- Claim:
  Suppose we have an alphabet Σ that is well ordered. Then the corresponding multiset path order is also well ordered.
- Proof:
  we know that is WQO. So we need to prove it's total. By induction:
- Let $f(s_1, \dots, s_n), g(t_1, \dots, t_m)$ be 2 trees. By induction we can assume that the subtrees s_i, t_j are totally ordered.

- If $g(t_1, \dots, t_m) \le s_i$ for some $i \Rightarrow t < s$
- If $f(s_1, \dots, s_n) \le t_j$ for some $j \Rightarrow s < t$
- Else by induction hypothesis we know that:
  - $g(t_1, \dots, t_m) > s_i$ for all i, and
  - $f(s_1, \dots, s_n) > t_j$ for all j
- So $f > g \Rightarrow s > t$ and $g > f \Rightarrow t > s$
- What about $f = g$?
  Also by induction hypothesis:
  - $\{s_1, \dots, s_n\} \le \{t_1, \dots, t_m\}$ or
    $\{t_1, \dots, t_m\} \le \{s_1, \dots, s_n\}$

- So either:
- $f(s_1, \dots, s_n) \leq g(t_1, \dots, t_m)$ or
- $g(t_1, \dots, t_m) \leq f(s_1, \dots, s_n)$

- So given a well ordered alphabet (Σ), we can ask what is the corresponding well order of the multiset path order (relevant to Σ)

The ordinal Γ_0

- Defining ordinals using fix points:
  - Suppose $C \subset \omega_1$ is close and unbound (club)
  - $C = \{\alpha_\gamma \mid \gamma < \omega_1\}$ an enumeration of C
  - $C' = \{\gamma \mid \alpha_\gamma = \gamma\}$ – all fixed points of C
  - Fact: C club $\Rightarrow$ C' is also club !!
  - $\Rightarrow \mathrm{C}'' \supseteq C''' \supseteq \ldots \supseteq C^n$ also club !!
  - At limit ordinal we intersect:
    $C^\omega = \cap_{n<\omega} C^n$ - club.

- Formally we define:
- $C^0 = C$. $C^{\alpha+1} = (C^\alpha)'$. $C^\alpha = \bigcap_{\beta<\alpha} C^\beta$
- For all $\alpha < \omega_1$ we define the function $\varphi^\alpha : \omega_1 \to \omega_1$ as the enumeration function of the club C^α.
  So $\varphi^\alpha(\beta)$ is the β element of the set C^α.

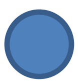
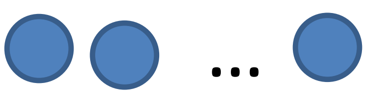
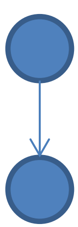
- Is there α such that $\min C^\alpha = \alpha$?!
  YES !
  Actually $\{\alpha \mid \min C^\alpha = \alpha\}$ is also a club !!

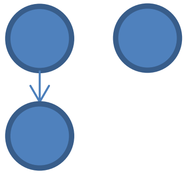
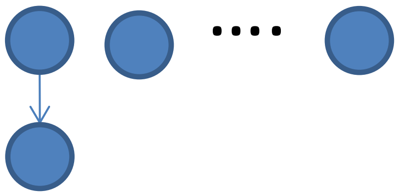
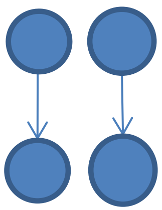
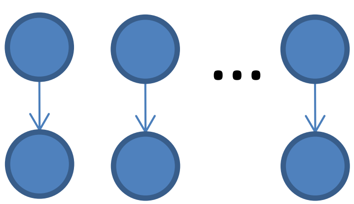
- To define Γ_0 we start with $C = C^0 = \{\omega^\alpha | \alpha < \omega_1\} = \{1, \omega, \omega^2, \ldots.\}$
- So $\varphi^0(\alpha) = \omega^\alpha$
- $C^1 = \{\alpha | \omega^\alpha = \alpha\} = \{\epsilon_0, \epsilon_1, \ldots..\}$ – the epsilon numbers
- $\varphi^1(\alpha) = \epsilon_\alpha$
- What is the first element of C^2 ?
- $\epsilon_0, \epsilon_{\epsilon_0}, \epsilon_{\epsilon_{\epsilon_0}}, \ldots = \zeta_0$

