Termination

Well-Quasi Orderings

CMU Exam [Floyd]

- $Dt = 1$
- $Dc = 0$
- $D(x+y) = Dx+Dy$
- $D(xy) = yDx+xDy$

- $D(D(tt)) = D(tDt+tDt) = D(tDt) + D(tDt) = \ldots$

CONTRIBUTIONS TO MECHANICAL MATHEMATICS

by

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Abstract : The thesis contains two parts which are self-contained units. In Part 1 we present several results on the relation between the problem of termination and equivalence of programs and abstract programs, and the first order predicate calculus. Part 2 is concerned with the relation between the termination of interpreted graphs, and properties of well-ordered sets and graph theory. (Author)

Descriptors : (*COMPUTER PROGRAMMING, ALGORITHMS), COMPUTER PROGRAMS, NUMERICAL ANALYSIS, SET THEORY, GRAPHICS, THEORY, FLOW CHARTING, SEQUENCES(MATHEMATICS), COMPATIBILITY, MATRICES(MATHEMATICS), THESES

Subject Categories : Theoretical Mathematics
Computer Programming and Software

Gremlins

Gremlins

Disjunctiveness

while c do

$A \mid B$

a,b wfo

$(A \cup B)^+ \subseteq a \cup b$

Disjunctiveness

while $x > 0$ and $y > 0$ do

$x := x-1$ $y := ?$	|	$y := y-1$

$x_i > x_j \vee y_i > y_j$ for $i>j$

need $x_i \geq x_j$

Jumping

while c do

A | B

while c do A while c do B

$BA \subseteq A(A \cup B)^* \cup B$

Jumping

while $x > 0$ and $y > 0$ do

$$\begin{array}{l|l} x := x-1 & y := y-1 \\ y := ? & \end{array}$$

$BA \subseteq A$

Jumping

while $x > 0$ and $y > 0$ do

$$\begin{array}{l} x := x-1 \\ y := x+y \end{array} \;\Big|\; y := y-1$$

$BA \subseteq AB$

Disjunctiveness

while $x > 0$ and $y > 0$ do

$$\begin{array}{l|l} x := x-1 & y := y-1 \\ y := xy & \end{array}$$

$BA \subseteq AB^*$

Fairness

```
s := true
n := 0
while s do
        n := n+1   |   s := false
```

Fairness

s := ?

