all center points reduces to computing the convex hull of $\mathcal{A}_k(\Gamma)$ for a given k. Matoušek [238] described an $O(n \log^2 n)$-time algorithm for computing the convex hull of $\mathcal{A}_k(\Gamma)$ for any $k \leq n$; recall, in contrast, that the best known upper bound for $\mathcal{A}_k(\Gamma)$ is $O(n(k+1)^{1/3})$.

Ham sandwich cuts. Let $S_1, S_2, \ldots, S_d$ be d sets of points in $\mathbb{R}^d$, each containing n points. Suppose n is even. A *ham sandwich cut* is a hyperplane h so that each open halfspace bounded by h contains at most $n/2$ points of S_i, for $i = 1, \ldots, d$. It is known [132, 328] that such a cut always exists. Let Γ_i be the set of hyperplanes dual to S_i. Then the problem reduces to computing a vertex of the intersection of $\mathcal{A}_{n/2}(\Gamma_1)$ and $\mathcal{A}_{n/2}(\Gamma_2)$. Megiddo [254] developed a linear-time algorithm for computing a ham sandwich cut in the plane if S_1 and S_2 can be separated by a line. For arbitrary point sets in the plane, a linear-time algorithms was later developed by Lo etal [235]. Lo *et al.*also described an algorithm for computing a ham sandwich cut in $\mathbb{R}^3$ whose running time is $O(\psi_{n/2}(n) \log^2 n)$, where $\psi_k(n)$ is the maximum complexity of the k-level in an arrangement of n lines in the plane. By Dey's result on k-levels [124], the running time of their algorithm is $O(n^{4/3} \log^2 n)$.

14.5 Robotics

As mentioned in the introduction, *motion planning* for a robot system has been a major motivation for the study of arrangements. Let B be a robot system with d degrees of freedom, which is allowed to move freely within a given two or three-dimensional environment cluttered with obstacles. Given two placements I and F of B, determining whether there exists a collision-free path between these placements reduces to determining whether I and F lie in the same cell of the arrangement of the family Γ of "contact surfaces" in $\mathbb{R}^d$, regarded as the configuration space of B (see the introduction for more details). If I and F lie in the same cell, then a path between I and F in $\mathbb{R}^d$ that does not intersect any surface of Γ corresponds to a collision-free path of B in the physical environment from I to F. If d is a part of the input, the problem is known to be PSPACE-complete [86, 288]. Canny [85, 87] gave an $n^{O(d)}$-time algorithm to compute the roadmap of a single cell in an arrangement $\mathcal{A}(\Gamma)$ of a set Γ of n surfaces in $\mathbb{R}^d$ provided that the cells in $\mathcal{A}(\Gamma)$ form a Whitney regular stratification of $\mathbb{R}^d$ (see [180] for the definition of Whitney stratification). Using a perturbation argument, he showed that his approach can be extended to obtain a Monte Carlo

algorithm to determine whether two points lie in the same cell of $\mathcal{A}(\Gamma)$. The algorithms was subsequently extended and improved by many researchers see [66, 212, 181]. The best-known algorithm, due to Basu *et al.* [66], can compute the roadmap in time $n^{d+1}b^{O(d^2)}$. Much work has been done on developing efficient algorithms for robots with a small number of degrees of freedom, say, two or three [196, 205, 221]. The result by Schwarzkopf and Sharir [295] gives an efficient algorithm for computing a collision-free path between two given placements for a fairly general robot system with three degrees of freedom.

It is impractical to compute the roadmap, or any other explicit representation, of a single cell in $\mathcal{A}(\Gamma)$ if d is large. A general Monte Carlo algorithm for computing a *probabilistic roadmap* of a cell in $\mathcal{A}(\Gamma)$ is described by Kavraki *et al.* [218]. This approach avoids computing the cell explicitly. Instead, it samples a large number of random points in the configuration space and only those configurations that lie in the free configuration space (FP) are retained (they are called *milestones*); we also add I and F as milestones. The algorithm then builds a 'connectivity graph' whose nodes are these milestones, and whose edges connect pairs of milestones if the line segment joining them in configuration space lies in FP (or if they satisfy some other "local reachability" rule). Various strategies have been proposed for choosing random configurations [39, 64, 215, 219]. The algorithm returns a path from I to F if they lie in the same connected component of the resulting network. Note that this algorithm may fail to return a collision-free path from I to F even if there exists one. This technique nevertheless has been successful in several real-world applications.

