Termination

2. Games

Readings

- Floyd, "Assigning Meaning to Programs"
- "Proving Termination with Multiset Orderings"

Robert W. Floyd

ASSIGNING MEANINGS TO PROGRAMS[1]

Introduction. This paper attempts to provide an adequate basis for formal definitions of the meanings of programs in appropriately defined programming languages, in such a way that a rigorous standard is established for proofs about computer programs, including proofs of correctness, equivalence, and termination. The basis of our approach is the notion of an interpretation of a program: that is, an association of a proposition with each connection in the flow of control through a program, where the proposition is asserted to hold whenever that connection is taken. To prevent an interpretation from being chosen arbitrarily, a condition is imposed on each command of the program. This condition guarantees that whenever a command is reached by way of a connection whose associated proposition

Invariants

```
r := 1
u := 1
loop    v := u                      1≤r≤n
        until r≥n
        s := 1
        loop   u := u+v
               s := s+1         1≤s≤r+1
               while s≤r
               repeat
        r := r+1
        repeat
```

Double Induction

- Inner loop
- Outer loop

Ackermann's Function

$$a(0,n) = n+1$$

$$a(m+1,0) = a(m,1)$$

$$a(m+1,n+1) = a(m,a(m+1,n))$$

Ackermann

- $a(4,4) = 2\uparrow 7-3$
- Computation is much longer
- Fact: $a(m,n) > m+n \geq m,n$

Double Induction

- Call by value termination
- Assume terminating for smaller m
  - Assume terminating for same m and smaller n

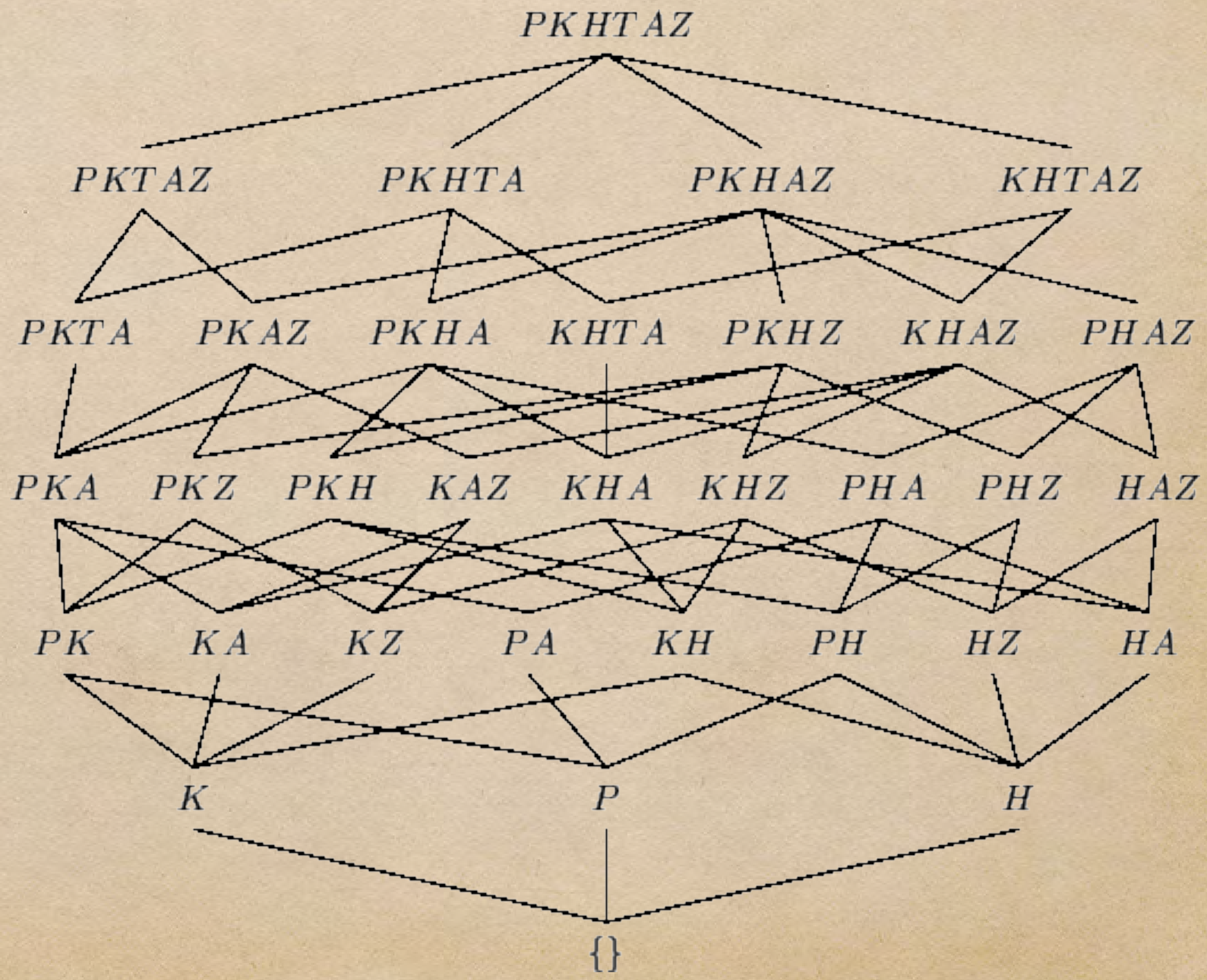
Basic A(m,n)

```
DIM s(tsize + 1)

 t = 1: s(t) = m
 DO
     c = c + 1
     m = s(t): t = t - 1
     IF m = 0 THEN
         n = n + 1
     ELSEIF n = 0 THEN
         t = t + 1: s(t) = m - 1
         n = 1
     ELSE
         t = t + 1: s(t) = m - 1
         t = t + 1: s(t) = m
         n = n - 1
     END IF
     IF t > d THEN
         d = t
         IF d > tsize THEN
             PRINT "failure": END
         END IF
     END IF
 LOOP UNTIL t = 0

A = n
END FUNCTION
```

Orderings

- Partial ordering
  - Irreflexive
  - Transitive
  - Asymmetric

Hasse Diagram



Orderings (Well-founded)

- Partial ordering
  - Irreflexive
  - Transitive
  - Asymmetric
- Well-founded
  - No infinite decreasing chains

Well-Founded Orderings

- $N, >$
- $Z^-, <$
- Z, ???
- Finite trees, subtree
- NxN, lexicographic
- Σ^*, subword
- Σ^*, lexicographic ???

Couples

$(a,b) > (a',b')$

- Component-wise: $a>a'$ & $b \geq b'$ or $a \geq a'$ & $b>b'$
- Lexicographic: $a>a'$ or $a=a'$ & $b>b'$
- Reverse lexicographic: $a>a'$ & $b=b'$ or $b>b'$
- Pairs of pairs: $(1,0) > (0,(1,0)) > \ldots$

Mixed Couples

If V and W are well-founded, then their pairs VxW are well-founded lexicographically.

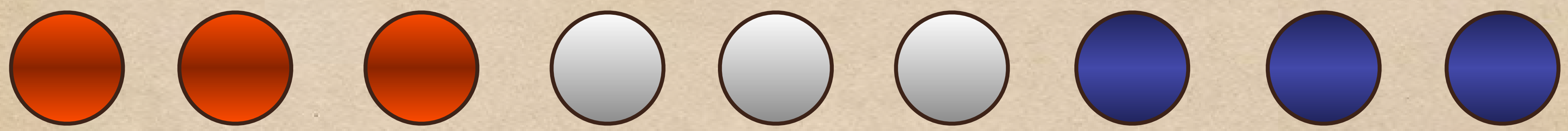
Ackermann

- Termination of recursion
    - Induction on (m,n)

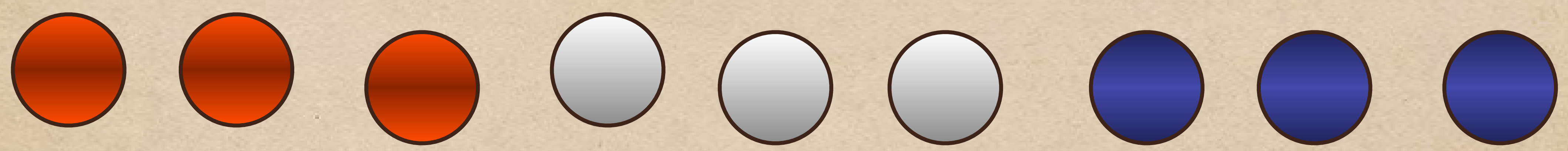
Turing's Program

```
r := 1
u := 1
loop   v := u
       until r≥n              (n-r,r-s)
       s := 1
       loop   u := u+v
              s := s+1
              while s≤r
              repeat
       r := r+1
       repeat
```

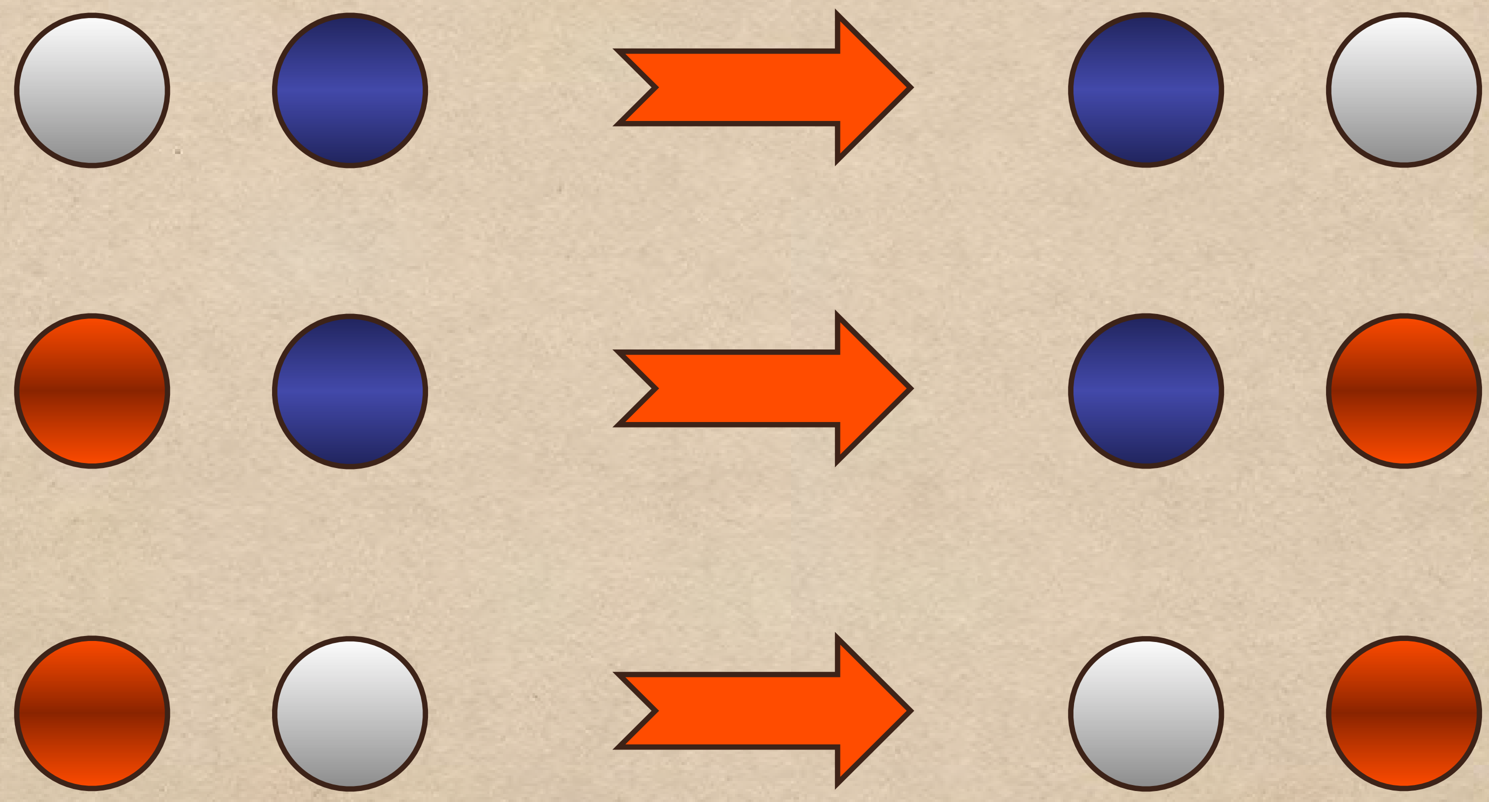
Dutch National Flag



Dutch National Flag



Flag Problem



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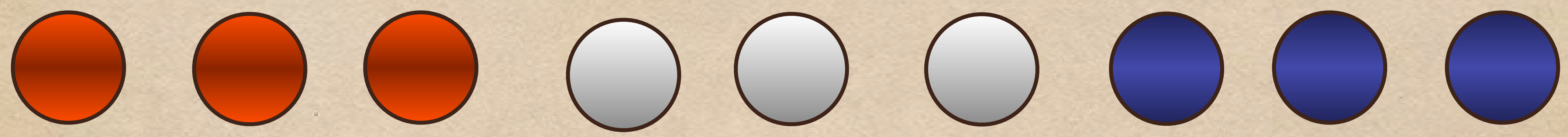