n := 0

while s>0 do

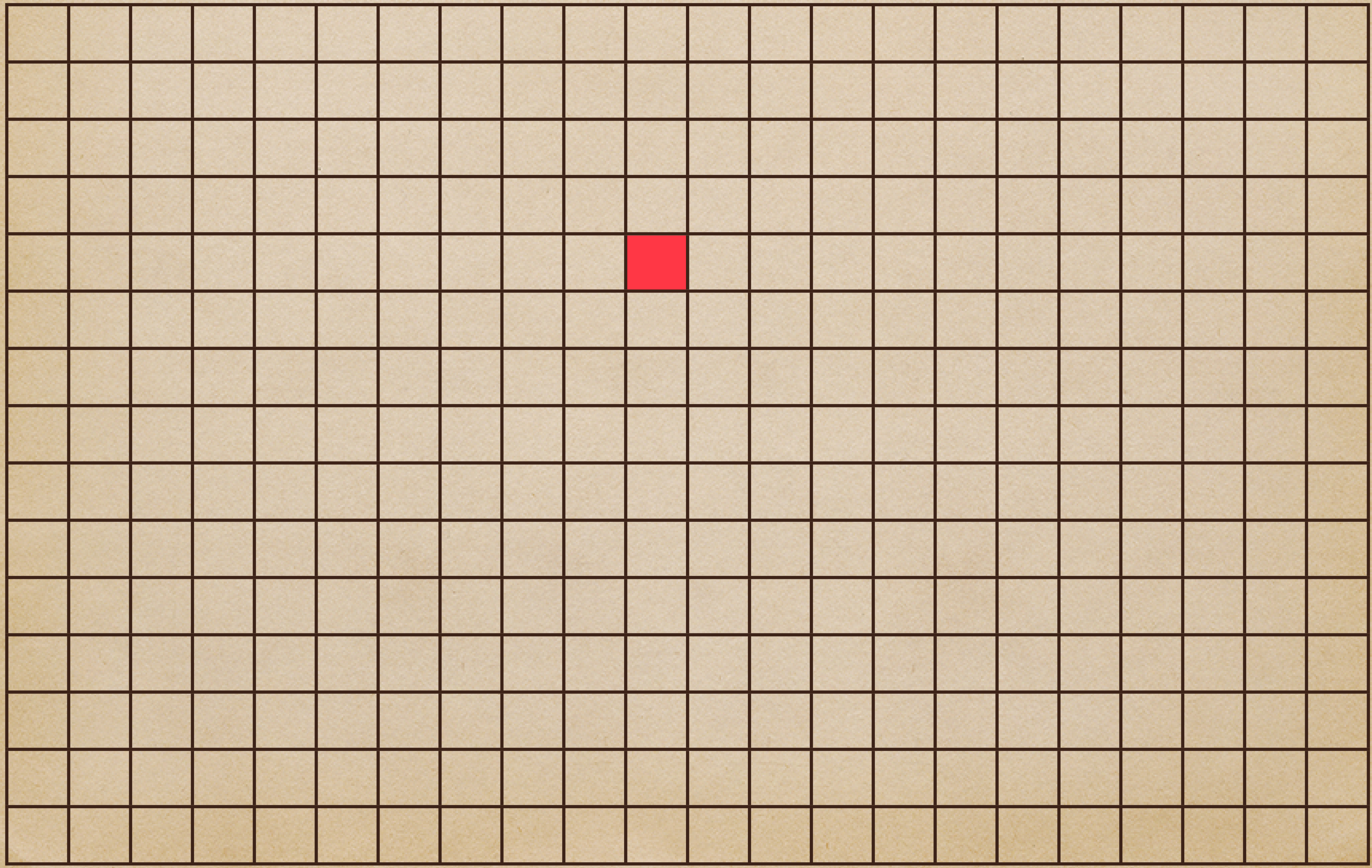
n := n+1 | s := s-1

fair ⇒ no AAAAAAAAAAAA...