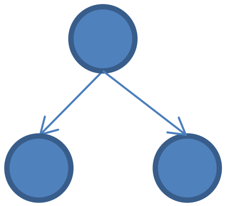
- Γ_0 is the first ordinal such that

$$\min\ C^{\Gamma_0} = \Gamma_0$$

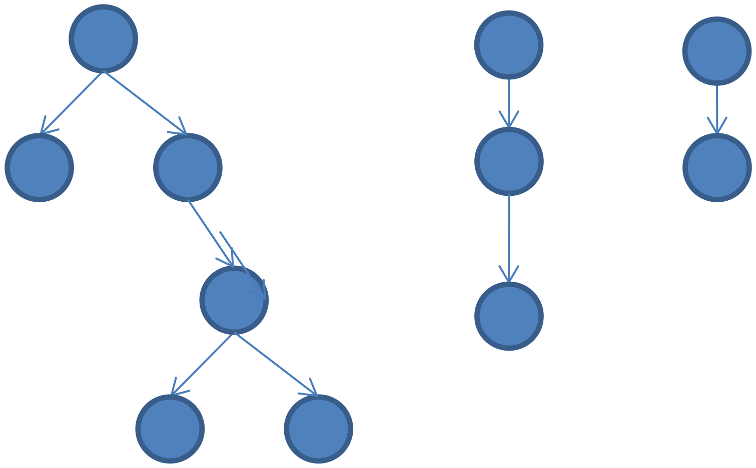
- Now we proceed to understand the connection between Γ_0 and multiset path order

- Back to multiset path order. Suppose we have an alphabet with a single element. What will be the corresponding tree order ?
- Instead of a single tree consider a (finite) multiset of trees:
- empty tree ⇒ ordinal 0
-  (singleton tree) ⇒ ordinal 1
-  (n singletons) ⇒ ordinal n
-  ⇒ the ordinal ω

-  $\Rightarrow$ ordinal $\omega + 1$
-  $\Rightarrow$ ordinal $\omega + n$
-  $\Rightarrow$ the ordinal $\omega + \omega$
-  $\Rightarrow$ the ordinal $\omega \cdot n$

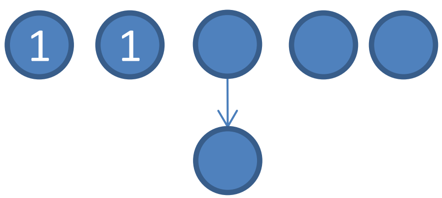
-  $\Rightarrow$ the ordinal $\omega^2 = \omega^{1+1}$
- Define function ψ: $\{MS\ trees\} \rightarrow ordinals$
- For multiset $\{t_1, \dots, t_n\}$ we define:
- $\psi\ (\{t_1, \dots, t_n)\} = \psi(t_1) + \cdots + \psi(t_n)$
- For a tree t $= f\,(t_1, \dots, t_n)$ we define:
- $\psi\ (t) = \omega^{\psi(t_1)+\psi(t_2)+\cdots+\psi(t_n)} = \omega^{\psi(\{t_1 \dots t_n\})}$
- the sum in descending order

- Example:



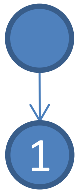
- $\omega^{\omega^{\omega^2}+1} + \omega^{\omega} + \omega^1$
- Can represent each ordinal $< \epsilon_0$ that way.

- So when the alphabet is $\{0\}$ the trees with multiset path order go up to ϵ_0.
- What happen with $\Sigma = \{0,1\}$?
- Definitely a singleton labeled with 1 is above all trees with only label $0 \Rightarrow \quad \epsilon_0$
- 

$\Rightarrow \quad \epsilon_0$
- 

$\Rightarrow \quad \epsilon_0 \cdot 2 + \omega + 2$
- 

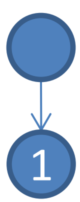
$\Rightarrow \quad \epsilon_0 \cdot n$

- 

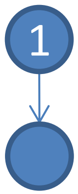
This must be $\epsilon_0 \cdot \omega$.

