Termination

Bigger & Bigger

Primitive Recursion

- 0
- +1
- projections
- composition
- f(x,n) := if n=0 then g(x) else h(f(x,n-1),x,n-1)

Ackermann's Function

- $A(0,n) = n+1$
- $A(m+1,0) = A(m,1)$
- $A(m+1,n+1) = A(m,A(m+1,n))$

$A(m,n) > m+n$

- Induction on (m,n)
  - $A(0,n) = n+1 > n$
  - $A(m+1,0) = A(m,1) > m+1$
  - $A(m+1,n+1) = A(m,A(m+1,n))$
    $> m+A(m+1,n) \geq m+n+2$

$x>y \Rightarrow A(m,x) > A(m,y)$

- Induction on (m,x)
  - $A(0,x) = x+1 > y+1 = A(0,y)$
  - $A(m+1,x+1) = A(m,A(m+1,x)) > A(m,A(m+1,y)) = A(m+1,y+1)$

$x > y \Rightarrow A(x,n) > A(y,n)$

- Induction on (x,n)
  - $A(x,n) > x+n > n = A(0,n)$
  - $A(x+1,0) = A(x,1) > A(y,1) = A(y+1,0)$
  - $A(x+1,n+1) = A(x,A(x+1,n))$
    $> A(y,A(x+1,n)) > A(y,A(y+1,n))$
    $= A(y+1,n+1)$

$A(m+n+2,x) > A(m,A(n,x))$

- Induction $(m+n,x)$
  - $A(n+2,x) > A(n+1,x) \geq A(n,x)+1 = A(0,A(n,x))$
  - $A(m+n+2,0) = A(m+n+1,1) > A(m,A(n-1,1)) = A(m,A(n,0))$
  - $A(m+n+2,x+1) = A(m+n+1,A(n+m+2,x)) > A(m,A(n,A(m,x))) > A(m,A(n,x+m)) \geq A(m,A(n,x+1))$