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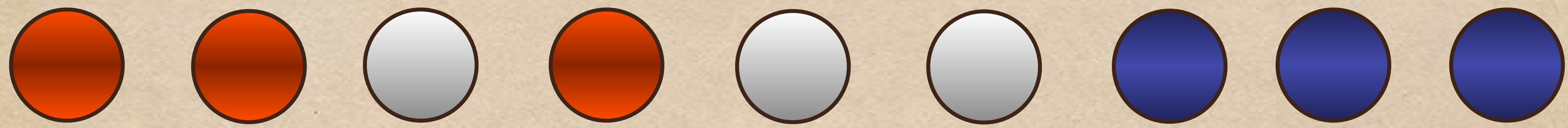

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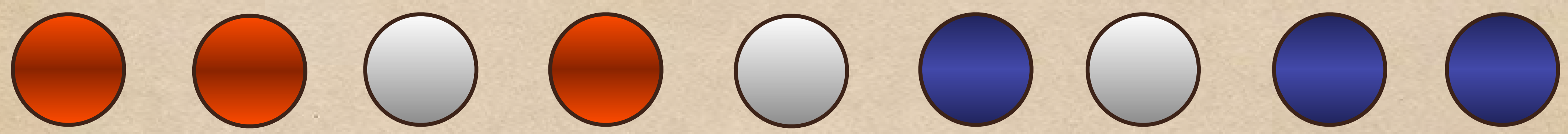
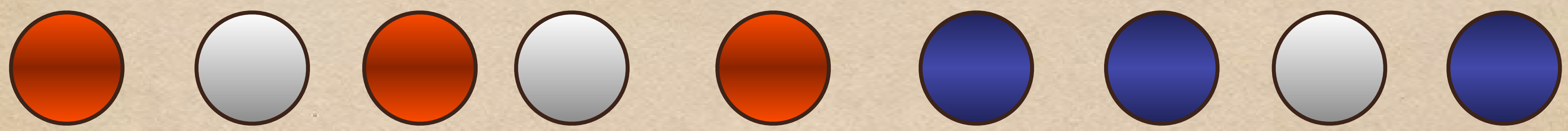
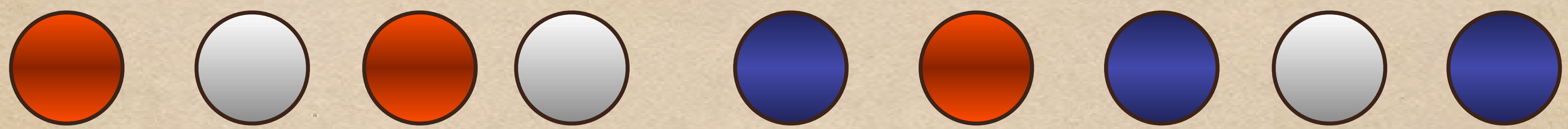


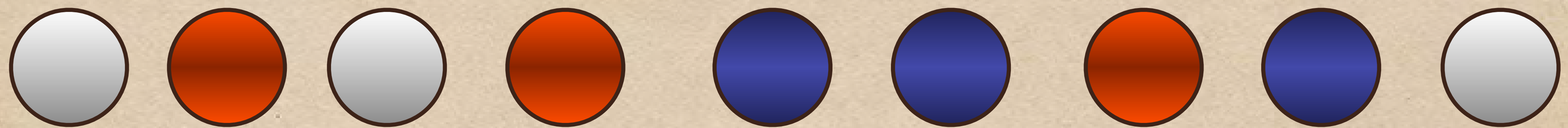
Dutch National Flag



Dutch National Flag



Dutch National Flag



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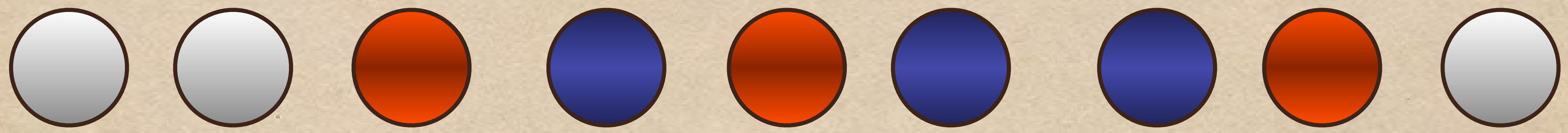
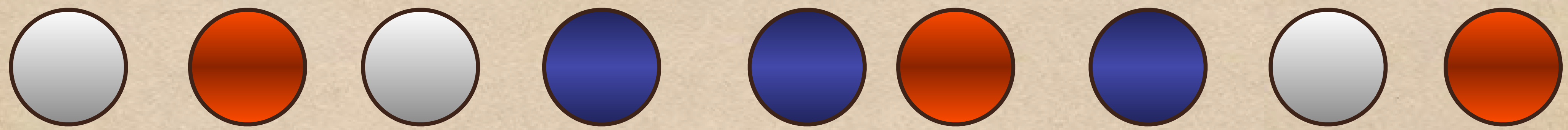
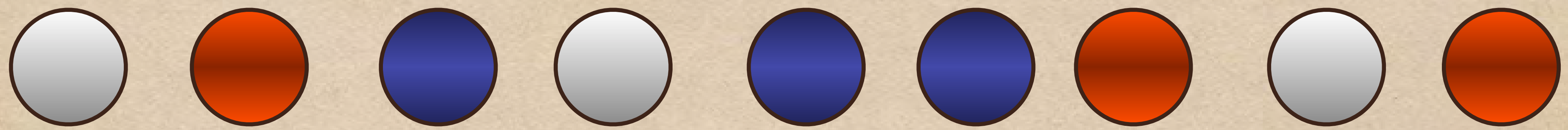


Dutch National Flag



Dutch National Flag



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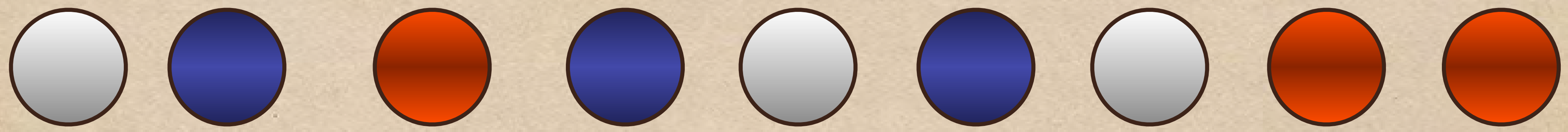
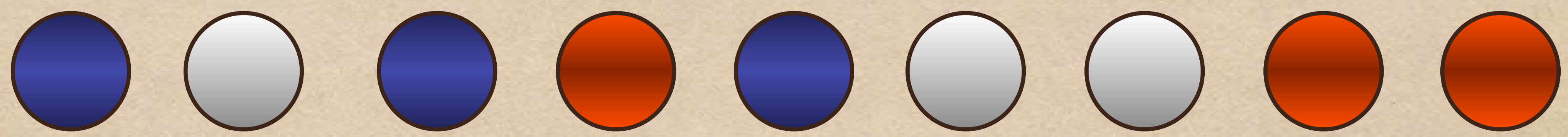
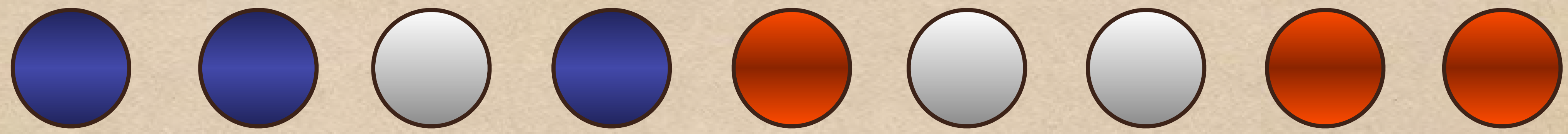


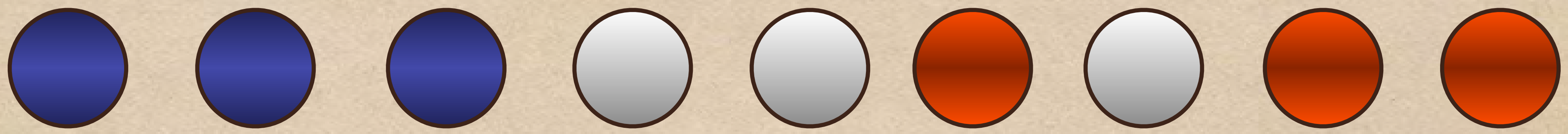
Dutch National Flag



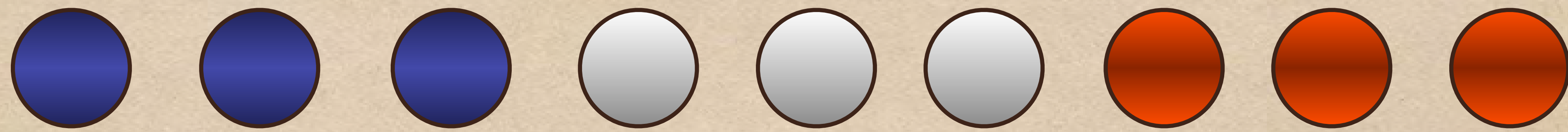
Dutch National Flag



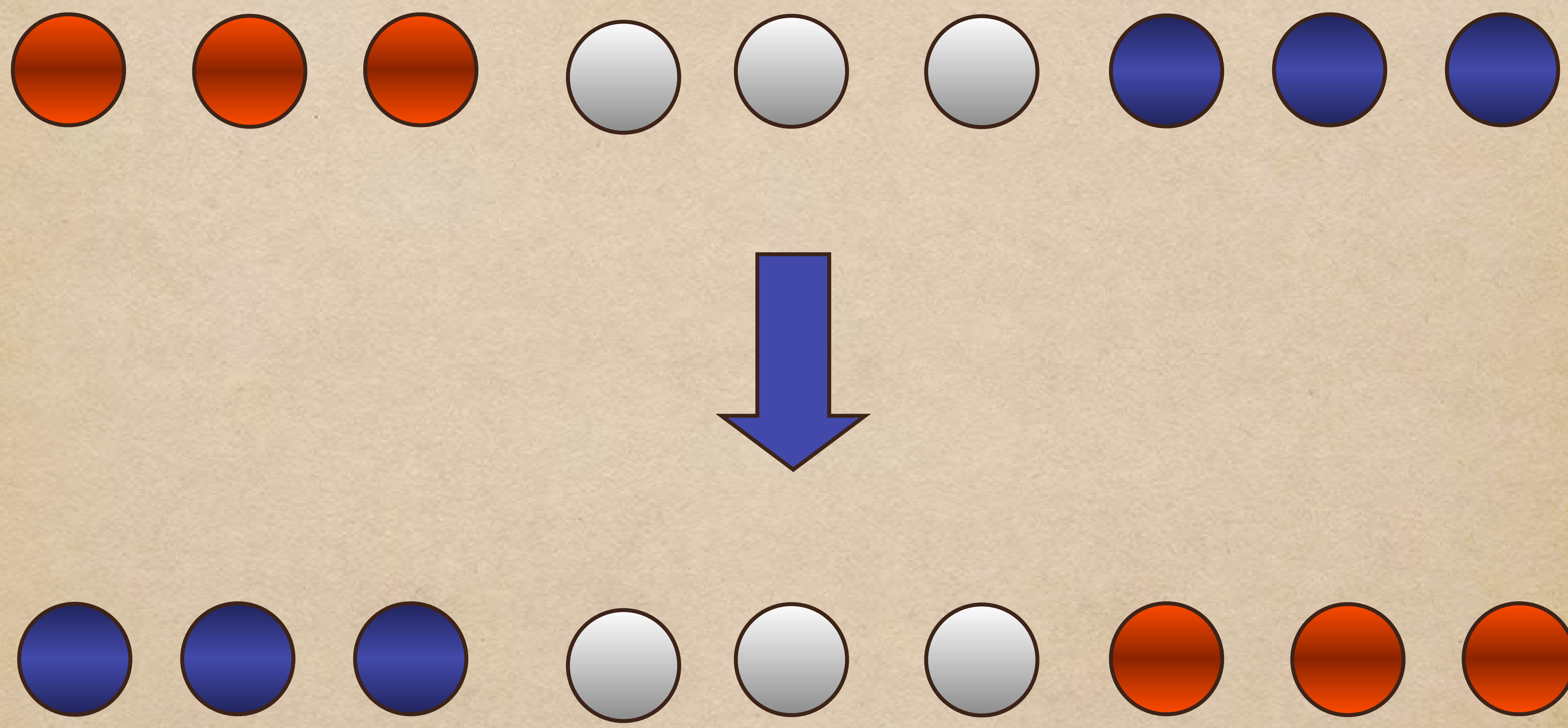
Dutch National Flag



Dutch National Flag



Dutch National Flag



Basic A(m,n)

```
DIM s(tsize + 1)

 t = 1: s(t) = m
 DO
     c = c + 1
     m = s(t): t = t - 1
     IF m = 0 THEN
         n = n + 1
     ELSEIF n = 0 THEN
         t = t + 1: s(t) = m - 1
         n = 1
     ELSE
         t = t + 1: s(t) = m - 1
         t = t + 1: s(t) = m
         n = n - 1
     END IF
     IF t > d THEN
         d = t
         IF d > tsize THEN
             PRINT "failure": END
         END IF
     END IF
 LOOP UNTIL t = 0

A = n
END FUNCTION
```

S(1:TSIZE)

LEXICOGRAPHICALLY

Sequences

(a,b,c,...) > (a',b',c',d',...)