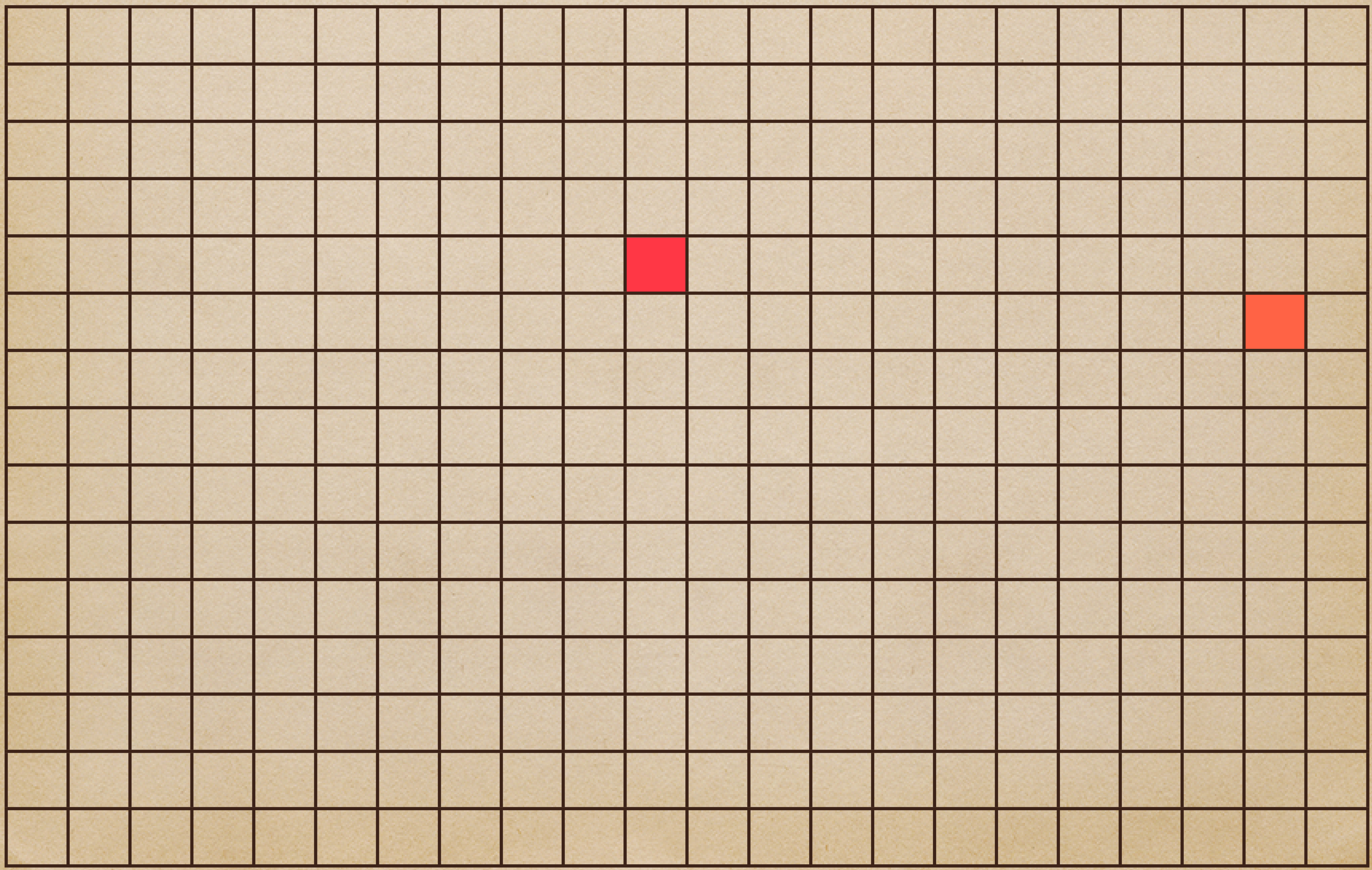
Grid Game

- Given (upper-right) grid coordinates ($x0$,$y0$)
- Choose (xi,yi) to prolong game s.t.
  - $xi < xj$ for all $j<i$ OR $yi < yj$ for all $j<i$