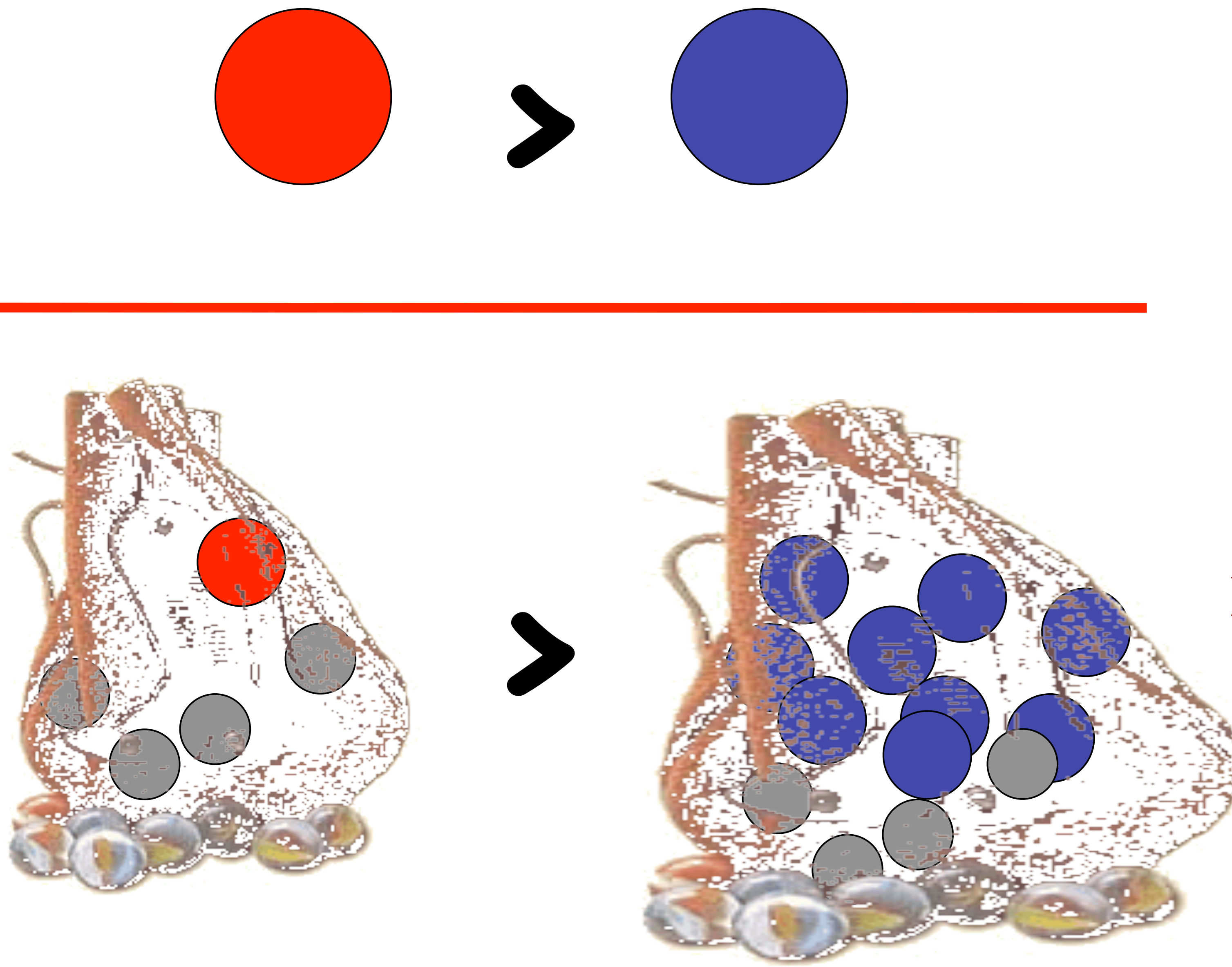
A isn't Primitive Recursive

- Denote $x = x_1,\ldots,x_k$ and $x_m = \max x_j$
- Say $A_i > g$ if $A(i,x_m) > g(x)$ for all x
- Easy: $A_0 > 0$; $A_1 > +1$; $A_0 > \text{proj}_i$
- Suppose $f(x) = h(g_1x,\ldots,g_kx)$, $A_s > g_1,\ldots,g_k,h$
  - $A_{2s+2} > f$: $A(2s+2,x) > A(s,A(s,x))$
    $> A(s,\max\{g_jx\}) > h(g_1x,\ldots,g_kx)$

A isn't Primitive Recursive

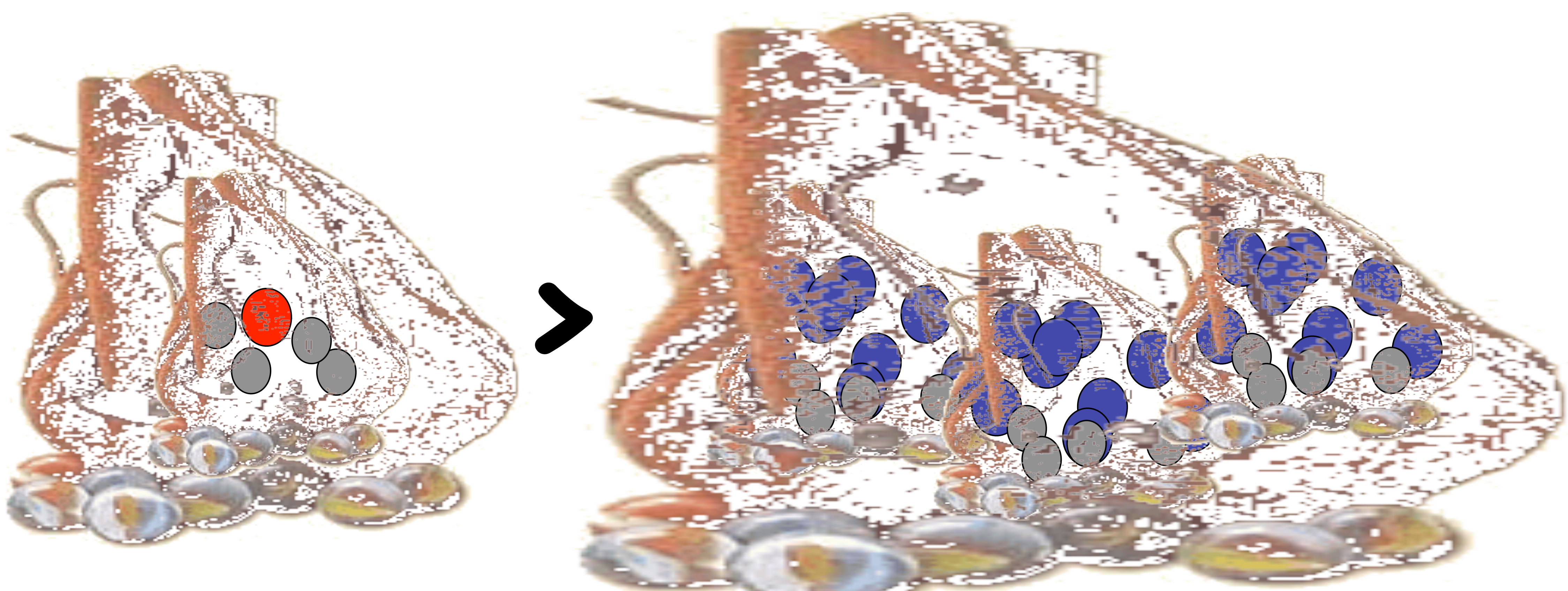
- Suppose $A_s > g,h$ and $f(x,n) = \text{if } n=0 \text{ then } g(x) \text{ else } h(f(x,n-1),x,n-1)$
- $A(r,n+x_m) > f(x,n)$, $r = 2s+1$, by induction on n:
  - $f(x,0) = g(x) < A(s,x_m) < A(r,0+x_m)$
  - $f(x,n+1) = h(f(x,n),x,n) < A(s,\max\{f(x,n),n,x_m\}) < A(s,A(r,n+x_m)) < A(2s,A(r,n+x_m)) = A(r,n+1+x_m)$
- $f(x,n) < A(r,n+x_m) < A(r,2N+3) = A(r,A(2,N)) < A(r+4,N)$ where $N = \max\{n,x_m\}$