- Lex is bad : 10 > 010 > 0010 > ...
- Length-lex: 0010 > 010 > 001 > 10 > 01

Unbounded Sequences

- Sorted-lex: 221 > 211110000 > 2111000000 > …
- Sorted-lex: ∞∞21 > ∞88880 > 9998888000 > …

Sorted Sequences

- $s11 \geq s12 \geq s13 \geq \ldots \geq s1j \geq \ldots$
- $s21 \geq s22 \geq s23 \geq \ldots \geq s2j \geq \ldots$
- etc. ...
- Let j be first unstable column, changing at i
- $s\_1,1 = s\_i,1 \geq s\_i,j > s\_i+1,j$
- Consider rest: s[i+1..∞,j..∞] and continue
- Gives infinite descending sequence of elements

Harder A(m,n)

```
t := 1
s[t] := m
loop
   c := c + 1
   m := s[t]
   t := t - 1
   if m = 0
   then
      n := n + 1
   elseif n = 0
   then
      t := t + 1
      s[t] := m - 1
      n := 1
   else
      t := t + 2
      s[t-1] := m - 1
      s[t] := m
      n := n - 1
   until t = 0
```

S CAN GROW AND GROW

(SORTED) LEX DOESN'T WORK

Harder A(m,n)

```
t := 1
s[t] := m
loop
   c := c + 1
   m := s[t]
   t := t - 1
   if m = 0
   then
      n := n + 1
   elseif n = 0
   then
      t := t + 1
      s[t] := m - 1
      n := 1
   else
      t := t + 2
      s[t-1] := m - 1
      s[t] := m
      n := n - 1
   until t = 0
```

N:=A(m,n)

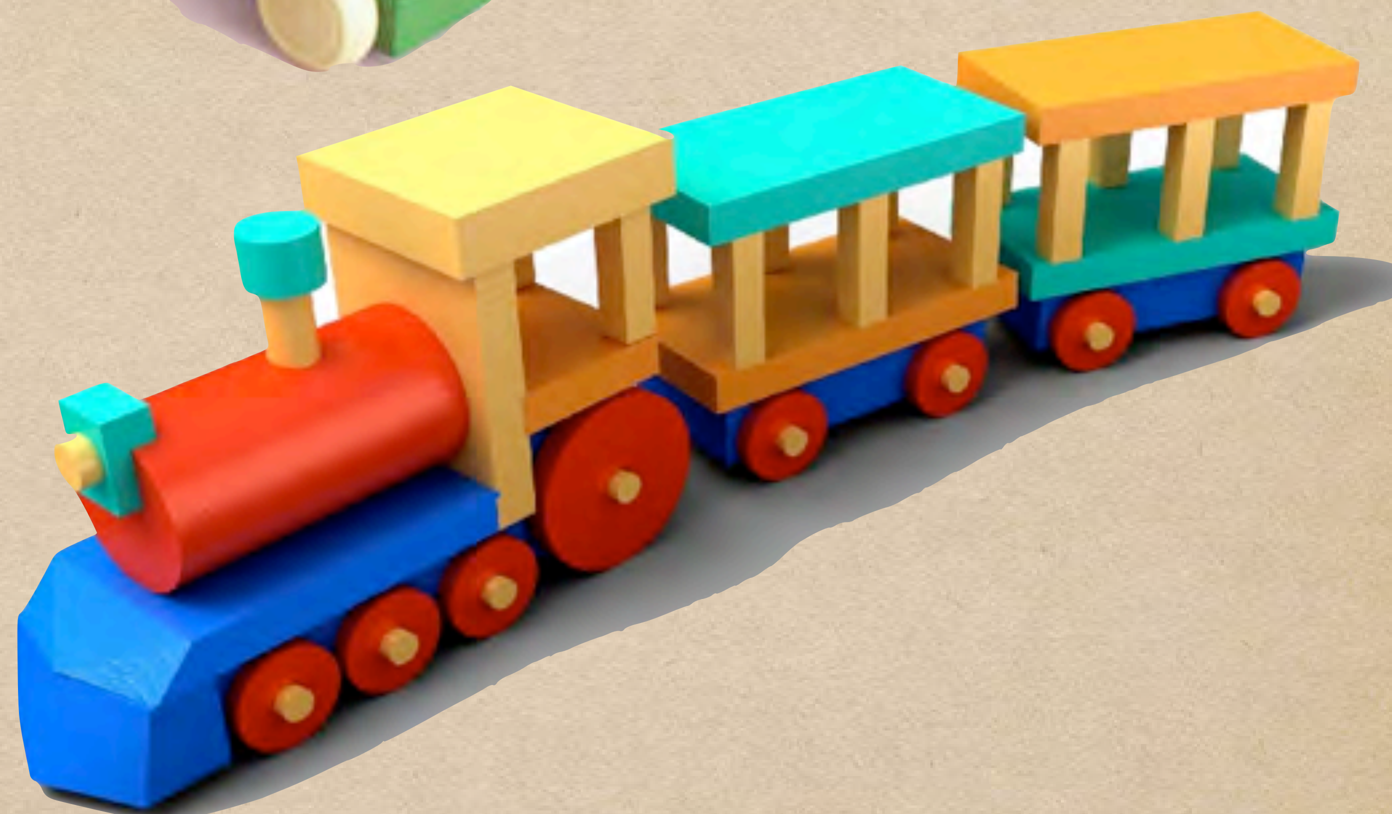
$$\sum 3 \quad Ns[J]+\left\{\begin{matrix} N \\ n \end{matrix}\right.$$

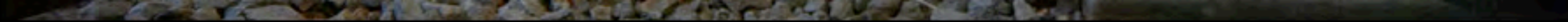
Well-Orderings

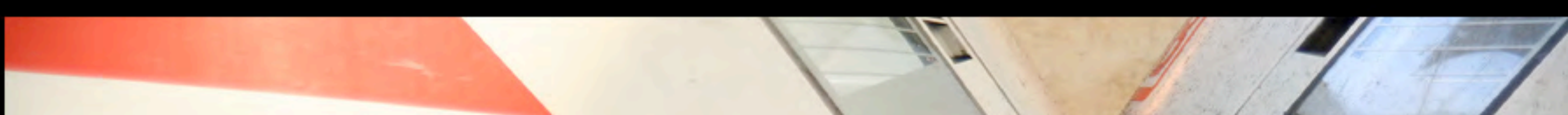
- a b c ...
- a b c ... ∞
- a b c ... 0 1 2 ...
- a0 a1 a2 ... b0 b1 b2 ... c0 c1 c2 ... ...
- 000 001 002 ... 010 011 ... 020 ... 100 101 ...

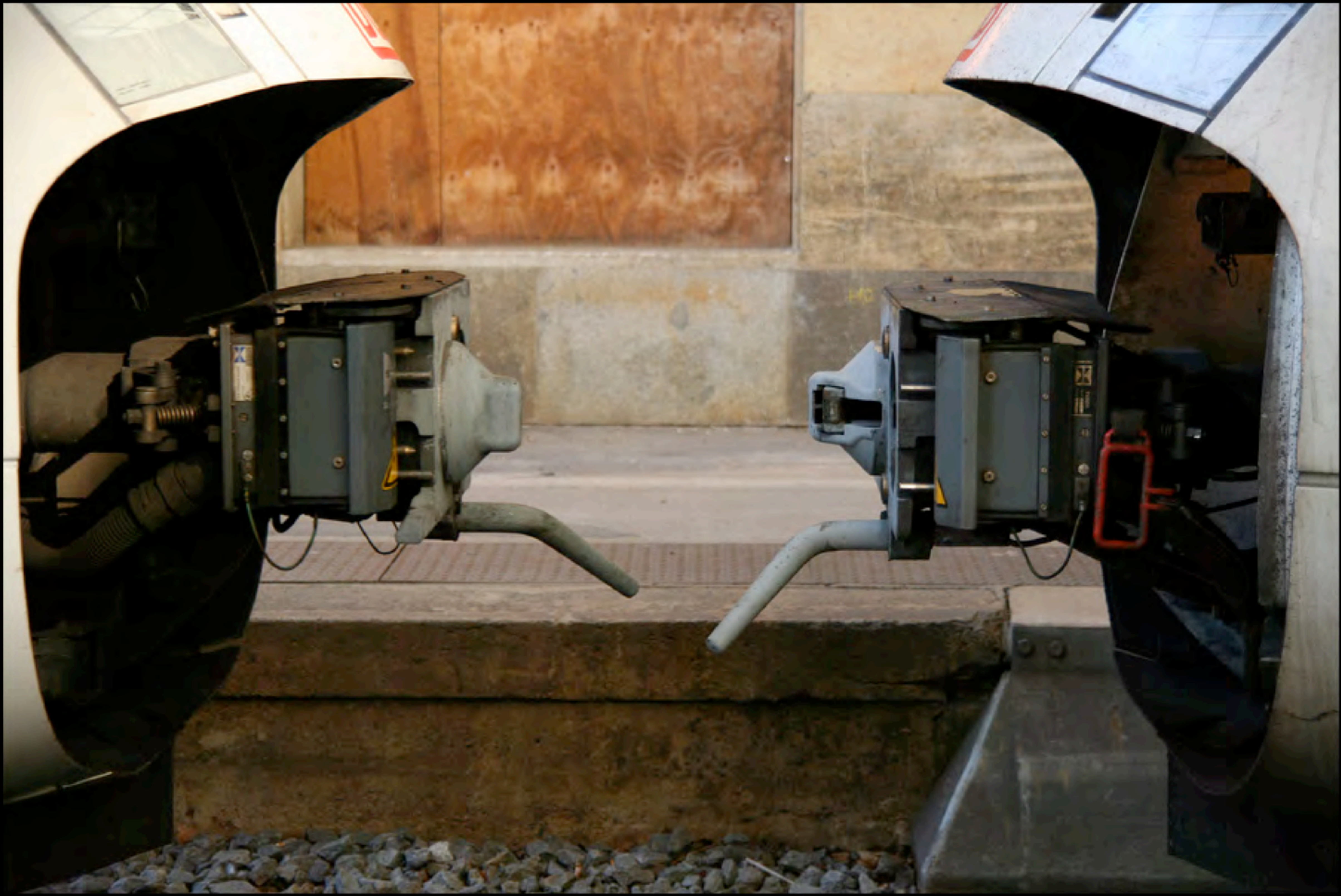
Chocolate Bar

- Yumm (click here)





Before & After

- $n \to \lfloor n/2 \rfloor$, $\lceil n/2 \rceil$ $(n>1)$

Before & After

- $1 \to$
- $n \to \lfloor n/2 \rfloor, \lceil n/2 \rceil \quad (n>1)$

Before & After

- 1 →
- n → 1 , n-1   (n>1)

Before & After

- 1 →
- n → i , n-i (n>1, i>0)

Before & After

- m →
- n → n-1 , n-1 (n>1, i>0)

Proof by Cases

A[x]

A[true], A[false]

