Termination

Dependencies

Assumption

- Simplification orders
- Assume fixed or bounded arity
- Otherwise need another condition
  - $f(\ldots s \ldots) \gtrsim f(\ldots\ \ldots)$

Substitutions

- substitution $\{x_i \mapsto u_i\}$
- apply $t\{x_i \mapsto u_i\}$, replace each occurrence of variable x_i in t with term u_i
- compose $\{x_i \mapsto u_i\}\sigma = \{x_i \mapsto u_i\sigma\}$

Unifiers

- substitution σ unifies terms s and t if $s\sigma = t\sigma$
- substitution μ more general than σ if there's a τ (not a renaming) such that $\sigma = \mu\tau$
- if there is a unifier, then there is a unique most general one μ (unique up to renaming)

Unifiers

- x, y distinct variables
- f, g distinct symbols

- $mgu(x,x) = \varnothing$; $mgu(x,y) = \{x \mapsto y\}$
- $mgu(x,t) = \{x \mapsto t\}$, t does not contain x
- $mgu(x,t) = fail$, t contains x (but isn't x)
- $mgu(f(\underline{s}),g(\underline{t})) = fail$; $mgu(f(),f()) = \varnothing$
- $mgu(f(u,\underline{s}),f(v,\underline{t})) = \mu \cup mgu(f(\underline{s}\mu),f(\underline{t}\mu))$ where $\mu = mgu(u,v)$

Non-termination

- Can use most general unifier to look for examples of nontermination
  - Given two derivations $s \dashrightarrow t$ and $u \dashrightarrow v$
    - renamed so that the two have distinct variables
    - rules are one-step derivations
  - extend (if possible) by mgu μ of u and <u>nonvariable</u> subterm of t
    - $s\mu \dashrightarrow t\mu = r\mu[u\mu] \dashrightarrow r\mu[v\mu]$

Jumping

- Let $P = R \cup B$
- If $s \; R \cup B \; t$
  - then $s \; R \; t$
  - or $s \; B \; v_1 \; P \; v_2 \; P \; \ldots \; P \; v_n \; P \; t$
- In short $RB \subseteq R \cup BP^*$
- Hence (induction) $RB^* \subseteq R \cup BP^*$

Jumping Union

- If B jumps over R
- then union well-founded iff both are
- s_1 BB...B t_1 RB* t_2 RB* t_2 RB* ...
  - s_1 BB...B t_1 R t_2 R t_3 R ...
- s_1 BB...B t_1 RB* t_2 RB* t_2 RBBBB...
  - s_1 BB...B t_1 R R R u_k BBBB...

Escaping

- For any immortal red chain
  s_1 R s_2 R s_3 R...
- there is also an immortal purple chain after some blue turn
  s_1 R s_2 R ... R s_k B t_1 P t_2 P...

Escaping Union

- If B jumps over R
- and B escapes from R
- then union well-founded iff B is
  - s_1 BB...B t_1 <u>R</u> t_2 <u>R</u> t_3 <u>R</u> ... XXX
  - s_1 BB...B t_1 <u>R</u> <u>R</u> <u>R</u> u_k BBBB...

Top & Not

- Two parts to rewriting $\Rightarrow$
  - instance of rule $\Rightarrow_{top}$
  - within a context $\Rightarrow_{in}$

Top | Not

- Immediate subterm: f(...t...) $\rhd$ t
- If s1 $\Rightarrow$ s2 $\Rightarrow$ s3 $\Rightarrow$ ...
  - Either si $\Rightarrow_{top}$ ... sj $\Rightarrow_{top}$ ... sk $\Rightarrow_{top}$
  - Or s1 $\Rightarrow$ ... $\Rightarrow$ sk $\rhd$ t1 $\Rightarrow$ t2 $\Rightarrow$ ...

Facts

- $f(\ldots s \ldots u \ldots) \Rightarrow_{in} f(\ldots t \ldots u \ldots) \rhd t$
  - $f(\ldots s \ldots u \ldots) \rhd s \Rightarrow t$
- $f(\ldots s \ldots u \ldots) \Rightarrow_{in} f(\ldots t \ldots u \ldots) \rhd u$
  - $f(\ldots s \ldots u \ldots) \rhd u$

Dependencies

- Let ➤ be $\Rightarrow_{top} \rhd^*$
- Rule $s \Rightarrow t[u]$
  - s ➤ u
  - exclude variable u

Dependency Pairs

- R rewrite step
- T top step
- I inner step (not at top)
- D dependency pair (includes top step)
- A subterm

Dependencies

- $B = D \cup I$
- $R \subseteq B$
- $DA \subseteq D \cup A^+ \subseteq B \cup A^+$
- $IA \subseteq A \cup AR \subseteq A \cup AB$
- $BA \subseteq B \cup A^+ \cup AB$
- A jumps over B ($D \cup I$)

Dependencies

- Show $B = D \cup I$ is terminating
- $D \subseteq >$
- $I \subseteq \gtrsim$
- $>$ well-founded
- $\gtrsim > \subseteq >$ “compatible”

Proof

- Infinite D & I, with infinitely many Ds
- A escapes from I and jumps over I
- Can't have infinite tail of only I
- So show I*D terminates
- $I^*D \subseteq \gtrsim > \subseteq >$

Advantage

- Must have infinitely many D steps at top
- So enough to show other steps $\gtrsim$

Quotient

- $x - 0 \Rightarrow x$
- $sx - sy \Rightarrow x - y$
- $0 \div sy \Rightarrow 0$
- $sx \div sy \Rightarrow s(\,[x-y] \div sy\,)$

Rules

- $x - 0 \gtrsim x$
- $sx - sy \gtrsim x - y$
- $0 \div sy \gtrsim 0$
- $sx \div sy \gtrsim s(\ [x-y] \div sy\ )$

Drop Subtrahend

LPO with only first argument of $-$

- $-x \gtrsim x$
- $-sx \gtrsim -x$
- $0 \div sy \gtrsim 0$
- $sx \div sy \gtrsim s(-x \div sy)$

Pairs

- $sx - sy > x - y$
- $sx \div sy > (x-y) \div sy$
- $sx \div sy > x-y$

Pairs

- $-sx > -x$
- $sx \div sy > -x \div sy$
- $sx \div sy > -x$

Dependency Graph