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A GEOMETRICAL APPROACH TO MULTISET ORDERINGS

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Abstract. We survey different ways of ordering multisets, and give a classification of multiset orderings based on the notion of a cone in $\mathbf{R}^n$. This enables us to derive new results about the dominance ordering and the standard multiset ordering.

1. Introduction

A multiset on a set S is an unordered sequence of elements of S. This paper is concerned with the construction and classification of well-founded partial orderings of $\mathcal{M}(S)$, the set of finite multisets on S. Many such orderings have appeared in the computer science and mathematical literature, for example in proofs of program termination [9, 3, 23], in equational reasoning algorithms based on term rewriting systems [8, 17, 15, 21], in computer algebra [5, 2] and in papers on invariant theory [6, 20, 12], ring theory [1] and the theory of partitions [11, 4, 16]. We describe these orderings in detail in Section 2.

Our classification is based on the notion of a "tame" ordering, which, roughly speaking, is an ordering with those properties that arise naturally in termination proofs, that is, an ordering on $\mathcal{M}(S)$ preserving both multiset union and an ordering on S. We define tame orderings precisely in Section 3, and prove that they are well-founded (Lemma 3.1). It turns out that all but a few of the orderings mentioned in Section 2 are tame.

The main purpose of the paper is to show that tame orderings can be classified geometrically, and that this geometric approach can be used to give a unified picture of different multiset orderings and to derive new results. For simplicity we restrict throughout to the case where S is finite, although a similar classification is possible for infinite S but would require the use of concepts drawn from functional analysis. We show in Section 3 (Lemma 3.2, Theorem 3.5) that any tame ordering on $\mathcal{M}(S)$, where $|S| = n$, arises from a particular subset of real n-dimensional space $\mathbf{R}^n$ called

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a positive cone, and conversely any such positive cone gives rise to a tame ordering on $\mathcal{M}(S)$. We discuss in particular positive cones that are described by matrices. As an example we classify in Section 4 all tame orderings on $\mathcal{M}(S)$ in the case when $|S|=2$.

The rest of the paper is concerned with demonstrating the power of this geometric approach. In Section 4 we describe two interesting new multiset orderings. The first is obtained by using a partial order on the underlying set S to generate a matrix and hence a cone. The resulting multiset ordering is a new example of a multiset ordering which inherits the orderings on the underlying set. The second is obtained by transforming an existing ordering by the action of an invertible matrix (Theorem 4.1).

The dominance ordering and the standard multiset ordering are tame orderings which have both received particular attention in the literature, and in Sections 5 and 6 respectively we give geometrical descriptions of these two orderings which enable us to characterise them as tame orderings. Our geometric approach makes it very easy to generalise the dominance ordering to the case when the underlying set S is partially ordered. We show that the dominance ordering is the unique weakest possible tame ordering on $\mathcal{M}(S)$ (Theorem 5.1), a generalisation of a result proved by Gerstenhaber [12]. We show that the strongest possible tame ordering on $\mathcal{M}(S)$ is either the standard multiset ordering, or an ordering got from it by the action of an invertible matrix (Theorem 6.2).

2. Background

We shall use $>$ to denote a partial order on a set T, that is, a relation on T which is irreflexive and transitive. If any two distinct elements of T are comparable, we call $>$ a *total order.* The notation $a \geqslant b$ means that either $a > b$ or $a = b$. If there is no infinite sequence $(s_1, s_2, \ldots)$ with $s_i > s_{i+1}$ for all i we call $>$ *well-founded.*

Let S be a set and let A be an element of $\mathcal{M}(S)$, that is, an unordered finite sequence of elements of S. We shall regard A as a function from S to N, the natural numbers (including 0), so that $A(x)$ represents the number of times the element x of S occurs in the unordered sequence A. We may thus regard A as a vector whose entry in the coordinate corresponding to x is $A_x = A(x)$. For example, if $A = \{a, a, a, b\}$ then $A(a) = A_a = 3$, $A(b) = 1$ and A is represented by the vector $(3, 1)$. Notice that the multiset consisting of the single element x corresponds to the coordinate vector E_x which has a 1 in the entry corresponding to x and 0s elsewhere. An ordering $\gg$ on $\mathcal{M}(S)$ is said to inherit an ordering $>$ on S if $E_x \gg E_y$ whenever $x > y$.