Before & After

- 1 →
- n → i , j   (0<i,j<n)

Before & After

- 1 →
- n → i , j , k   (0<i,j,k<n)

Before & After

- 1 →
- n → n1 , n2 , ..., nk (0<ni<n)

Konig's Lemma

- A tree is finite (has finitely many edges)

  if and only if

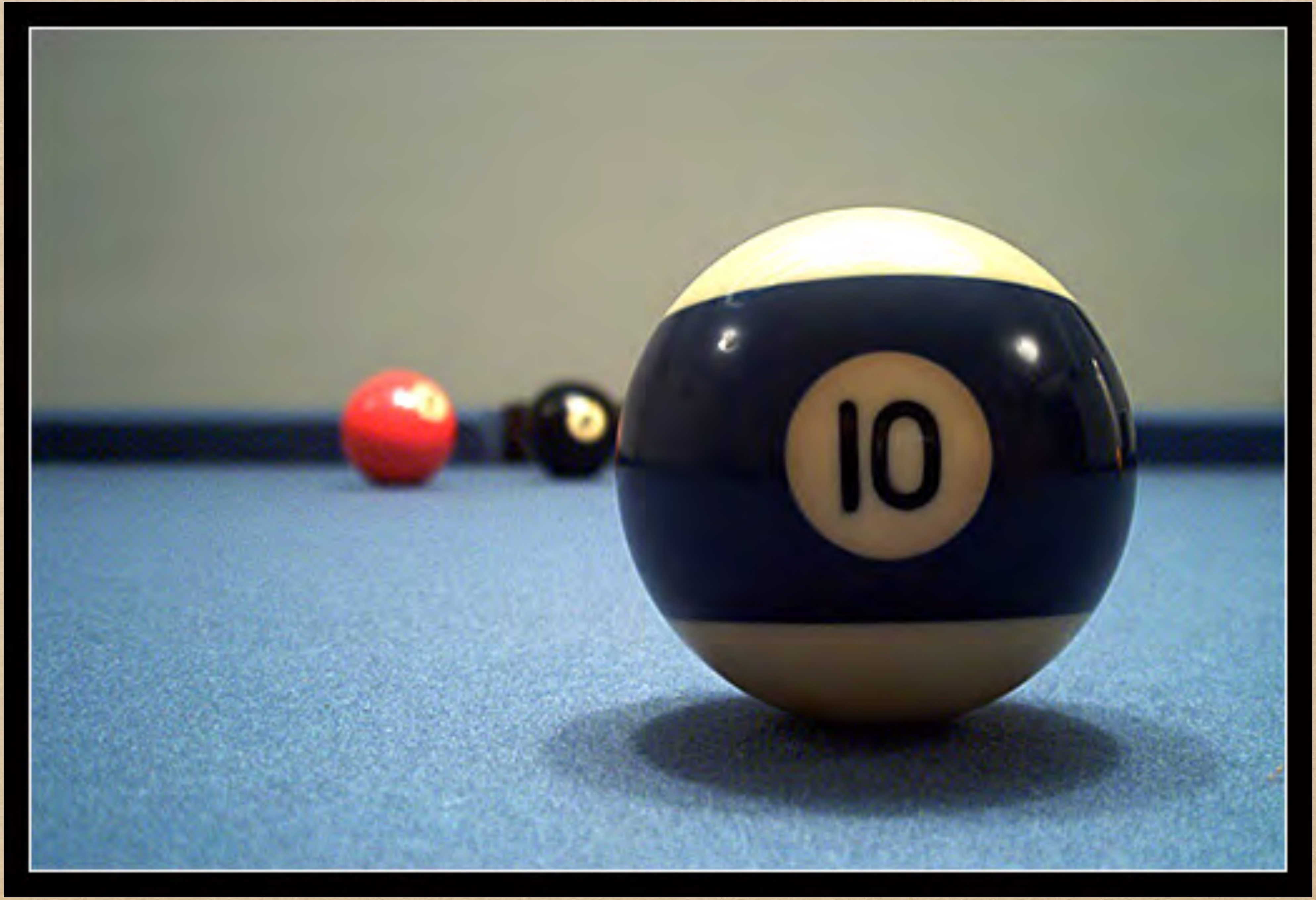
  - all nodes have finite degree

  and

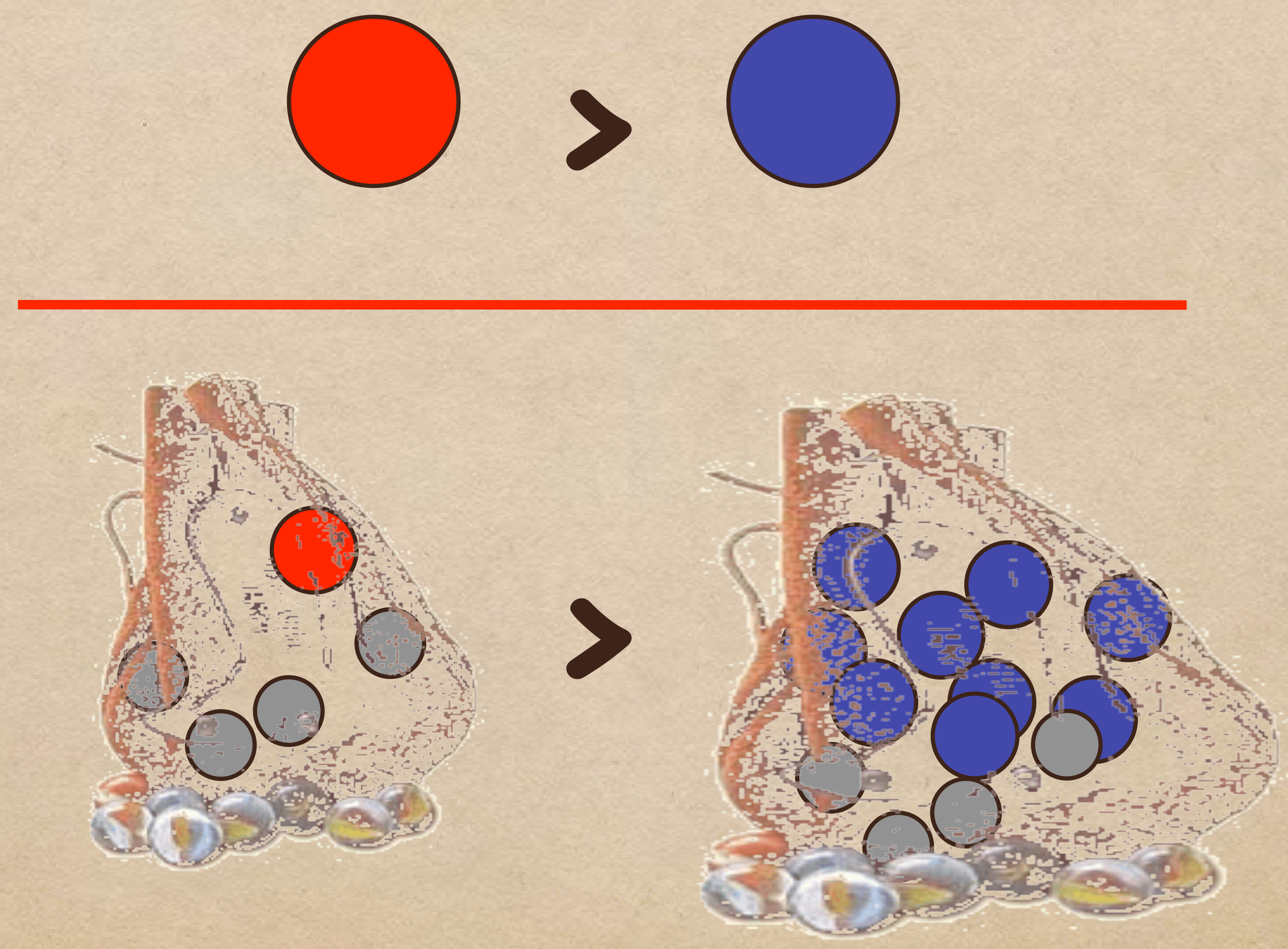
  - all branches (simple paths) have

Billiards

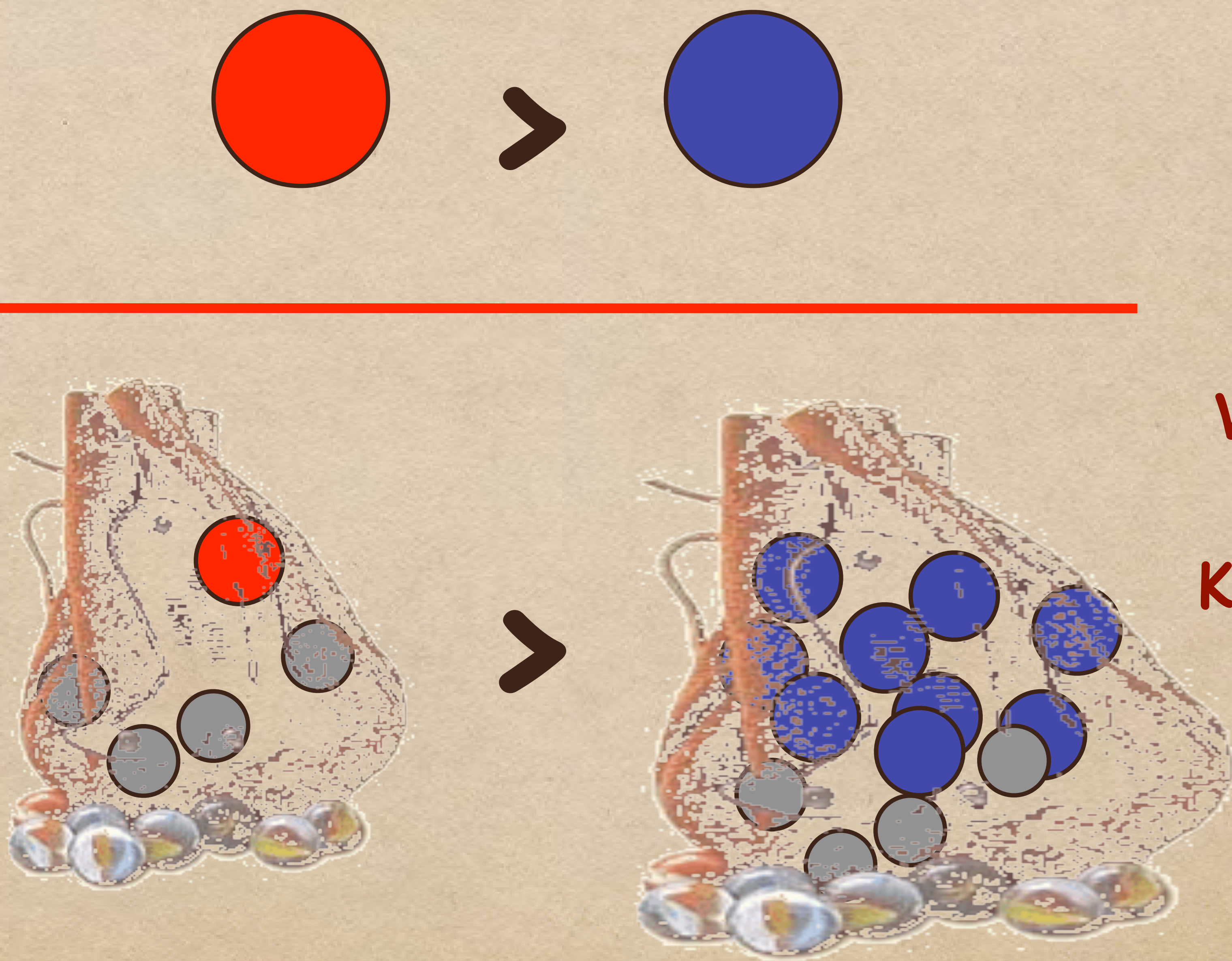
Smullyan's Billiards



Multiset (Bag) Ordering



Multiset (Bag) Ordering



Well-founded
by
König's Lemma

Harder A(m,n)

```
t := 1
s[t] := m
loop
   c := c + 1
   m := s[t]
   t := t - 1
   if m = 0
   then
      n := n + 1
   elseif n = 0
   then
      t := t + 1
      s[t] := m - 1
      n := 1
   else
      t := t + 2
      s[t-1] := m - 1
      s[t] := m
      n := n - 1
   until t = 0
```