Multiset Ordering



Well-founded
by
König's Lemma

Nested Ordering



Well-Founded Induction

$$\frac{\forall x.\ [\forall y{<}x.\ P(y)] \Rightarrow P(x)}{\forall x.\ P(x)}$$

Ordinals

$0 < 1 < 2 < \cdots$

$< \omega < \omega+1 < \omega+2 < \cdots$

$< \omega 2 < \omega 2+1 < \cdots < \omega 3 < \cdots < \omega 4 < \cdots$

$< \omega^2 < \omega^2+1 < \cdots < \omega^2+\omega < \omega^2+\omega+1 < \cdots$

$< \omega^3 < \omega^3+1 < \cdots < \omega^4 < \cdots < \omega^5 < \cdots$

$< \omega^\omega < \cdots < \omega^{\omega^\omega} < \cdots < \omega^{\omega^{\omega^\omega}} < \cdots$

Bags of Bags

- An empty bag is worth 0
- A bag containing bags worth α_i, is worth $\Sigma\omega^{\alpha_i}$

Goodstein Step

- Increment base & decrement number
  - $4 : 2^2$
  - $26 : 3^3 - 1 = 27 - 1 = 26 = 3^2 + 3^2 + 3 + 3 + 2$
  - $41 : 4^2 + 4^2 + 4 + 4 + 1$

Goodstein Step

- Base is a bag (and the whole thing is in a bag)
  - 2^2 is {{{}}}
  - $3^2 + 3^2 + 3 + 3 + 2$ is {{2},{2},{},{},2}
  - $4^2 + 4^2 + 4 + 4 + 1$ is {{2},{2},{},{},1}

Goodstein 16

$$g_{16}(2) = 16 = 2^{2\text{\textasciicircum}2}$$

$$g_{16}(3) = 3^{3\text{\textasciicircum}3}-1 = 2\cdot3^{2\cdot3\text{\textasciicircum}2+2\cdot3+2} + 2\cdot3^{2\cdot3\text{\textasciicircum}2+2\cdot3+1}$$

$$+ 2\cdot3^{2\cdot3\text{\textasciicircum}2+2\cdot3} + 2\cdot3^{2\cdot3\text{\textasciicircum}2+1\cdot3+2} + 2\cdot3^{2\cdot3\text{\textasciicircum}2+1\cdot3+1}$$