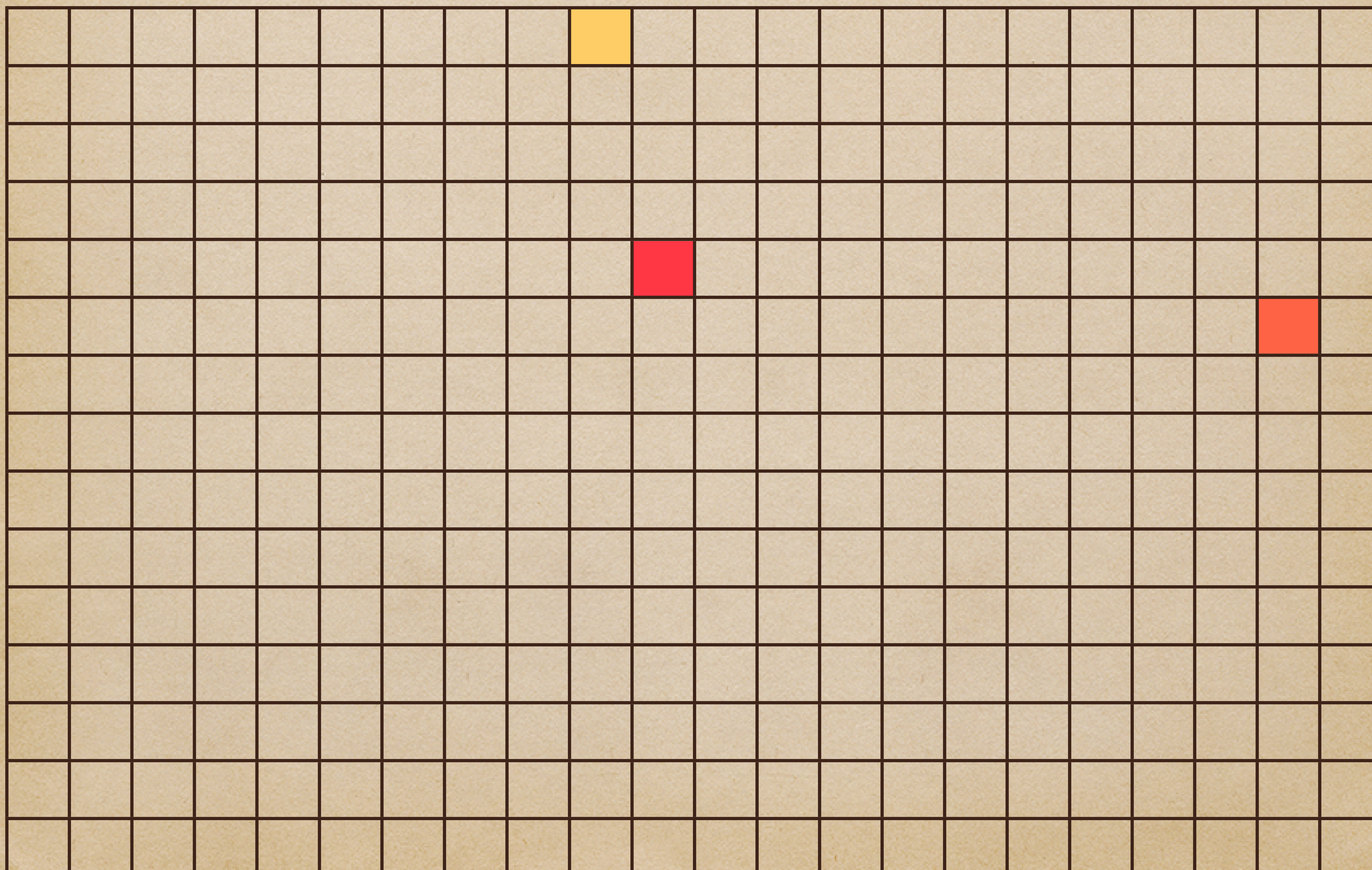
Grid Game



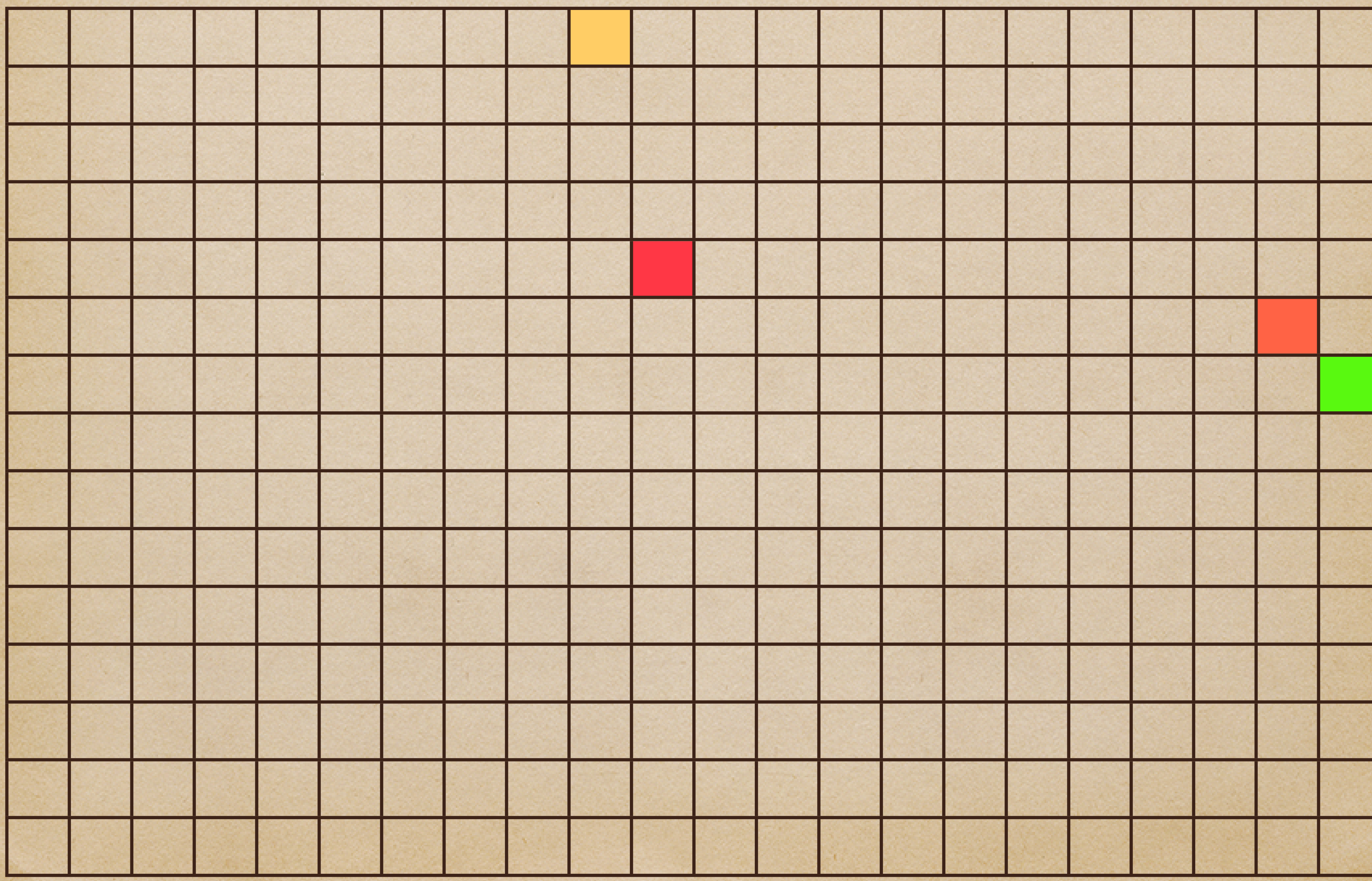
Grid Game



Grid Game



Grid Game