Bag of pairs

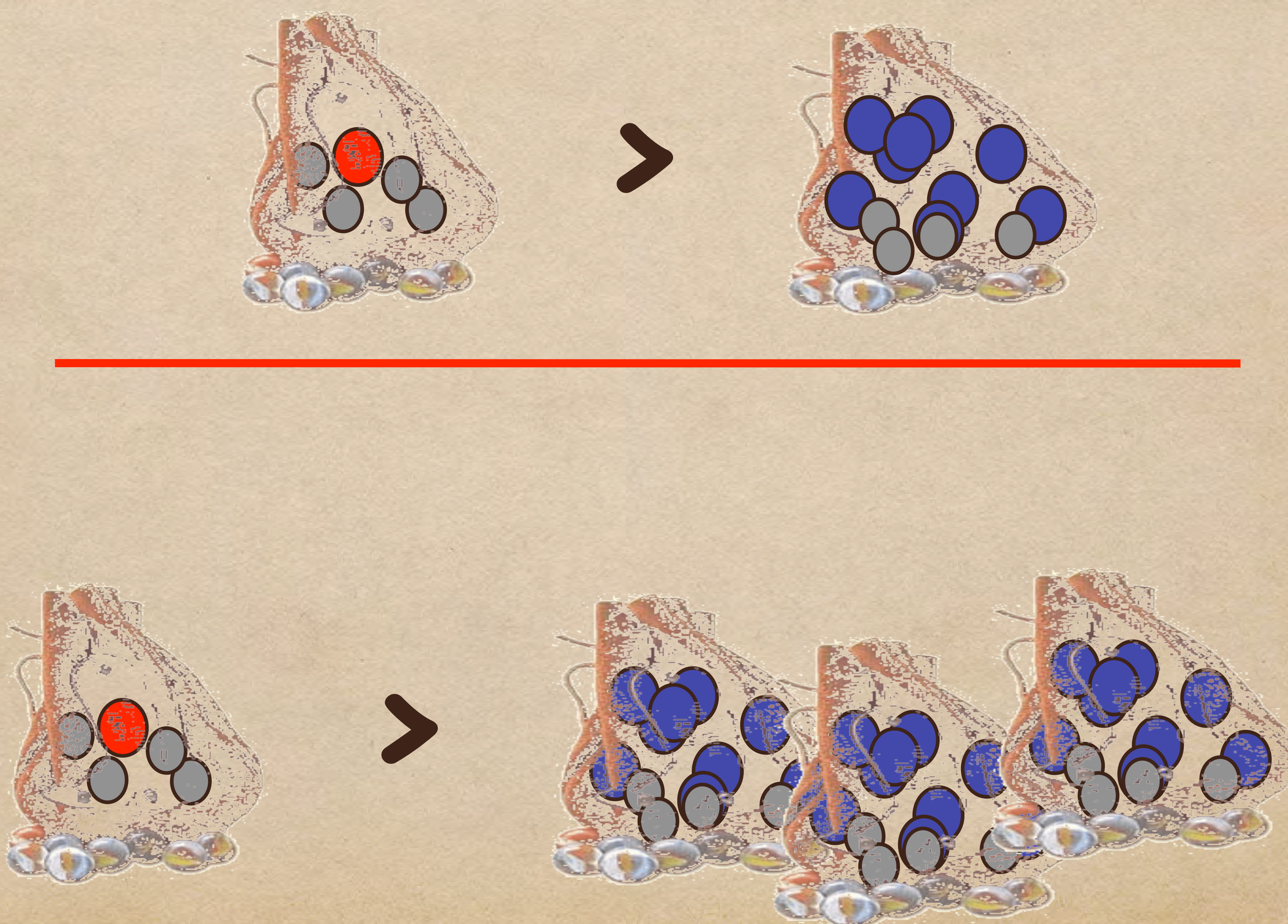
(s[i],∞)   i<t

(s[t],n)

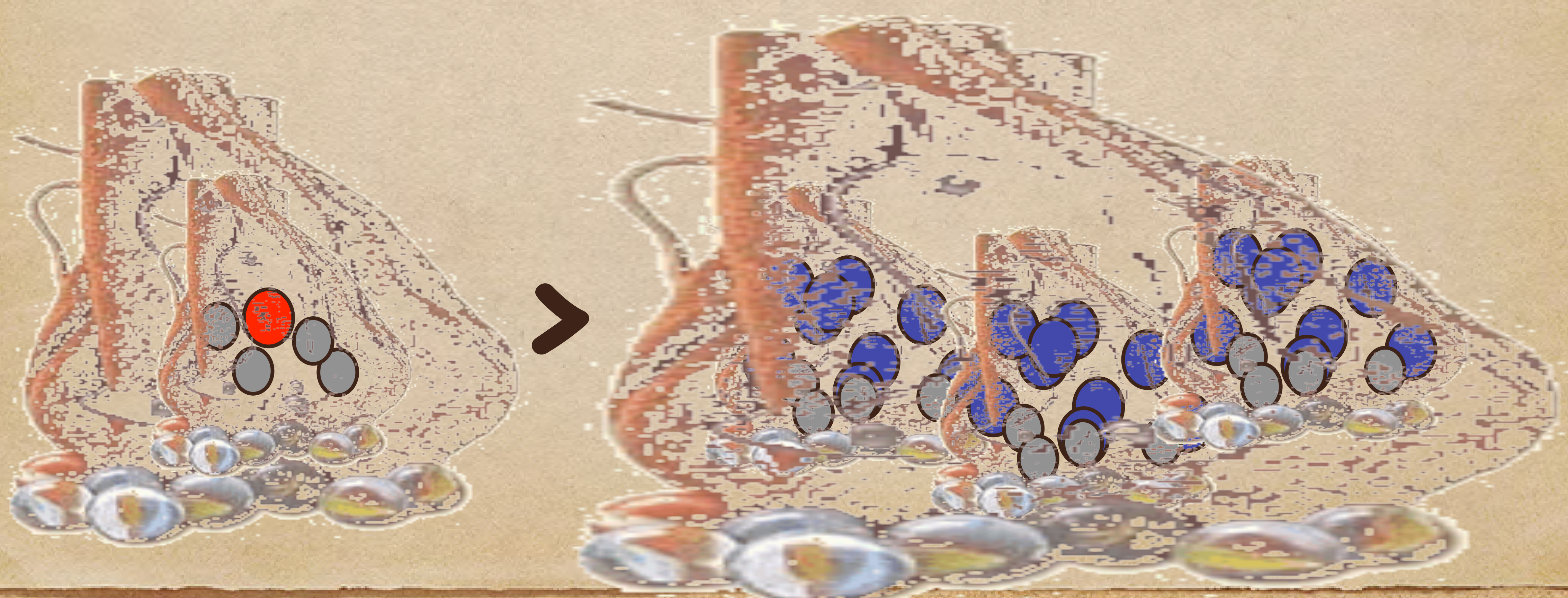
Nested Matryoshka Dolls

Nested Bags

Nested Ordering



Nested Ordering



Goodstein 4

4, 26, 41, 60, 83, 109, 139, 173, 211, 253, 299, 348, 401, 458, 519, 584, 653, 726, 803, 884, 969, 1058, 1151, 1222, 1295, 1370, 1447, 1526, 1607, 1690, 1775, 1862, 1951, 2042, 2135, 2230, 2327, 2426, 2527, 2630, 2735, 2842, 2951, 3062, 3175, 3290, 3407,..., 11115, 11327,..., 40492,40895,..., 154349,

16212958578003 1489, 1621295865853 37855,

$3 \cdot 2^{402653210} - 1$, ..., 2, 1, 0

Goodstein 19

- 19, 7625597484990, ~1.3×10^{154}, ...
-

Goodstein Step

- Increment base & decrement number
  - $4 : 2^2$
  - $26 : 3^3 - 1 = 27 - 1 = 26 = 3^2 + 3^2 + 3 + 3 + 2$
  - $41 : 4^2 + 4^2 + 4 + 4 + 1$

Goodstein Step

- Base is a bag (and the whole thing is in a bag)
  - 2^2 is {{{}}}
  - $3^2 + 3^2 + 3 + 3 + 2$ is {{2},{2},{},{},2}
  - $4^2 + 4^2 + 4 + 4 + 1$ is {{2},{2},{},{},1}

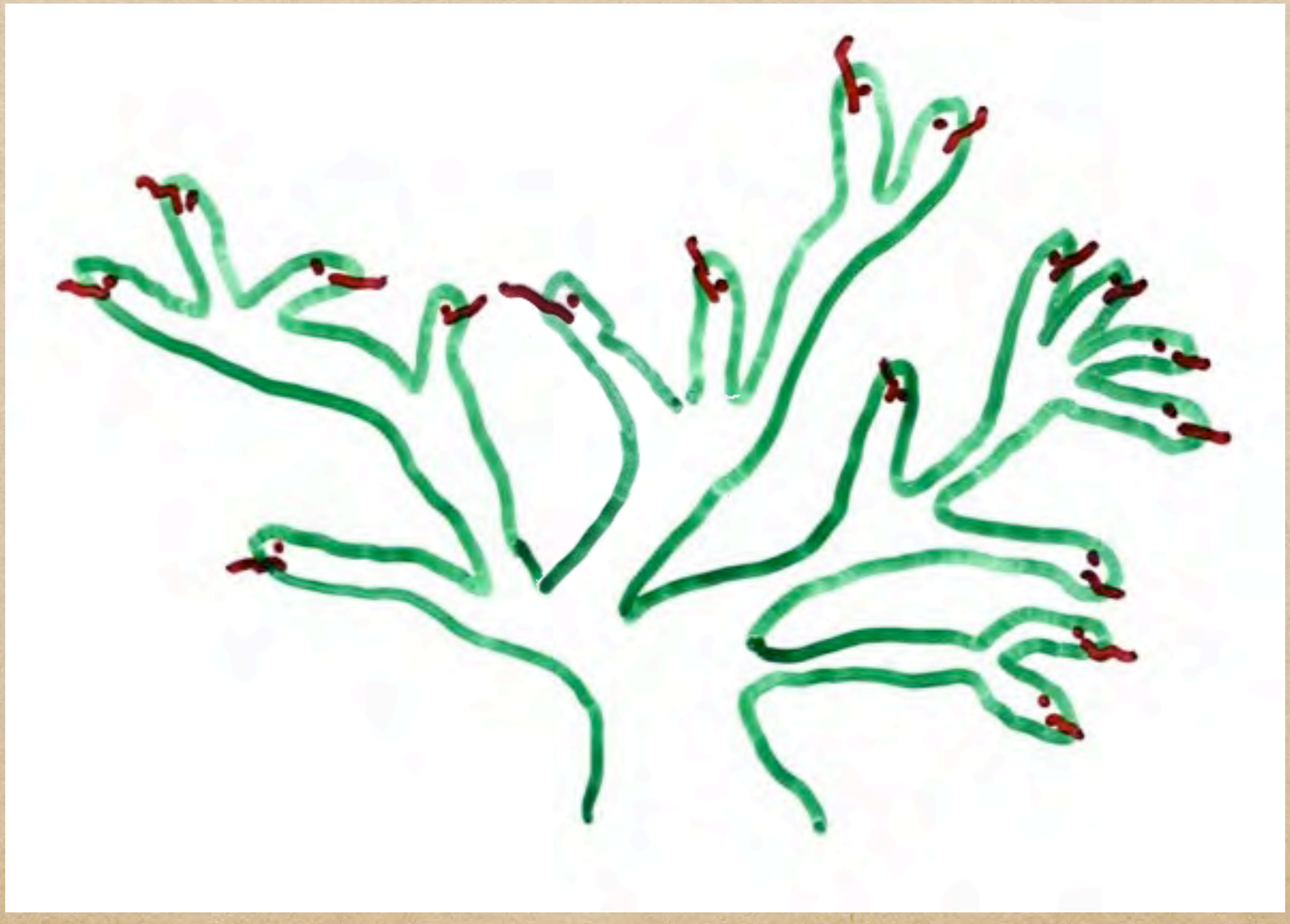
Hydra

Hercules' Second Labor

Each time Hercules bashed one of Hydra's heads, Iolaus held a torch to the headless neck.