$$+ 2\cdot3^{2\cdot3\text{\textasciicircum}2+1\cdot3} + 2\cdot3^{2\cdot3\text{\textasciicircum}2+2} + 2\cdot3^{2\cdot3\text{\textasciicircum}2+1} + 2\cdot3^{2\cdot3\text{\textasciicircum}2}$$

$$+ 2\cdot3^{3\text{\textasciicircum}2+2\cdot3+2} + 2\cdot3^{3\text{\textasciicircum}2+2\cdot3+1} + 2\cdot3^{3\text{\textasciicircum}2+2\cdot3} + 2\cdot3^{3\text{\textasciicircum}2+1\cdot3+2}$$

$$+ 2\cdot3^{3\text{\textasciicircum}2+1\cdot3+1} + 2\cdot3^{3\text{\textasciicircum}2+1\cdot3} + 2\cdot3^{3\text{\textasciicircum}2+2} + 2\cdot3^{3\text{\textasciicircum}2+1} + 2\cdot3^{3\text{\textasciicircum}2}$$

$$+ 2\cdot3^{2\cdot3+2} + 2\cdot3^{2\cdot3+1} + 2\cdot3^{2\cdot3} + 2\cdot3^{1\cdot3+2} + 2\cdot3^{1\cdot3+1} + 2\cdot3^{1\cdot3}$$

$$+ 2\cdot3^{2} + 2\cdot3^{1} + 2 = 7625597484986$$

Goodstein 16

$g_{16}(2) = 16 = 2^{2\hat{}2}$

$g_{16}(3) = 3^{[1000]}-1 = 2\cdot3^{[222]} + 2\cdot3^{[221]} + 2\cdot3^{[220]} +$
$2\cdot3^{[212]} + 2\cdot3^{[211]} + 2\cdot3^{[210]} + 2\cdot3^{[202]} + 2\cdot3^{[201]} +$
$2\cdot3^{[200]} + 2\cdot3^{[122]} + 2\cdot3^{[121]} + 2\cdot3^{[120]} + 2\cdot3^{[112]} + 2\cdot3^{[111]} +$
$2\cdot3^{[110]} + 2\cdot3^{[102]} + 2\cdot3^{[101]} + 2\cdot3^{[100]} + 2\cdot3^{[022]} + 2\cdot3^{[021]}$
$+ 2\cdot3^{[020]} + 2\cdot3^{[012]} + 2\cdot3^{[011]} + 2\cdot3^{[010]} + 2\cdot3^{[002]} +$
$2\cdot3^{[001]} + 2\cdot3^{[000]} = 7625597484986$

where [abc] is the base 3 representation

Goodstein 16

$g_{16}(2) = \omega^{\omega\hat{}\omega}$

$g_{16}(3) = 2\cdot\omega^{2\cdot\omega\hat{}2 + 2\cdot\omega + 2} + 2\cdot\omega^{2\cdot\omega\hat{}2 + 2\cdot\omega + 1} + 2\cdot\omega^{2\cdot\omega\hat{}2 + 2\cdot\omega}$

$+ 2\cdot\omega^{2\cdot\omega\hat{}2 + 1\cdot\omega + 2} + 2\cdot\omega^{2\cdot\omega\hat{}2 + 1\cdot\omega + 1} + 2\cdot\omega^{2\cdot\omega\hat{}2 + 1\cdot\omega} +$

$2\cdot\omega^{2\cdot\omega\hat{}2 + 2} + 2\cdot\omega^{2\cdot\omega\hat{}2 + 1} + 2\cdot\omega^{2\cdot\omega\hat{}2} + 2\cdot\omega^{\omega\hat{}2 + 2\cdot\omega + 2} +$

$2\cdot\omega^{\omega\hat{}2 + 2\cdot\omega + 1} + 2\cdot\omega^{\omega\hat{}2 + 2\cdot\omega} + 2\cdot\omega^{\omega\hat{}2 + 1\cdot\omega + 2} +$

$2\cdot\omega^{\omega\hat{}2 + 1\cdot\omega + 1} + 2\cdot\omega^{\omega\hat{}2 + 1\cdot\omega} + 2\cdot\omega^{\omega\hat{}2 + 2} + 2\cdot\omega^{\omega\hat{}2 + 1} +$