Tricolor

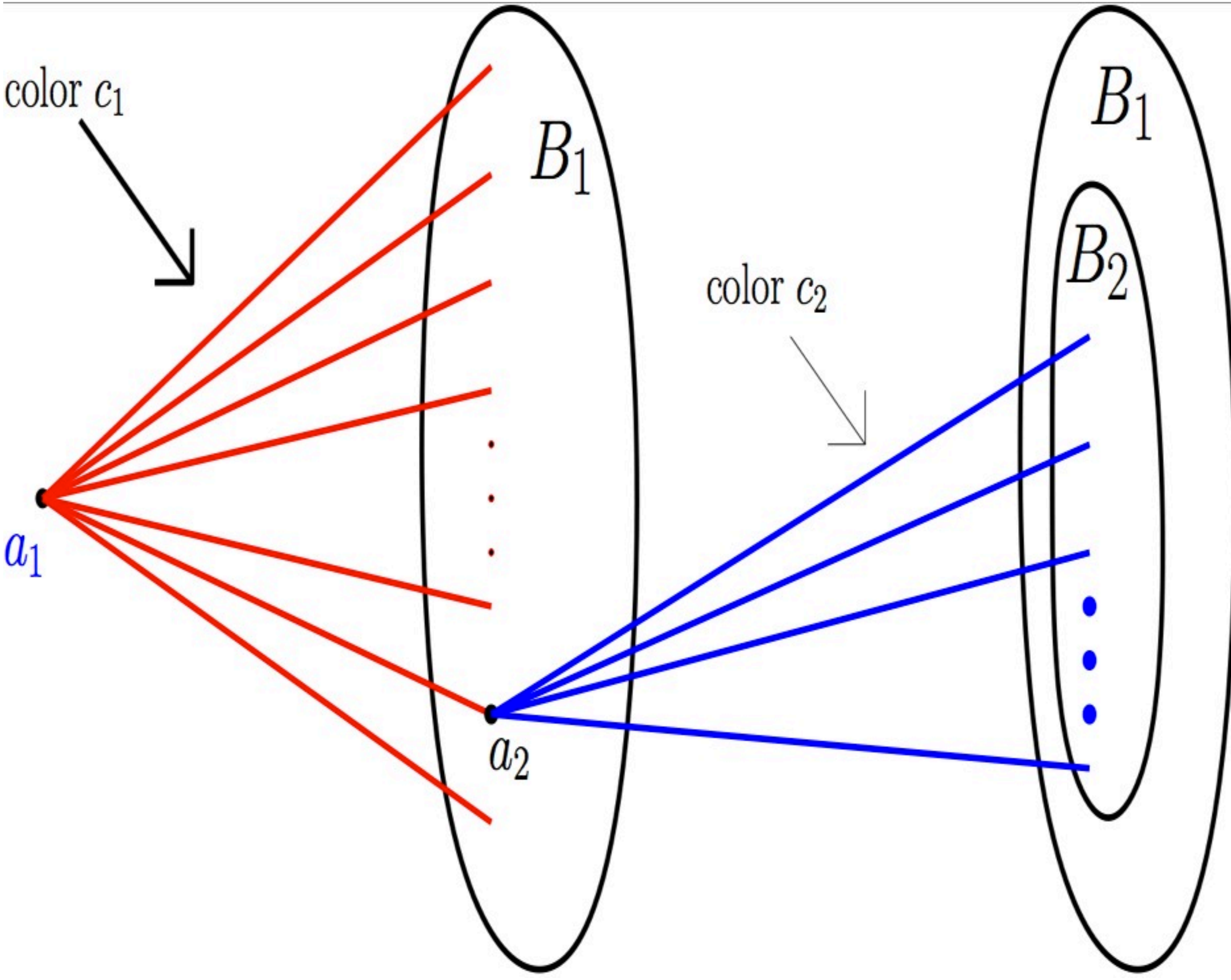
- Color pairs i,j of points
    - Purple if $x_i > x_j$ and $y_i > y_j$
    - Blue if only $x_i > x_j$
    - Red if only $y_i > y_j$
- Consider sequence of points
    - Ramsey contradicts well-foundedness