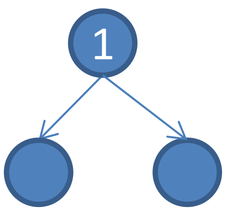
- But applying the function $\psi(\epsilon_0)$ we get :

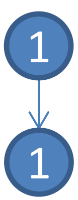
$$\psi\,(\epsilon_0) = \omega^{\epsilon_o} = \epsilon_0$$

- So we need a fix here. For $\epsilon_\alpha + n$ ordinals we define: $\psi\,(\epsilon_\alpha + n) = \omega^{\epsilon_\alpha + n + 1}$
- So: 

$\Rightarrow \omega^{\epsilon_0 + 1} = \epsilon_0 \cdot \omega$

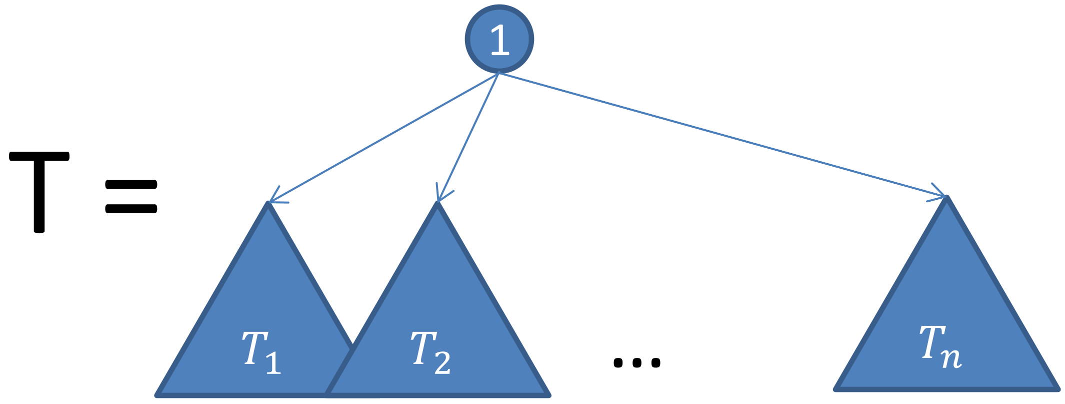
- Using all trees with 1 only as leafs we can reach up to ϵ_1
- 

$\Rightarrow \epsilon_1$
- 

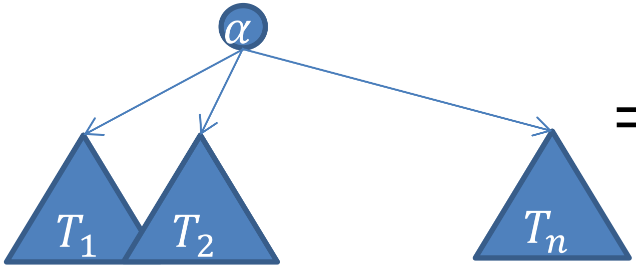
$\Rightarrow \epsilon_2$
- 

$\Rightarrow \epsilon_{\epsilon_0}$

- We can define φ for trees like:



- As $\psi\ (T) = \epsilon_{\psi(T_1)+\cdots\psi(T_n)}$

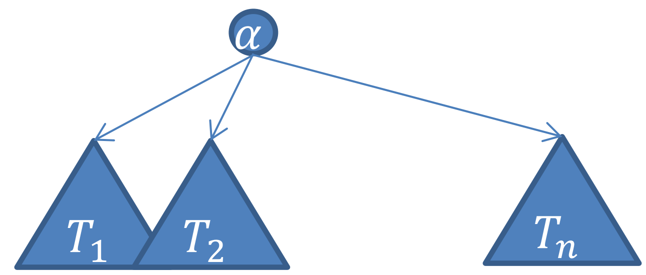
- Back to the function φ^{α} we defined before.
- For trees with root labeled 0 the function ψ is (almost) the same as φ^{0}.
- For trees with root labeled 1 the function ψ is the same as φ^{1}.
- So it's tempting to define the following mapping:
- ψ: 

$\Rightarrow \varphi^{\alpha}(\psi(T_1) + \cdots + \psi(T_n))$

- But then we have problems in all fixed points where $\varphi^{\alpha}(\beta) = \beta$. So we need a fix. Define:

- $\psi(\alpha, \beta) = \begin{cases} \varphi^{\alpha}(\beta' + n + 1) & if\ \beta = \beta' + n \quad for\ \varphi^{\alpha}(\beta') = \beta' \\ \varphi^{\alpha}(\beta) & else \end{cases}$

- Consider the tree: $\alpha(T_1, \dots, T_n)$



- Define

$$\Theta\big(\alpha(T_1, \dots, T_n)\big) = \psi(\alpha, \Theta(T_1) + \cdots + \Theta(T_n))$$

- Define also for multiset of trees:
- $\Theta(\{T_1, \dots, T_n\}) = \Theta(T_1) + \cdots + \Theta(T_2)$

- Consider the Θ as function from (MS of) labeled trees, labeled by ordinals up to Γ_0
- So: $\Theta : MS(TREE) \rightarrow O$rdinals
- The following holds:
  1. $\Theta : MS(TREE) \rightarrow \Gamma_0$
  2. Θ is one to one
  3. Θ is onto Γ_0
  4. Θ is order preserving

1. $\boldsymbol{\Theta : MS(TREE) \rightarrow \Gamma_0}$

This follows from the fact that:

For every $\alpha, \beta < \Gamma_0$

$\Rightarrow \quad \alpha + \beta < \Gamma_0$ and $\varphi^{\alpha}(\beta) < \Gamma_0$

2. **Θ is onto** $\boldsymbol{\Gamma_0}$

To prove that, we need the following theorem (recursive definition for ordinal up to Γ_0):

For every $\gamma < \Gamma_0$ either:

1. $\gamma = 0$

2. $\gamma = \alpha + \beta$ for some $\alpha, \beta < \gamma, \quad \alpha \leq \beta$

3. $\gamma = \varphi^{\alpha}(\beta)$ for some $\alpha, \beta < \gamma$

- Take $\gamma < \Gamma_0$.
- $\gamma = \alpha + \beta$ .By induction we have MS for α and MS for β. Unify them to single MS
- More interesting is $\gamma = \varphi^{\alpha}(\beta)$.

We'll see that there is a single tree T such that: $\Theta(T) = \gamma$

If we weren't have to fix ψ it would work flawlessly. But:

$$\psi(\alpha, \beta) = \begin{cases} \varphi^{\alpha}(\beta' + n + 1) & if\ \beta = \beta' + n \quad for\ \varphi^{\alpha}(\beta') = \beta' \\ \varphi^{\alpha}(\beta) & else \end{cases}$$

- Break to cases.

$$\beta = \beta' + n \text{ (possibly } n = 0)$$

1. Simple: $\varphi^{\alpha}(\beta') \neq \beta' \Rightarrow \quad \Psi(\alpha, \beta) = \gamma$
2. $\varphi^{\alpha}(\beta') = \beta'$ and $n > 0 \Rightarrow \quad \Psi(\alpha, \beta - 1) = \gamma$
3. $\varphi^{\alpha}(\beta') = \beta'$ and $n = 0$ (so $\beta = \beta'$)

Take $\delta' = \min\{\delta |\ \varphi^{\delta}(\beta) \neq \beta\}$

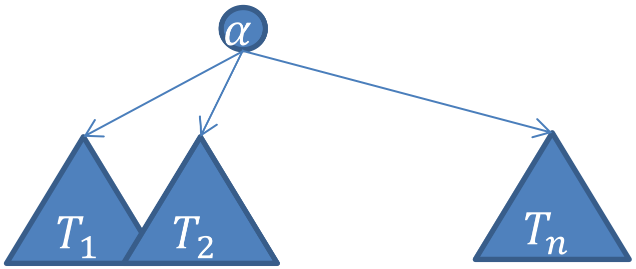
So we must have $\tau \neq \beta$ such that $\varphi^{\delta'}(\tau) = \beta$ (β is a fix point of clubs $C^{\delta}\ for\ \delta < \delta'$).