Dershowitz and Manna [9] seem to have been the first to use multiset orderings to prove program termination. They introduced an ordering which we shall call the standard multiset ordering, defined as follows. Let $>$ be any partial order on S, and A and B multisets. Then $A \gg B$ if and only if B can be obtained from A by removing one or more elements and replacing them with a set of elements each of which is strictly smaller than one of the ones which was removed. We describe this ordering in more detail in Section 6. It is easy to see that $\gg$ inherits $>$. If we take S to be finite and $>$ to be total, and regard elements of $\mathcal{M}(S)$ as elements of $\mathbb{N}^{|S|}$, this is just the usual lexicographic or "dictionary" ordering. Related orderings were constructed by Jouannaud and Lescanne in [14]. In [18] the present author showed how to generalise the standard multiset ordering using a matrix action. The standard multiset ordering has also been used in termination proofs in [3, 23].

Multiset orderings have been used by several authors to construct orderings on term algebras, leading to proofs of the termination of term-rewriting systems. A survey of recent developments in this area is given in [8].

For example, if we replace a term in the term algebra $T(\Sigma)$ by the multiset consisting of the function symbols that appear in it, we may order terms using an ordering on $M(\Sigma)$. In [15] Knuth and Bendix did this by assigning an integer weight $w(f)$ to each function symbol f and then to the corresponding multiset by adding up the weights of the function symbols appearing in it. Thus $A \gg B$ if and only if $\sum_f w(f)A(f) > \sum_f w(f)B(f)$. This technique was revived in [17], where a technique for automatically choosing the weights, and hence the appropriate multiset ordering, was described. Madlener and Otto [19] have obtained results characterising semi-Thue systems which can be proved terminating with an ordering of this kind.

We may also get a multiset from a term $f(s_1, \ldots, s_k)$ by considering its maximal subterms $s_1, \ldots, s_k$, and in [7] Dershowitz described the recursive path ordering, which at one stage compares two terms by comparing these multisets using the standard multiset ordering on the infinite set $M(T(\Sigma))$. In [21] Okada points out that a weaker ordering than this, in fact the ordering usually called the dominance ordering (see below), can also be used.

Orderings on multisets are also used in computing both Gröbner bases and straightening laws in polynomial rings, since any monomial $x_1^{a_1} \ldots . x_n^{a_n}$ can be regarded as a multiset over the finite set $S = \{x_1, \ldots, x_n\}$. A survey of work on Gröbner bases appears in [5]. The effect of using different orderings is investigated in [2], where it is shown that the reverse lexicographic ordering on $\mathbb{N}^n$, defined by $A \gg B$ if and only if the last non-zero element of the vector $A - B$ is negative, is in some sense the most efficient ordering to use in implementations. Several other orders of a lexicographic nature were defined by Baclawski in [1], who studied the concept of a straightening law in commutative algebras, that is, a method of expressing elements of the algebra in terms of other elements which precede it in some ordering.

A partition of the positive integer n is an unordered sequence of positive integers whose sum is n, so that $\{4, 2, 2, 1\}$ is a partition of 9. Thus a partition gives rise to a multiset, and in this guise ordering on multisets have been studied for many years—references may be found in Macdonald's book [16]. Since the underlying set S is just $\mathbb{N}$ in this case, a partition can be regarded as an integer vector directly, as well as by counting multiplicities. This gives rise to a certain duality which we shall not attempt to consider in the general case. Interesting examples of orderings on partitions which are not tame can be found in articles by Martin Gardner [11] and Brandt [4].

Another ordering which has been frequently used is the dominance ordering. In Macdonald's book it is used to study symmetric functions, and DeConcini, Eisenbud and Procesi [6] extended it to an ordering on tableaux, which are combinatorial objects related to partitions, which they then used to prove a straightening law. It also appears in [12].

However, this ordering goes back at least as far as 1903, when Muirhead used it in the following result, which seems worth stating in full as an unusual application of multiset orderings. A proof is given in [13].

Theorem 2.1. *Let $F = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ where the a_i are nonnegative real numbers and let $[[a]] = [[(a_1, \dots, a_n)]]$ denote the average of the $n!$ terms obtained from F by the possible permutations of the x_i. Define an ordering $\gg$ on $\mathbf{R}^n$ by $a \gg b$ if and only if $a \neq b$ and*

$$a_1 + \cdots + a_n = b_1 + \cdots + b_n,$$

$$a_1 + \cdots + a_i \geqslant b_1 + \cdots + b_i, \quad 1 \leqslant i \leqslant n-1.$$

Then $[[a]] \geqslant [[b]]$ for all positive real values of the x_i if and only if $a = b$ or $a \gg b$.