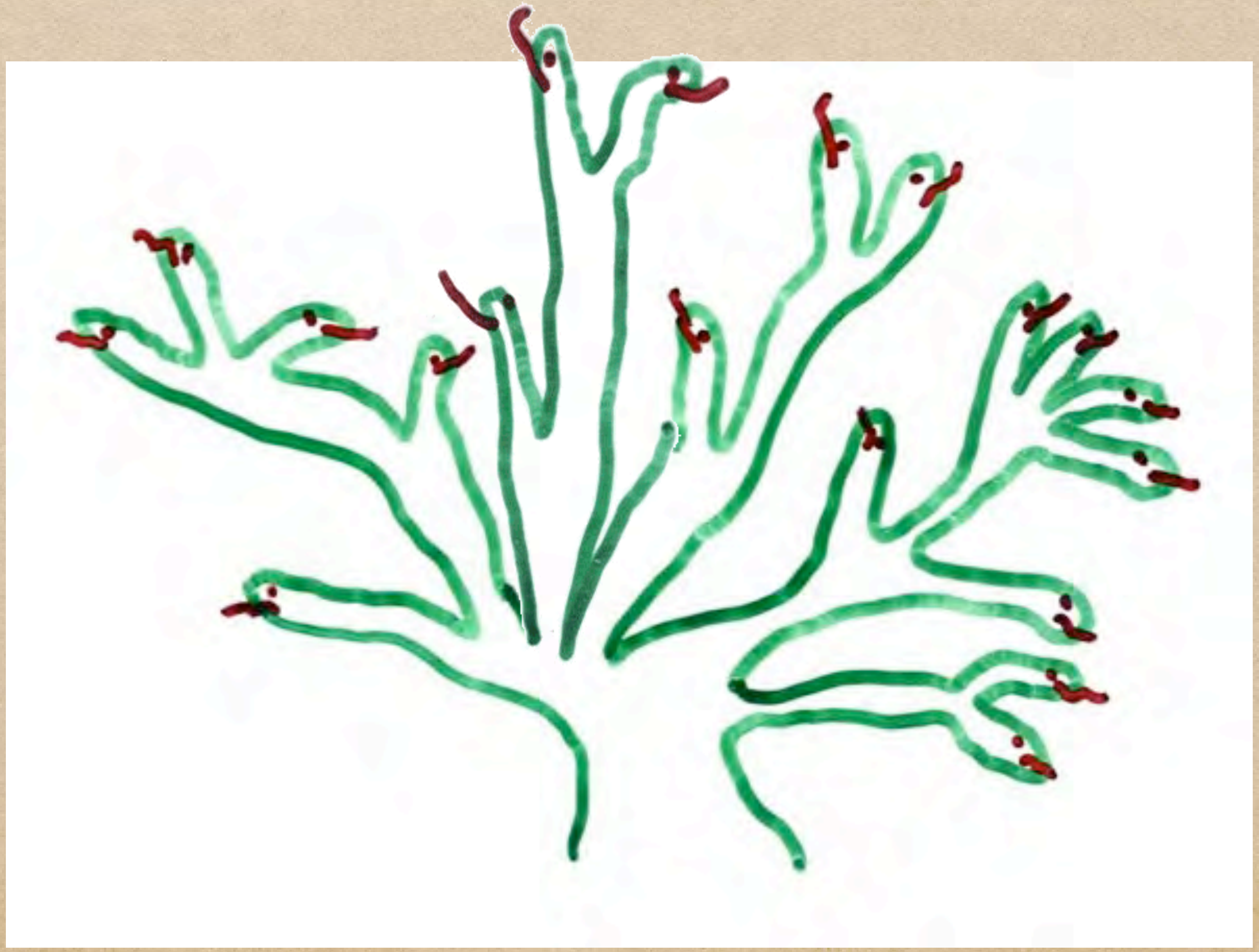
After destroying eight mortal heads, Hercules chopped off the ninth, immortal head, which he buried at the side of the road from Lerna to Elaeus, and covered with a heavy rock.

Hydra vs. Hercules

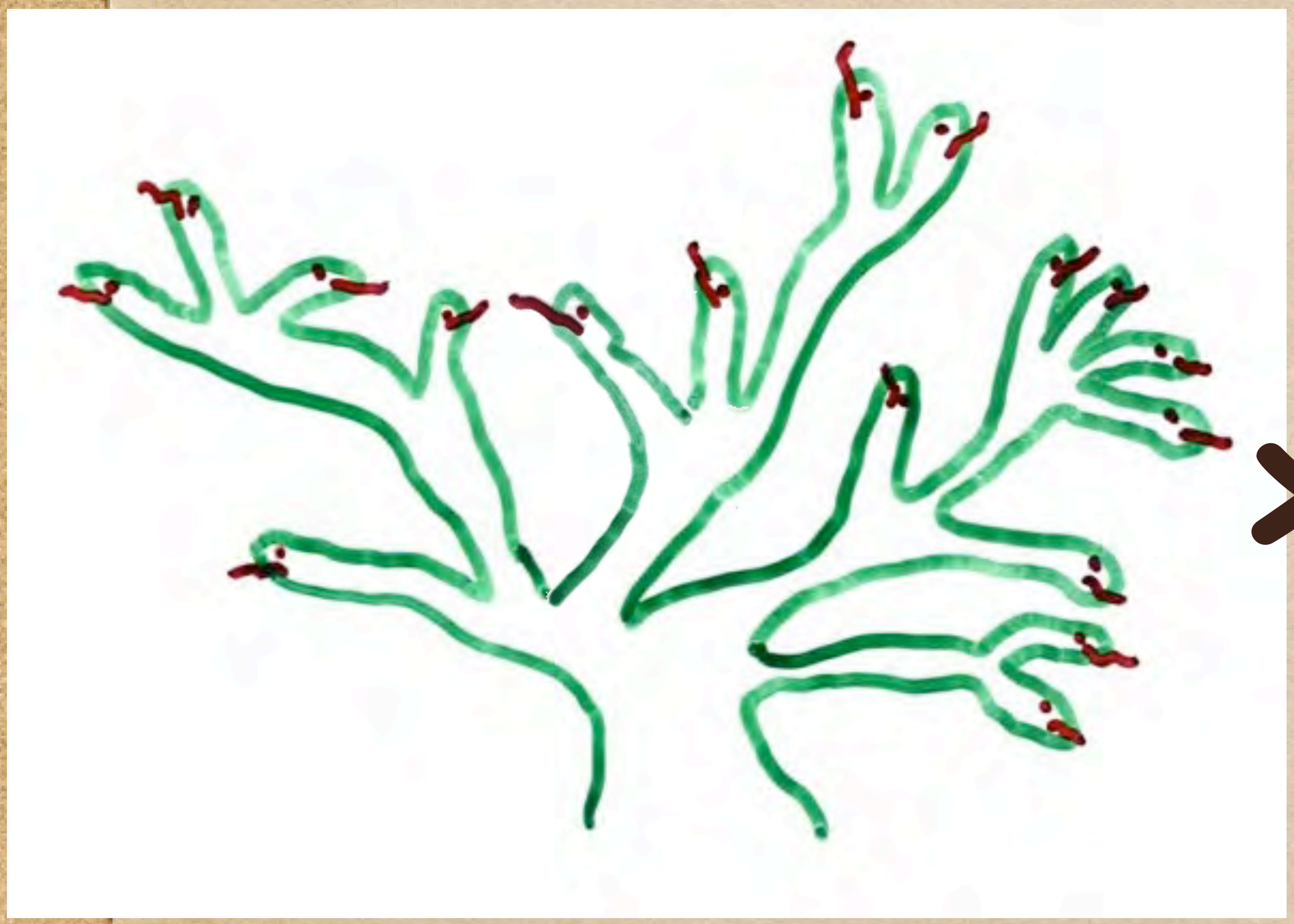


Hydra vs. Hercules

Hydra vs. Hercules



Hercules > Hydra



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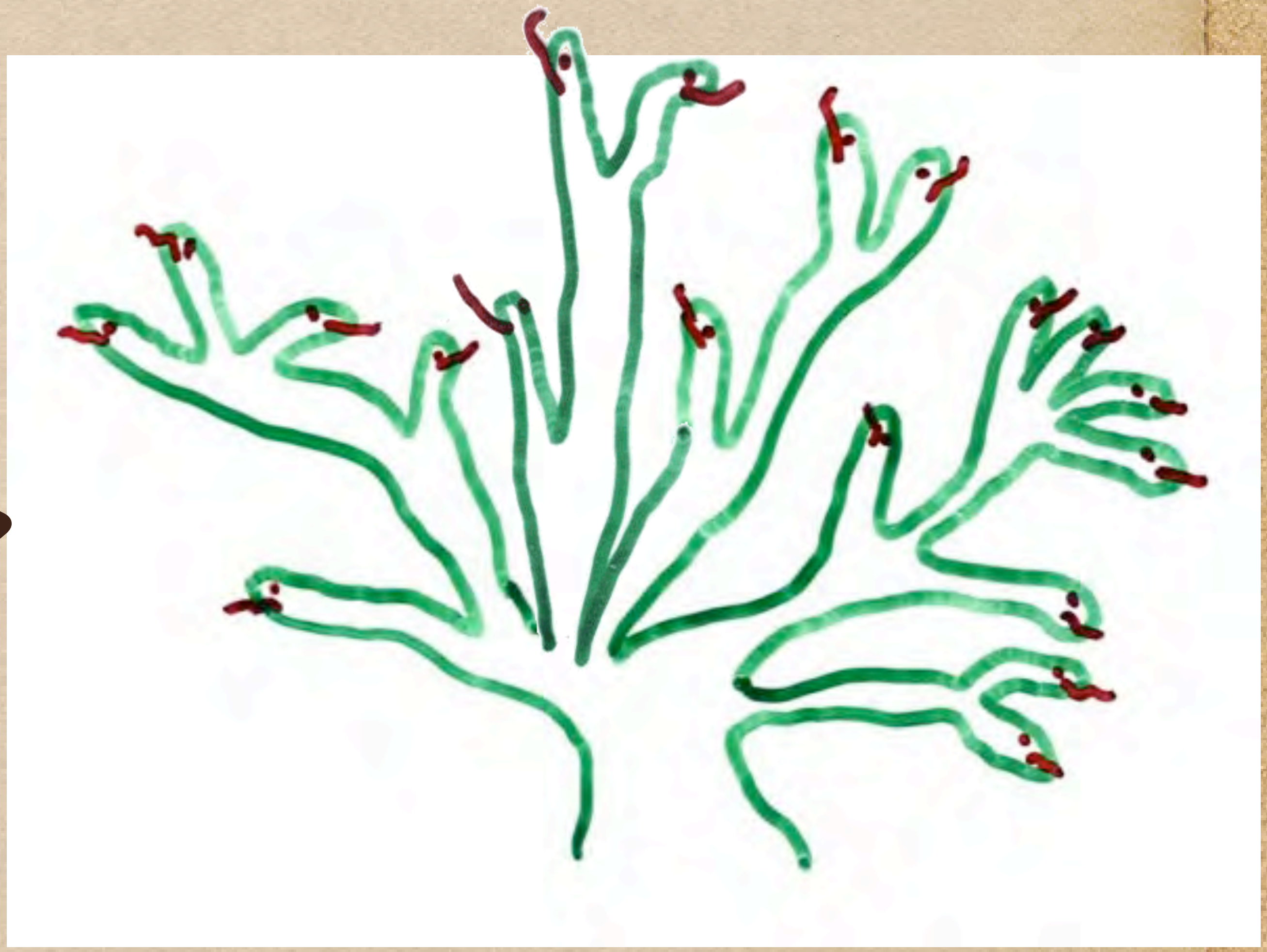
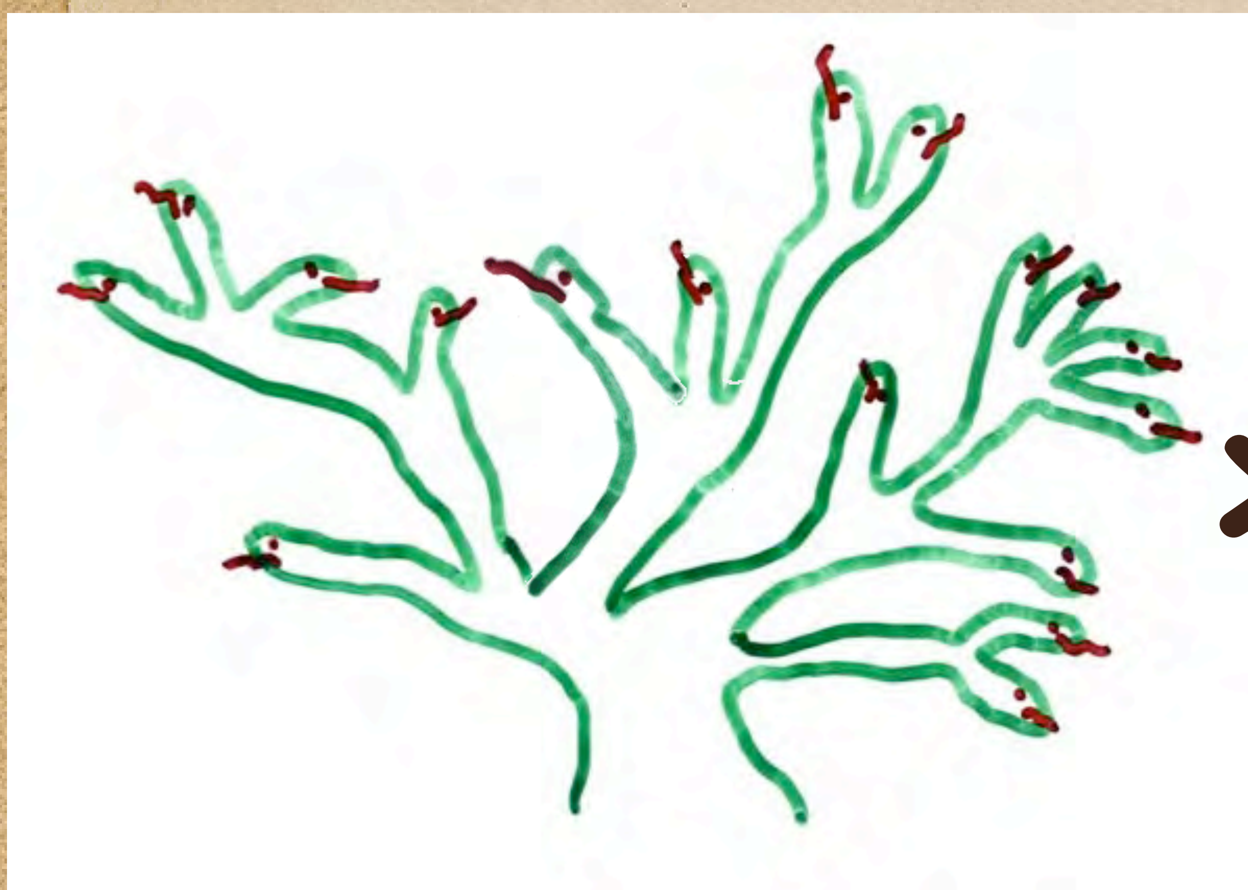