$$\Rightarrow \Psi(\delta', \tau) = \beta = \gamma$$

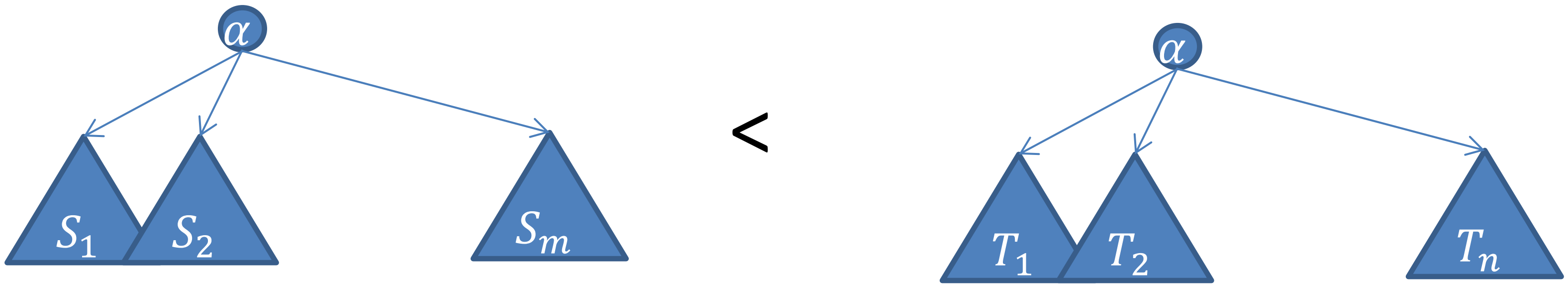
$\Rightarrow \Theta$ is onto Γ_0

- To prove the order preserving we need to use the following facts about the φ^α functions.
- $\beta' < \beta'' \Rightarrow \ \varphi^\alpha(\beta') < \varphi^\alpha(\beta'')$
- $\varphi^{\alpha'}(\beta') < \varphi^{\alpha''}(\beta'')$ iff
  1. $\alpha' = \alpha''$ and $\beta' < \beta''$
  2. $\alpha' < \alpha''$ and $\beta' < \varphi^{\alpha''}(\beta'')$
  3. $\alpha' > \alpha''$ and $\varphi^{\alpha'}(\beta') < \beta''$

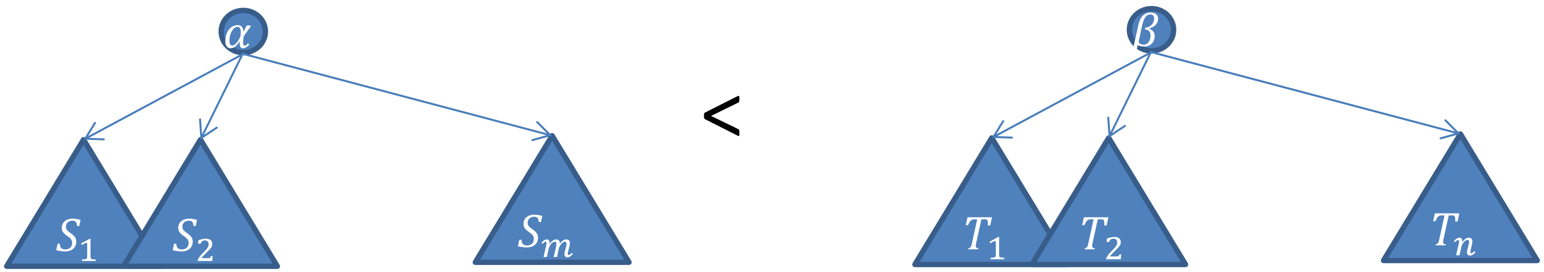
- Using them it’s easy to prove analog proposition about ψ:
  a) $\Psi(\alpha, \beta) > \beta$
  b) $\beta' < \beta'' \Rightarrow \Psi(\alpha, \beta') < \Psi(\alpha, \beta'')$
  c) $\alpha' < \alpha'',\ \beta' < \Psi(\alpha', \beta'') \Rightarrow \Psi(\alpha', \beta') < \Psi(\alpha'', \beta'')$
- (a) implies that: for each tree $T = \alpha(T_1 \ldots T_n)$



- $\Theta(T) > \Theta(T_i)$

- (b) implies that:
- 

- $\Rightarrow \quad \Theta(S) < \Theta(T)$

- (c) implies that:
- 

- $\alpha < \beta$ and $\Theta(S_i) < \Theta(T)$
- $\Rightarrow \ \Theta(S) < \Theta(T)$
- So Θ is order preserving, hence also 1:1.