For example,

$$[[(1, 0, 0, 0, \dots, 0)]] = \frac{1}{n}(x_1 + \cdots + x_n),$$

the arithmetic mean of $x_1, \dots, x_n$, and

$$\left[\left[\left(\frac{1}{n}, \dots, \frac{1}{n}\right)\right]\right] = x_1^{1/n} \dots x_n^{1/n},$$

the geometric mean of the x_i. Since, for $n \geqslant 2$, $(1, 0, 0, 0, \dots, 0) \gg (1/n, \dots, 1/n)$, we deduce the well-known result that the arithmetic mean is always larger than the geometric mean.

3. Cones and orderings

We assume from now on that S is finite and $|S| = n$, so that we may regard $\mathcal{M}(S)$ as N^n. In this section we first of all define tame orderings on N^n and $\mathbf{R}^n$ and show that any tame ordering on N^n is well founded (Lemma 3.1). We also show that any tame ordering on N^n gives rise to one on $\mathbf{R}^n$ and vice versa (Lemma 3.2).

We then define positive cones in $\mathbf{R}^n$, and show how a positive cone always gives rise to an ordering (Lemma 3.4). We pay particular attention to pointed cones, which consist of the solutions to some set of linear inequalities. Finally, we establish the bijection between tame orderings on $\mathcal{M}(S)$ and positive cones in $\mathbf{R}^n$ (Theorem 3.5).

3.1. Properties of tame orderings

In this section we define tame orderings and prove that they are well-founded on $\mathbf{N}^n$, and that any tame ordering on $\mathbf{R}^n$ induces a tame ordering on $\mathbf{N}^n$ and vice versa.

First of all we want orderings which in some way reflect an underlying ordering on S. If $>$ is an order on S then the order $\gg$ on $\mathbf{N}^n$ or $\mathbf{R}^n$ is said to *inherit* $>$ if $E_x \gg E_y$ whenever x and y are elements of S with $x > y$. If the ordering $>$ is total we shall often assume that the coordinates of M are arranged in descending order with the first coordinate greater than the second and so on; in this case we call the coordinate system *canonical.*

We shall also want orderings which are preserved under multiset union, in other words if A, B and C are multisets of S with $A > B$ we want $A \cup C > B \cup C$. In our vector notation union is just vector addition, so we shall call an ordering $>$ on $\mathbf{N}^n$ or $\mathbf{R}^n$ *additive* if whenever M, N and P are elements of $\mathbf{N}^n$ or $\mathbf{R}^n$ we have $M > N$, if and only if $M + P > N + P$.

We want the empty multiset, which corresponds to the zero vector $\mathbf{0}$, to be smaller than all other multisets, so we call an ordering on $\mathbf{N}^n$ or $\mathbf{R}^n$ *bounded* if $E_x > \mathbf{0}$ for all $x \in S$.

An ordering on a set Q is called *total* if any two elements of Q are related in it. Notice that this means that the order induces a total order on any subset of Q. Thus any total ordering on $\mathbf{N}^n$ or $\mathbf{R}^n$ inherits some total order on S.

An order on $\mathbf{N}^n$ or $\mathbf{R}^n$ will be called *scalar* if $\lambda M > \lambda N$ whenever $M > N$ and λ is a positive natural or real number respectively. Of course, scalarity for $\mathbf{N}^n$ follows from additiveness. Finally, an order on $\mathbf{N}^n$ will be called *rational* if $M > N$ whenever $\lambda M > \lambda N$ where λ is a positive natural number. (The corresponding notion for orderings on $\mathbf{R}^n$ is equivalent to scalarity.)

Combining these properties, an ordering $\mathbf{N}^n$ or $\mathbf{R}^n$ will be called *tame* if it is additive, bounded, scalar and rational.

We show first that any tame ordering on $\mathcal{M}(S)$ is well-founded.

Lemma 3.1. *Let S be a finite set, and let $\gg$ be a tame ordering on $\mathcal{M}(S)$. Then $\gg$ is well-founded.*

Proof. We identify $\mathcal{M}(S)$ with $\mathbf{N}^n$. Observe first that any infinite sequence $n_1, n_2, \ldots$ of elements of $\mathbf{N}$ contains an infinite nondecreasing subsequence, $n_{i_1}, n_{i_2}, \ldots$; we choose n_{i_1} to be the least element of the n_i, and for each $k > 1$, $n_{i_{k+1}}$ to be the least element of $n_{i_k+1}, n_{i_k+2}, \ldots$.