Hercules > Hydra



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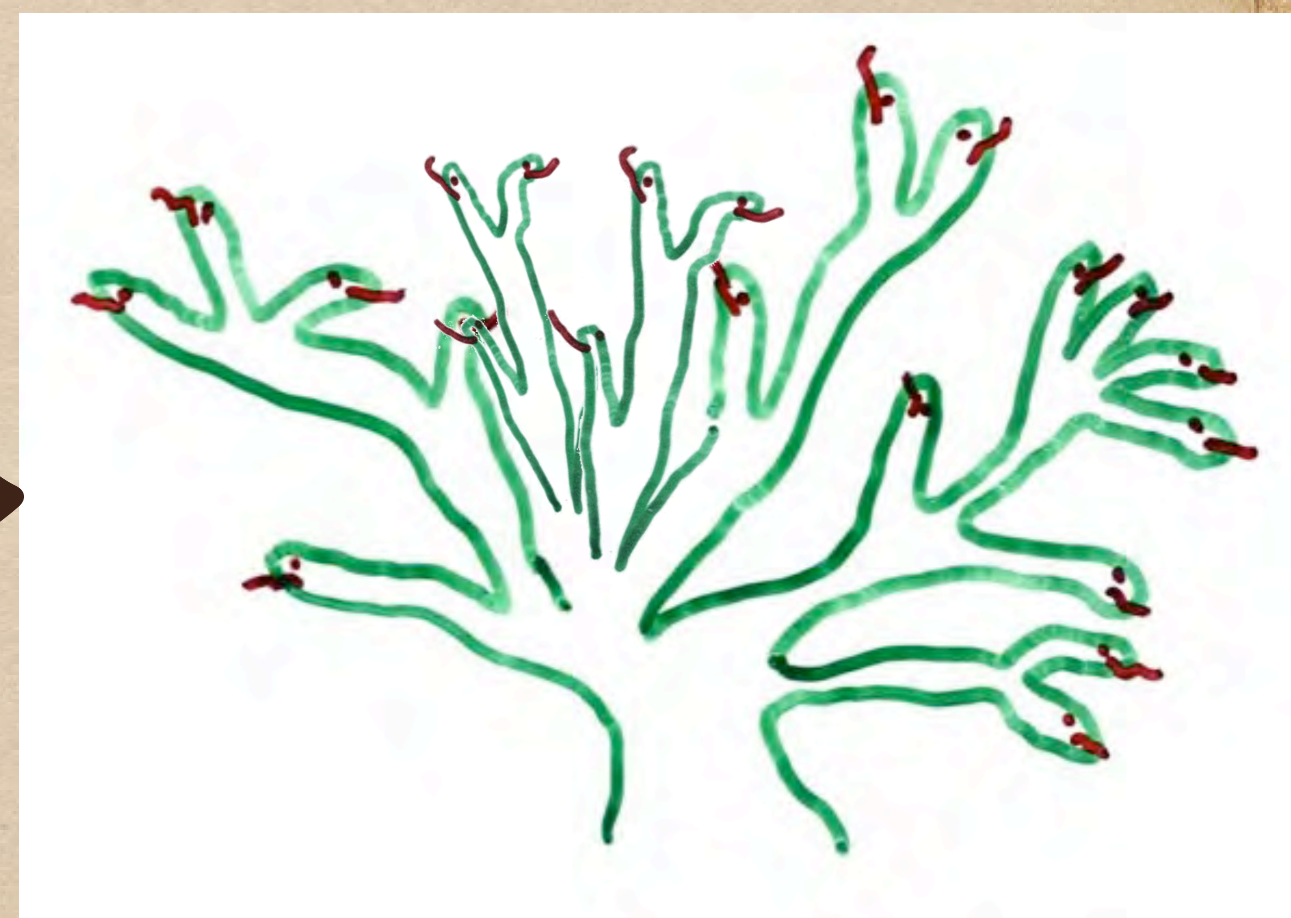
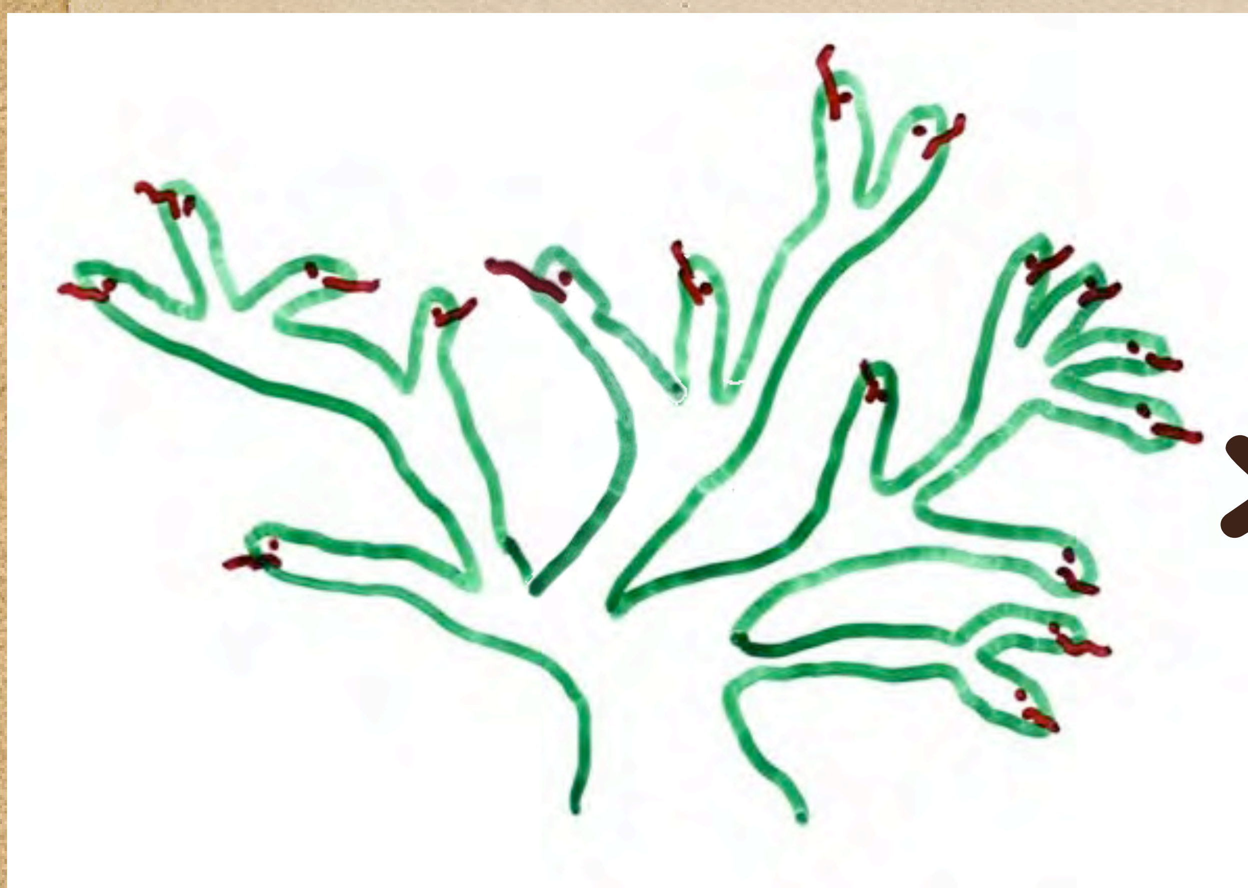