Assembly planning is another area in which the theory of arrangements has led to efficient algorithms. An *assembly* is a collection of objects (called parts) placed rigidly in some specified relative positions so that no two objects overlap. A *subassembly* of an assembly A is a subset of objects in A in their relative placements in A. An assembly operation is a motion that merges some subassemblies of A into a new and larger subassembly. An *assembly sequence* for A is a sequence of assembly operations that starts with the individual parts separated from each other and ends up with the full assembly A. The goal of assembly planning is to compute an assembly sequence for a given assembly. A classical approach to assembly sequencing is *disassembly sequencing*, which separates an assembly into its individual parts [214]. The reverse order of a sequence of disassemblying operations yields an assembly sequence. Several kinds of motion have been considered in separating parts of an assembly, including translating a subassembly along a straight line, arbitrary translational motion, rigid motion, etc. A common approach to generate a disassembly sequence is the so-called *non-directional blocking graph* approach. It partitions the space of all allowable motions of separation into a finite number of cells so that within each cell the set of "blocking relations" between all pairs of parts remains fixed. The problem is then reduced to computing representative points in cells of the arrangement of a family of surfaces. This approach has been successful in many instances, including polyhedral assembly with infinitesimal rigid motion [186]; see also [199, 198].

Other problems in robotics that have exploited arrangements include fixturing [287], MEMS (micro electronic mechanical systems) [72], path planning with uncertainty [119],

and manufacturing [29].

14.6 Molecular modeling

In the introduction, we described the Van der Waals model, in which a molecule M is represented as a collection Γ of spheres in $\mathbb{R}^3$. (See [115, 135, 255] for other geometric models of molecules.) Let $\Sigma = \partial(\bigcup \Gamma)$. The boundary of Σ is called the "surface" of M. Many problems in molecular biology, especially those which study the interaction of a protein with another molecule, involve computing the molecular surface, a portion of the surface (e.g., the so-called *active site* of a protein), or various features of the molecular surface [137, 200, 231, 321]. We briefly describe two problems in molecular modeling that can be formulated in terms of arrangements.

The chemical behavior of solute molecules in a solution is strongly dependent on the interactions between the solute and solvent molecules. These interactions are critically dependent on those molecular fragments that are accessible to the solvent molecules. Suppose we use the Van der Waals model for the solute molecule and model the solvent by a sphere S. By rolling S on the molecular surface Σ, we obtain a new surface Σ', described by the center of the rolling sphere. If we enlarge each sphere of Γ by the radius of S, Σ' is the boundary of the union of the enlarged spheres.

As mentioned above, several mthods have been proposed to model the surface of a molecule. The best choice of the model depends on the chemical problem the molecular surface is supposed to represent. For example, the Van der Waal model represents the space requirement of molecular conformations, while isodensity contours and molecular electrostatic potential contour surfaces [255] are useful in studying molecular interactions. An important problem in molecular modeling is to study the interrelations among various molecular surfaces of the same molecule. Let $\Sigma = \{\Sigma_1, \ldots, \Sigma_m\}$ be a family of molecular surfaces of the same molecule. We may want to compute the arrangement $\mathcal{A}(\Sigma)$, or we may want to compute the subdivision of Σ_i induced by $\{\Sigma_j \cap \Sigma_i \mid 1 \leq j \neq i \leq m\}$.

Researchers have also been interested in computing "connectivity" of a molecule, e.g., computing voids, tunnels, and pockets of Σ. A *void* of Σ is a bounded component of $\mathbb{R}^3 \setminus \Sigma$; a *tunnel* is a hole through Σ that is accessible from the outside, i.e., an "inner" part of a non-contractible loop in $\mathbb{R}^3 \setminus \Sigma$; and a *pocket* is a depression or cavity on the boundary of Σ. Pockets are not holes in the topological sense and are not well defined; see [116, 137] for some of the definitions proposed so far.

Efficient algorithms have been developed for computing Σ, connectivity of Σ, and the arrangement $\mathcal{A}(\Gamma)$ [135, 200, 321]. Halperin and Shelton [206] describe an efficient perturbation scheme to handle degeneracies while constructing $\mathcal{A}(\Gamma)$ or Σ. Some applications require computing the measure of different substructures of $\mathcal{A}(\Gamma)$, including the volume of Σ, the surface area of Σ, or the volume of a void of Σ. Edelsbrunner *et al.* [136] describe an efficient algorithm for computing these measures; see also [134, 135].

15 Conclusions

In this survey we reviewed a wide range of topics on arrangements of surfaces. We mentioned a few old results, but the emphasis of the survey was on the tremendous progress made in this area during the last fifteen years. We discussed combinatorial complexity of arrangements and their substructures, representation of arrangements, algorithms for computing arrangements and their substructures, and several geometric problems in which arrangements play pivotal roles. Although the survey covered a broad spectrum of results, many topics on arrangements were either not included or very briefly touched upon. For example, we did not discuss arrangements of pseudo-lines and oriented matroids, we discussed algebraic and topological issues very briefly, and we mentioned a rather short list of applications that have exploited arrangements. There are numerous other sources where more details on arrangements and their applications can be found; see e.g. the books [70, 272, 304] and the survey papers [178, 197, 261, 271].

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