Ramsey's Theorem

- Two colors: yes and no
- Extend yes as long as possible
- If can forever, then done (all yes)
- If not, then repeat
- If repeats forever, then done (all no)

Ramsey's Theorem

- Reduce more than 2 colors to 2 (color-blindness). Repeat.
- For 2: Form sequence of nodes a1 a2 a3 ... by repeatedly taking monochromatically-connected subsets



color c_1
B_1
a_1
a_2
color c_2
B_1
B_2

$S := V$

$R := \emptyset$

do forever

$$x :\in S$$

$$R := R \cup \{x\}$$

$$S := S \setminus \{x\}$$

$$W := \{s \in S \mid c(x, s) = \text{white}\}$$

$$S := \begin{cases} W & \textbf{if } |W| = \infty \\ S \setminus W & \textbf{otherwise} \end{cases}$$

$$W := \{x \in R \mid \forall y \in R.\, y \neq x \to c(x, y) = \text{white}\}$$

$$\textbf{return} \begin{cases} W & \textbf{if } |W| = \infty \\ R \setminus W & \textbf{otherwise} \end{cases}$$

Ramsey's Theorem

Infinite complete multi-graph

Finitely colored multi-edges

Monochrome infinite clique

can have multiple multi-edges

Quasi-ordering

- Greater or equivalent
  - Transitive
  - Reflexive

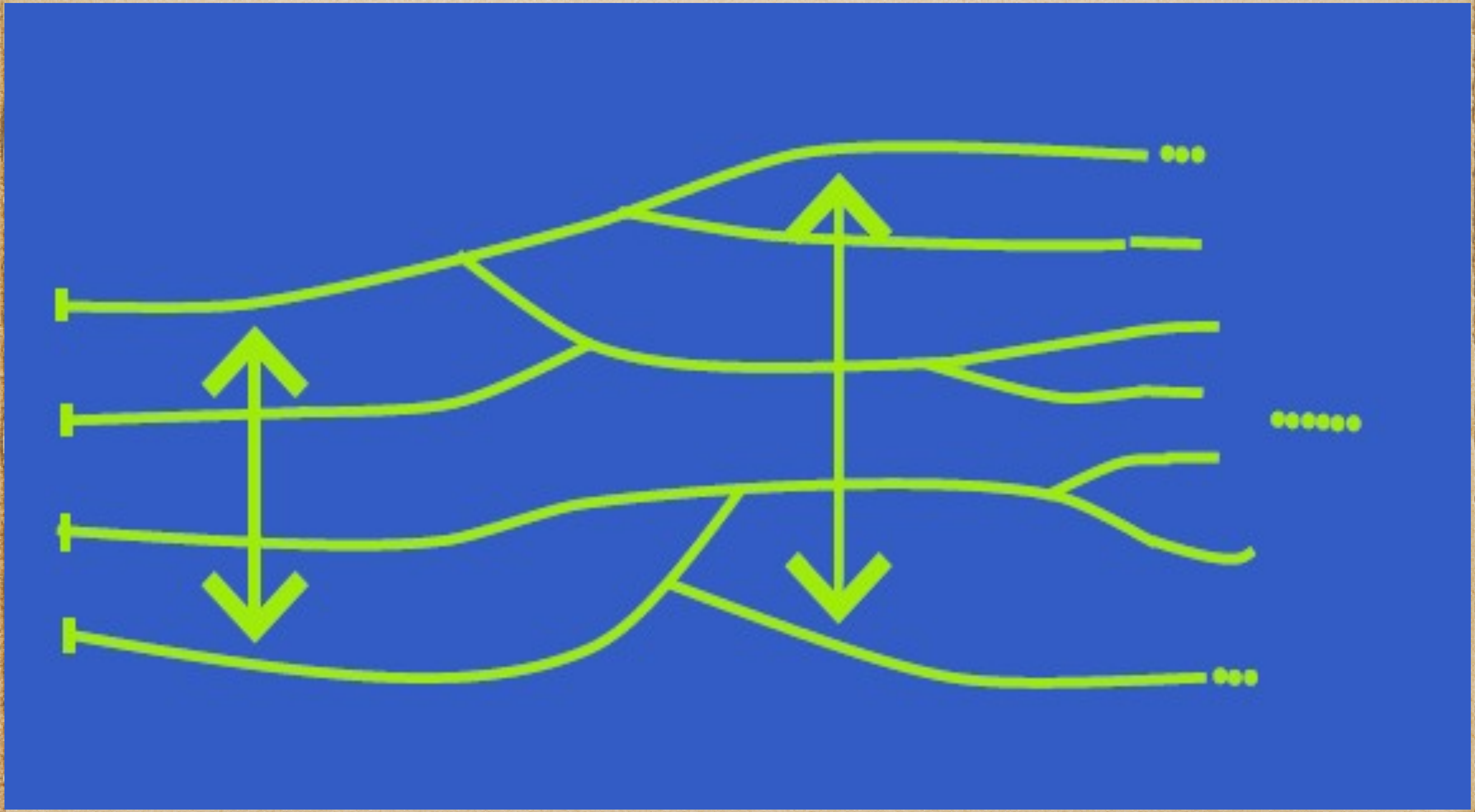
Quasi-ordering

- Equivalence (both directions)
- Strict part (only one)

Well-quasi-ordering

- Well-founded
  - no infinite strictly-descending sequences
- No infinite anti-chains

Wqo



A THEOREM ON PARTIALLY ORDERED SETS (Summary)

Michael Rabin

In the following note we give a condition for the finiteness of a partially ordered set. This theorem was established in order to prove the finiteness of certain classes of ideals.

Theorem.

Assumption: Let the partially ordered set M satisfy the following conditions:

a) The maximum condition (that is, the ascending chain condition).

b) The minimum condition (that is, the descending chain condition).

c) Every subset of M in which all pairs of elements are uncomparable, is finite.

Conclusion: M is finite.

The crucial point of the proof lies in the following general principle.

Equivalent Properties

- Wqo
- Every infinite sequence has an ordered pair

Well-Quasi-Order

Definition. *A set A is Well Quasi Ordered under $\precsim$ if for all infinite sequences from A:*

$$a_1, a_2, a_3, \ldots$$

there exists some $i < j$ such that $a_i \precsim a_j$.

Equivalent Properties

- Standard: wf and no inf antichain
- Simple: Every infinite sequence has an ordered pair
- Useful: Every infinite sequence contains an infinite non-decreasing chain
  - Why? -- Ramsey

Properties

- Every refinement (more order) is also wqo
- Every linearization (refinement s.t. all equivalence classes are comparable) is well-ordered

Dickson's Lemma

- Order (n-) tuples in product ordering
  - All components are in order
- Tuples of wqos are wqo

Good

- A pair is good if it is ordered
- A sequence is good if it has a good pair
- A set is good (wqo) if all sequences are good

Bad

- A sequence is bad if there is no good pair
- It is good if it has at least one pair