Hercules > Hydra



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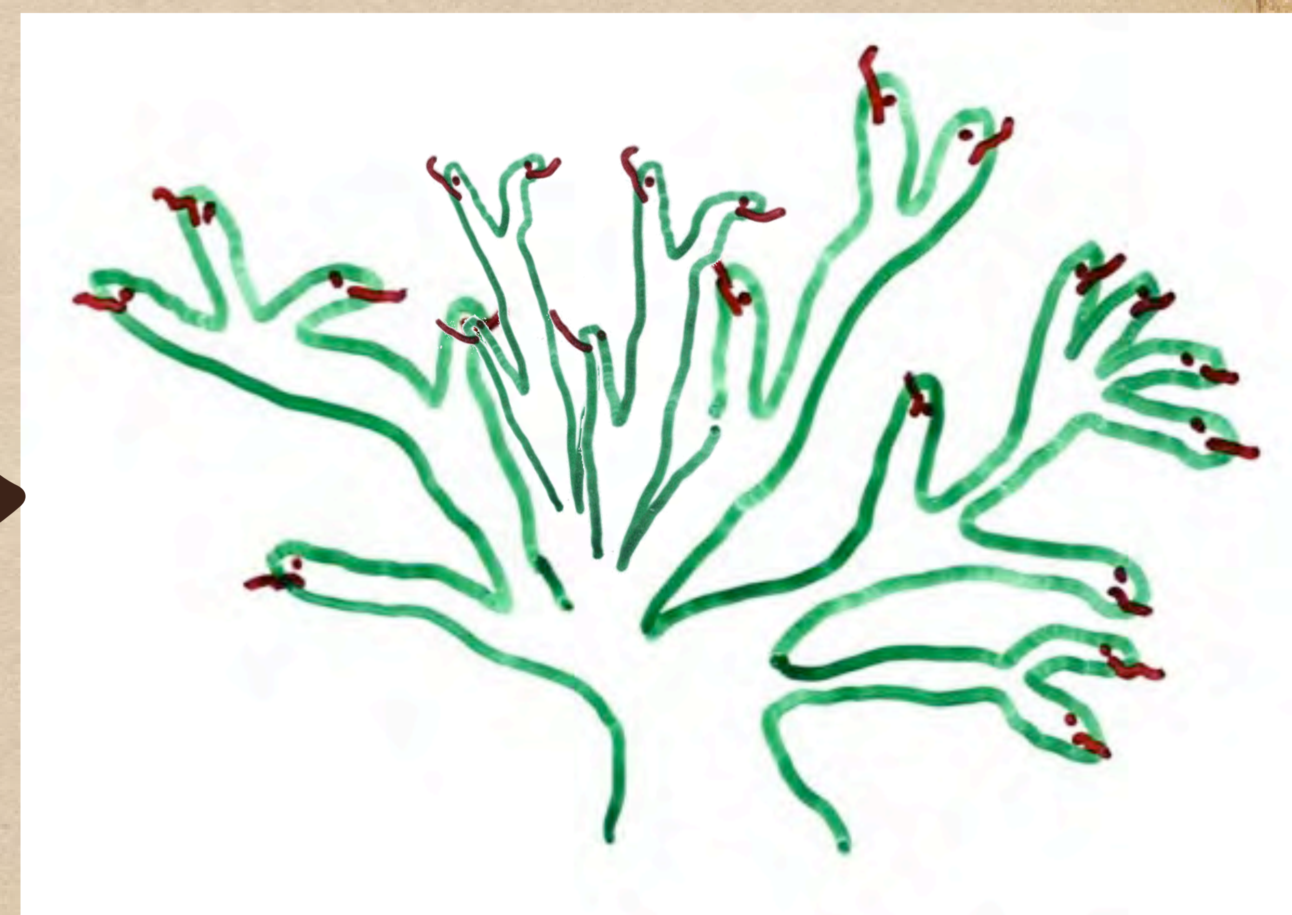
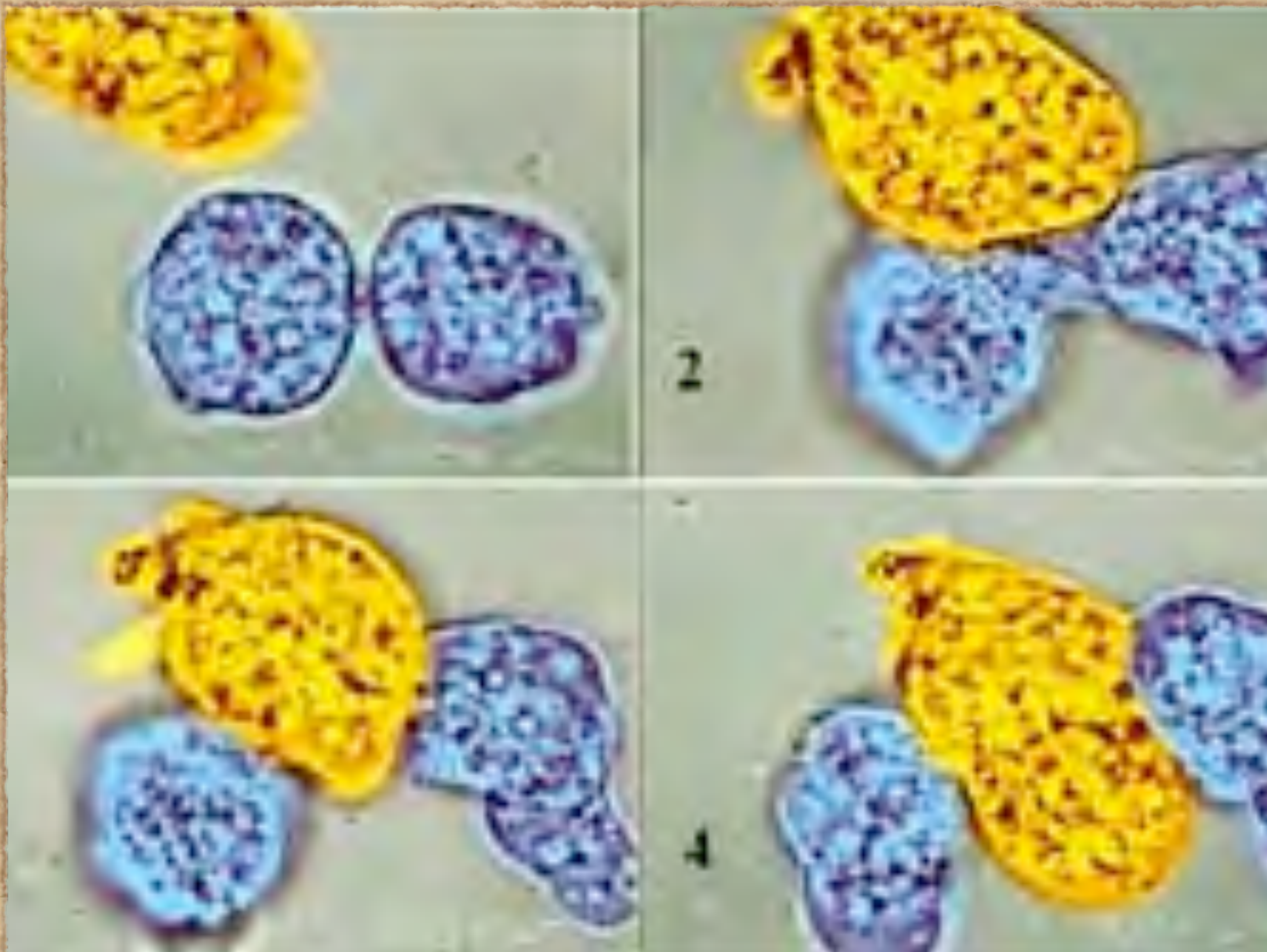


{{o{{ooo}o} {oo{oo}} {o{oooo}o} {oo}} >
{{o{{ooo}o} {o{oo}} {o{oo}} {o{oo}} {o{oooo}o} {oo}}

Hercules Defeats Hydra

- Cannot be proved in Peano Arithmetic [Paris & Kirby]
- Requires induction up to ε_0
  - Natural numbers do not suffice
  - Sophisticated variants require more powerful systems [Friedman]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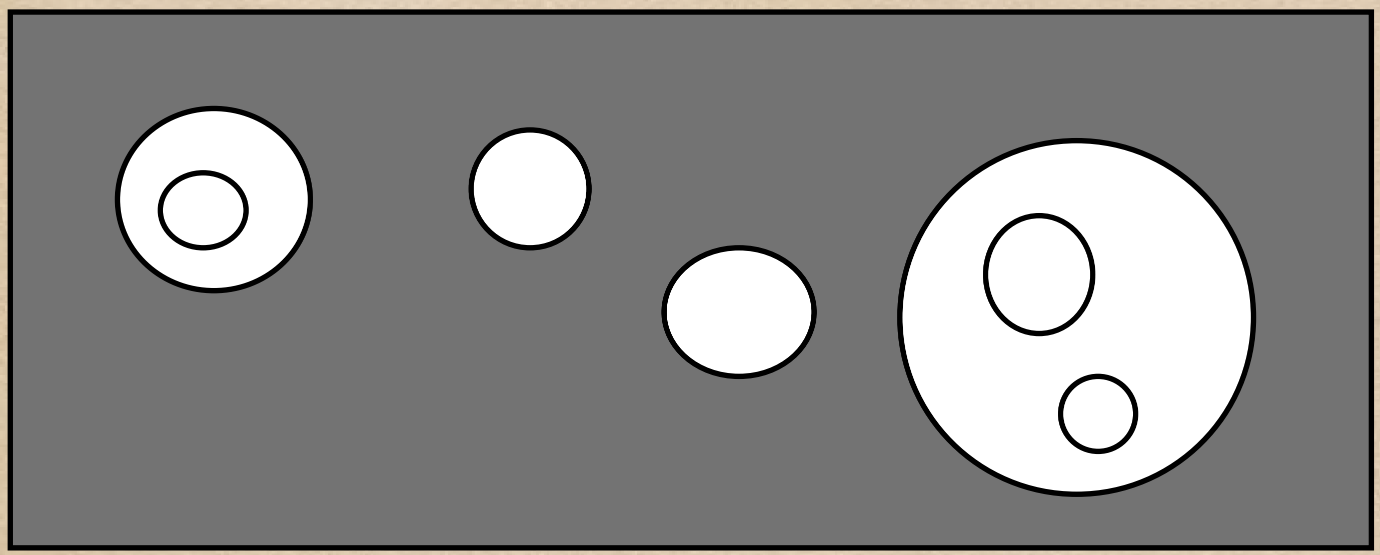
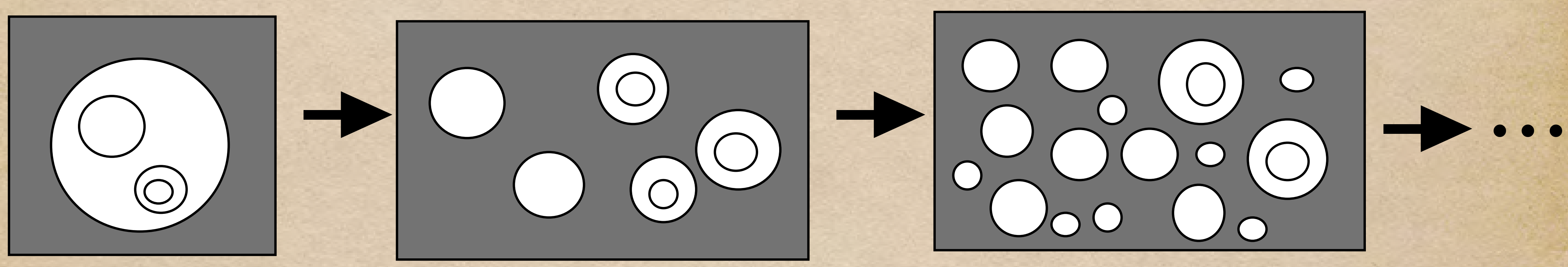
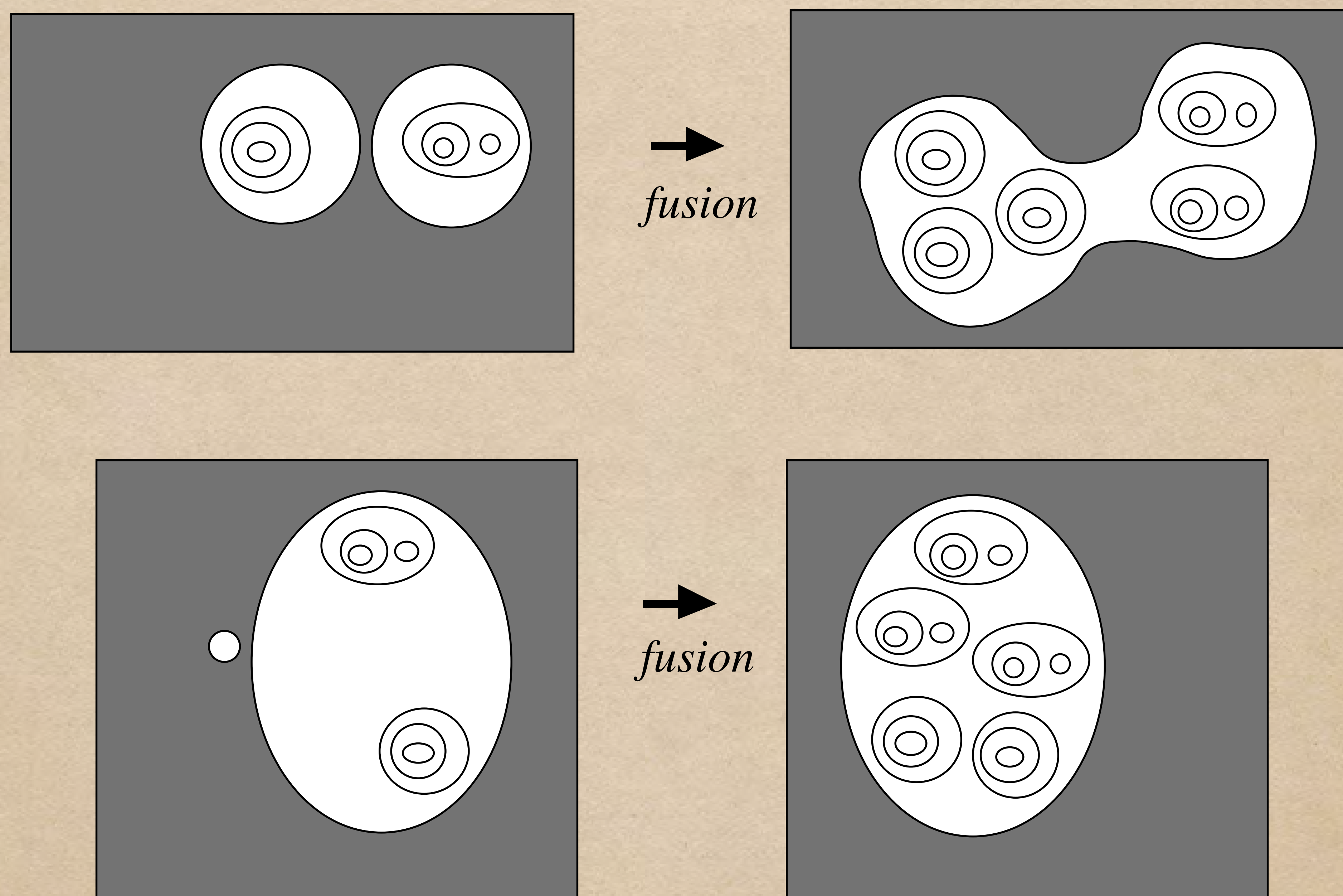
4

Amoebae



Amoebae




fusion
fusion