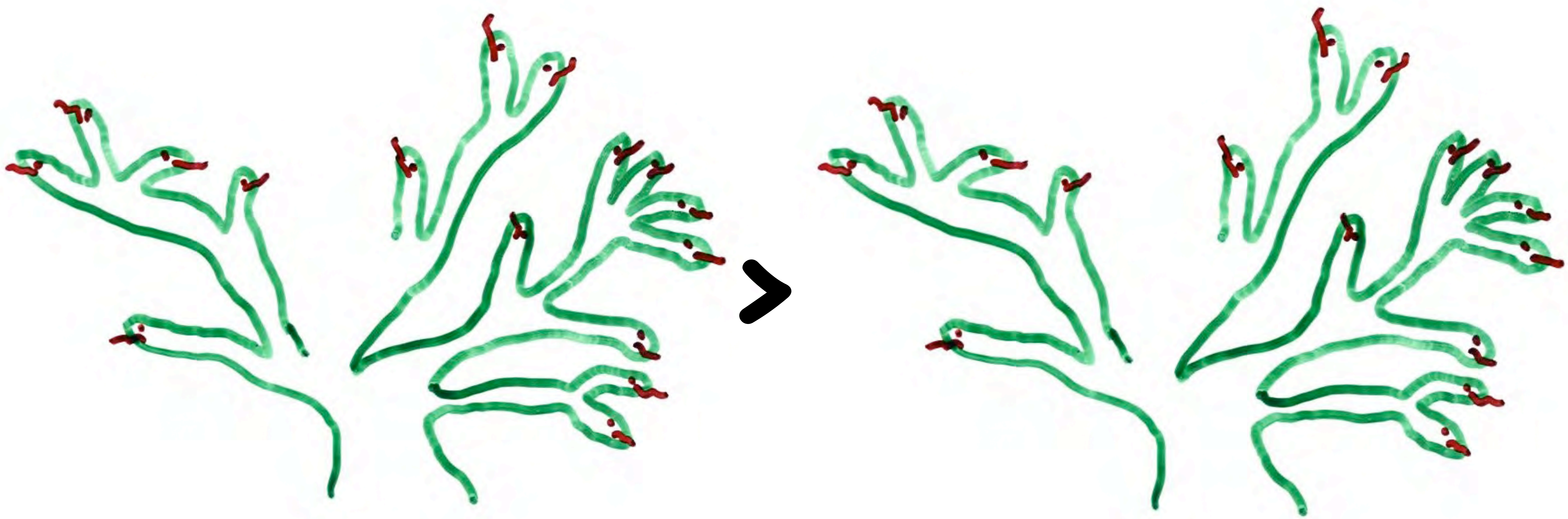
$2\cdot\omega^{\omega\hat{}2} + 2\cdot\omega^{2\cdot\omega + 2} + 2\cdot\omega^{2\cdot\omega + 1} + 2\cdot\omega^{2\cdot\omega} + 2\cdot\omega^{1\cdot\omega + 2} +$

$2\cdot\omega^{1\cdot\omega + 1} + 2\cdot\omega^{1\cdot\omega} + 2\cdot\omega^{2} + 2\cdot\omega^{1} + 2$

Goodstein

- Cannot be proved terminating in Peano Arithmetic

Hercules > Hydra



Hercules Defeats Hydra

- Cannot be proved in Peano Arithmetic [Paris & Kirby]
- Requires induction up to ε_0
- Natural numbers do not suffice
- Sophisticated variants require more powerful systems [Friedman]

Hydra Step

- Every head is an empty bag
- Every node (including the ground) is a bag of its children
- Each step replaces some internal bag with some number of smaller bags

Hydra Step

- Heads are worth 0
- Every node (including the ground), with children worth α_i, is worth $\Sigma\omega^{\alpha_i}$
- The kth step replaces a term $\omega^{\alpha+1}$ with $\omega^{\alpha}k$
- But if a head sprouting from the ground is cut, the total decreases by 1

Hercules Defeats Hydra

- Cannot be proved in Peano Arithmetic [Paris & Kirby]
- Requires induction up to ε_0
  - Natural numbers do not suffice
  - Sophisticated variants require more powerful systems [Friedman]