Ordinal notation for $\gamma < \Gamma_0$

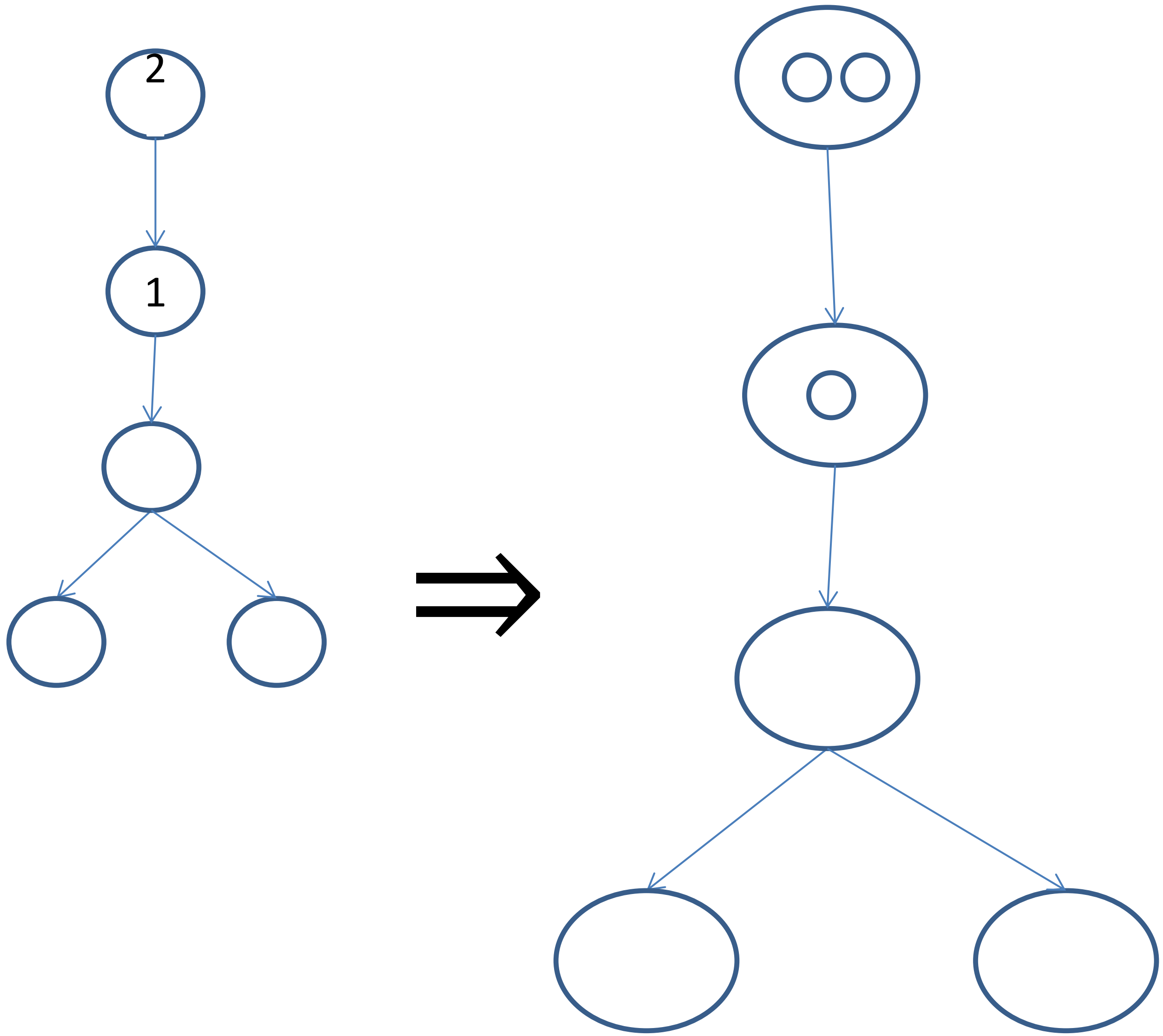
We saw the recursive definition for ordinal up to Γ_0:

For every $\gamma < \Gamma_0$ either:

1. $\gamma = 0$
2. $\gamma = \alpha + \beta$ for some $\alpha, \beta < \gamma, \quad \alpha \leq \beta$
3. $\gamma = \varphi^{\alpha}(\beta)$ for some $\alpha, \beta < \gamma$

- Since always $\alpha, \beta < \gamma$ for each γ we get a corresponding MS of trees. The labels in the nodes, are also MS of trees (recursively).