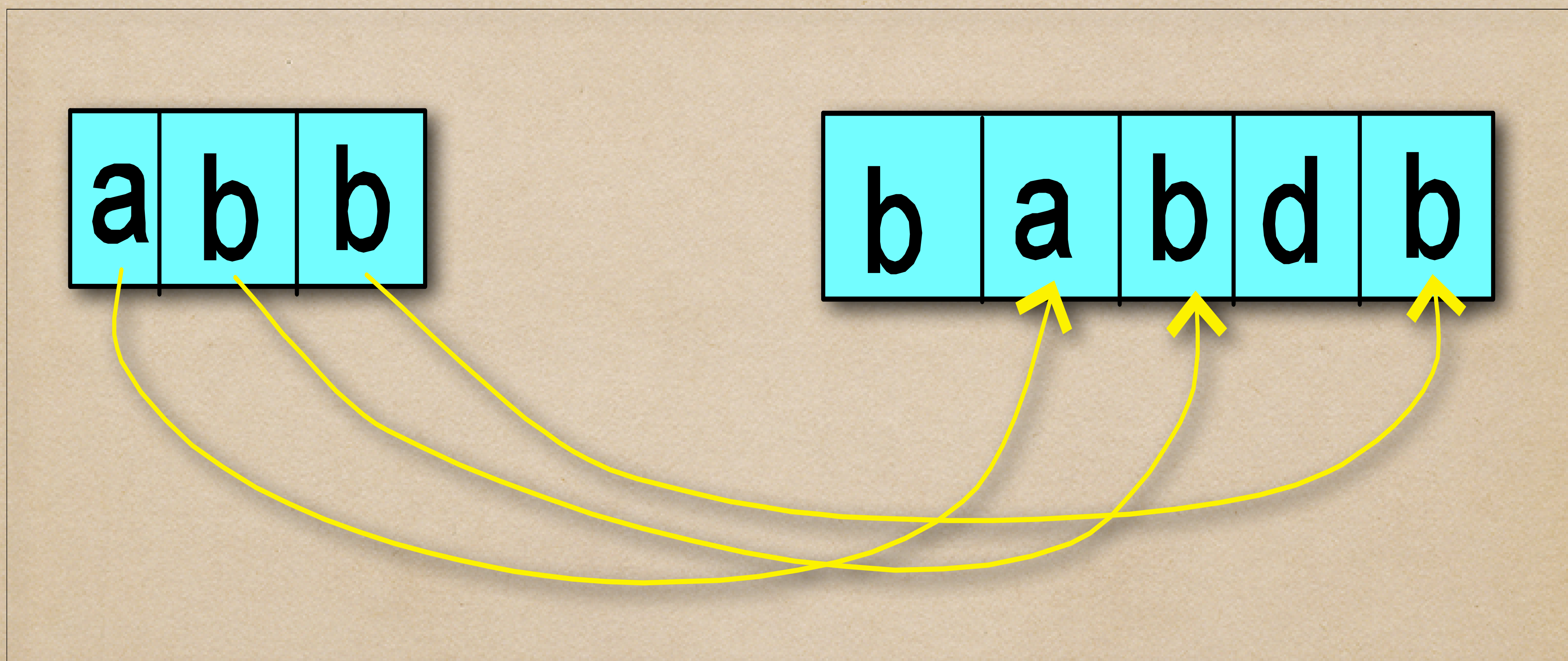
Good & Bad

- A qo is a wqo if all sequences are good
- A sequence is bad if it is not good
- If a set is not good, then there is a minimal counterexample (bad sequence)

Higman's Lemma

- Every infinite sequence of words (over a finite alphabet) includes an embedding.

Homeomorphic Embedding



Higman's Lemma

- Suppose a finite or infinite alphabet is wqo
- Extend order to string embedding
  - letters map in order to bigger or equivalent ones
- Strings are wqo

Precedence

- Example, Σ

  $a_0<a_1<a_2<\ldots$

  $b_0<b_1<b_2<\ldots$

  …

  $z_0<z_1<\ldots$

Minimal Bad Sequence

- acd eef afda …
- afda ab acd …
- ab eef afda …
- ab acd eef afda …
- ab afda acd …
- …

Minimal Bad Sequence

- acd eef afda …
- afda ab acd …
- ab eef afda …
- ab acd eef afda …
- ab afda acd …
- …

Minimal Bad Sequence

- ab eef afda …
- ab acd eef afda …
- ab afda acd …
- …

Minimal Bad Sequence

- ab eef afda …
- ab acd eef afda …
- ab afda acd …
- …

Minimal Bad Sequence

- ab acd eef afda …

Minimal Bad Sequence

- ab acd eef afda ...

Minimal Bad Sequence

- ab acd afda ...

Proof

- Consider minimal bad sequence
  - $\alpha_1 x_1 \; \alpha_2 x_2 \; \alpha_3 x_3 \; \ldots \; \alpha_i x_i \; \ldots \; \alpha_j x_j \; \ldots$
- Extract subsequence with first letters $\alpha_{i1} \; \alpha_{i2} \; \alpha_{i3} \; \ldots$ ordered
- Consider rests $x_{i1} \; x_{i2} \; x_{i3} \; \ldots$

- Tails (or substrings) of minimal bad sequence are good
  - Why?
  - Suppose bad tails $x_9 \ldots x_3\ x_{18} \ldots$
  - Consider $x_3\ x_{18} \ldots$ (where 3 min index)
  - $\alpha_1 x_1\ \alpha_2 x_2\ x_3\ x_{18} \ldots$ would be smaller than min bad

Contradiction

- ab acd   afda ... aacafad ...

Corollary: Bag Ordering

- Given wfo > on elements X, consider bag order
- Extend (by Zorn's Lemma) to total well-order >; X is wqo by ≥
- By Higman, sequences X* are wqo
- Were there an infinite descending sequence {bi} of multisets wrt >, it would be decreasing wrt >
- By Higman, there's a pair bj ≤ bk; by bag order bj > bk