- Example:
- $\zeta_{\epsilon_{\omega^2}} \Rightarrow$
-



Lexicographic path order

- Same as we proved with multiset, if the alphabet Σ is well ordered, than also the trees compared with lexicographic order are well ordered.
- To analyze it we need to go farther after Γ_0 ordinal.

- We saw the Veblen hierarchy:
- $\varphi(\alpha, \beta) = \varphi^{\alpha}(\beta)$.
- We can try to define more fixed point by adding another argument:
- $\varphi(0, \alpha, \beta) = \varphi(\alpha, \beta)$
- $\varphi(1,0,\tau)$ is the τ-th fixed point of the functions $\xi \ \rightarrow \ \varphi(\xi, 0)$
- So $\varphi(1,0,0) = \Gamma_0 \ , \ \ \varphi(1,0,\tau) = \Gamma_\tau$
- $\varphi(1,1,\tau)$ enumerates the fixed points of $\varphi(1,0,\tau)$ (that is of $\xi \ \rightarrow \Gamma_\xi$)
- $\varphi(2,0,\tau)$ enumerates the fixed points of $\varphi(1, \tau, 0)$

- The major property we had in Veblen functions that we used to prove order preserving was:

$\alpha' < \alpha''$ and $\beta' < \varphi(\alpha'', \beta'') \quad \Rightarrow \varphi(\alpha', \beta') < \varphi(\alpha'', \beta'')$

$\alpha' = \alpha''$ and $\beta' < \beta'' \quad \Rightarrow \varphi(\alpha', \beta') < \varphi(\alpha'', \beta'')$

$\Rightarrow$ first argument is more dominant

- A similar property can be proved for Veblen function with 3 arguments:

$$\alpha' < \alpha'' \; , \; \beta' < \varphi(\alpha'',\beta'',\gamma'') \; , \gamma' < \varphi(\alpha'',\beta'',\gamma'') \Rightarrow$$
$$\varphi(\alpha',\beta',\gamma') < \varphi(\alpha'',\beta'',\gamma'')$$

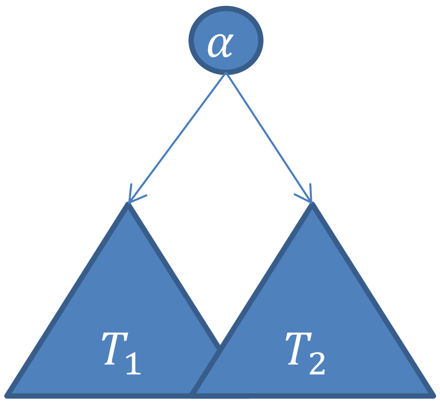
Also:

$$\alpha' = \alpha'' \; , \; \beta' < \beta'' \; , \gamma' < \varphi(\alpha'',\beta'',\gamma'') \Rightarrow$$
$$\varphi(\alpha',\beta',\gamma') < \varphi(\alpha'',\beta'',\gamma'')$$

- And

$$\alpha' = \alpha'' \; , \; \beta' = \beta'' \; , \gamma' < \varphi(\alpha'',\beta'',\gamma'') \Rightarrow$$
$$\varphi(\alpha',\beta',\gamma') < \varphi(\alpha'',\beta'',\gamma'')$$

- The last property fits exactly to the lexicographic order for trees, where each node has (at most) 2 child's



$$\Rightarrow \qquad \varphi(\alpha, T_1, T_2)$$

(need some fixing for fix point cases, as we did with multiset)

- We can go on and on and define recursively veblen functions for n arguments.
- Such a φ with $n+1$ arguments bring us to an ordinal large enough to contain the trees with node up to n Childs, while the node themself can be labeled (recursively) with such trees.