Now suppose that $\gg$ is not well-founded, so that there is an infinite sequence of elements $s_1, s_2, \ldots$ of $\mathbf{N}^n$ with $s_{i+1} \ll s_i$ for each i. We may choose an infinite subsequence $s_{i_1}, s_{i_2}, \ldots$ with $i_1 < i_2 < \cdots$ so that the first coordinates form a non-decreasing sequence of natural numbers. From this subsequence we may choose a subsequence such that the second coordinates also form a nondecreasing sequence of natural numbers. Repeating this for each coordinate in turn we obtain a subsequence $t_1, t_2, \ldots$ of $s_1, s_2, \ldots$ such that for each $i \geqslant 1$ each entry of t_i is not greater than the corresponding entry of t_{i+1}, that is, the vector $t_{i+1} - t_i$ has no negative entries. Now $t_{i+1} \not\gg t_i$ by hypothesis, and so $t_{i+1} \neq t_i$. Since $\gg$ is bounded, it follows that $(t_{i+1} - t_i) \gg \mathbf{0}$, and hence since $\gg$ is additive, that $(t_{i+1} - t_i) + t_i \gg t_i$, that is, $t_{i+1} \gg t_i$ which is a contradiction. Thus $\gg$ is well-founded. $\square$

(In fact it is easy to see, essentially by proving a multiset version of Higman's lemma [24], that the result is true for infinite S if we assume that $>$ is well-founded and $\gg$ inherits $>$.)

Observe that if $>$ is a tame ordering on $\mathbf{R}^n$ then its restriction to $\mathbf{N}^n$ is a tame ordering on $\mathbf{N}^n$. Conversely, any tame ordering on $\mathbf{N}^n$ can be extended linearly to a tame ordering on $\mathbf{R}^n$ in the following way (essentially just "tensoring up" to $\mathbf{R}$).

Lemma 3.2. *Let $>$ be a tame ordering on N^n. Define a relation $>_*$ on $\mathbf{R}^n$ by $r >_* s$ if and only if $r \neq s$ and there exists a $k \geqslant 1$, real positive scalars $\lambda_1, \ldots, \lambda_k$ and vectors $n_1, \ldots, n_k$, $m_1, \ldots, m_k$ in $\mathbf{N}^n$ with $n_i > m_i$ for each i such that*

$$r - s = \lambda_1(n_1 - m_1) + \cdots + \lambda_k(n_k - m_k).$$

Then

(i) *$>_*$ is a tame ordering on $\mathbf{R}^n$, and its restriction to $\mathbf{N}^n$ is just $>$;*

(ii) *$>_*$ is a the weakest tame ordering on $\mathbf{R}^n$ whose restiriction to $\mathbf{N}^n$ is $>$, in the sense that if $>_{**}$ is another ordering with this property and $x >_* y$, then $x >_{**} y$.*

Proof. (i): The only delicate point in proving that $>_*$ is a tame ordering lies in showing that it is irreflexive. So suppose that for some r we have $r >_* r$, that is, that for some choice of λ_i, n_i and m_i we have

$$\mathbf{0} = \lambda_1(n_1 - m_1) + \cdots + \lambda_k(n_k - m_k).$$

This means that the integer vectors $n_1 - m_1, \ldots, n_k - m_k$ are linearly dependent over the reals, and hence over the rationals and hence over the integers. (To see this consider the integer matrix whose rows are the k vectors. The row rank of this matrix is the row rank of its row echelon form, which is obtained by performing rational row operations. Thus the rank is the same over the reals and over the rationals. Thus if the vectors are linearly dependent over the reals they are linearly dependent over the rationals.)

Thus there are positive integers $c_1, \ldots, c_k$ with

$$\mathbf{0} = c_1(n_1 - m_1) + \cdots + c_k(n_k - m_k).$$

Let $N_i = c_i n_i$, $M_i = c_i m_i$, so that

$$\mathbf{0} = (N_1 - M_1) + \cdots + (N_k - M_k)$$

and hence

$$N_1+\cdots+N_k=M_1+\cdots+M_k.$$

But since $n_i>m_i$ and $>$ is scalar we have $N_i>M_i$ for each i, so that, since $>$ is additive,

$$N_1+\cdots+N_k>M_1+\cdots+M_k,$$

which is impossible as $>$ is irreflexive. Thus $>_*$ is irreflexive, and hence is a tame ordering on $\mathbf{R}^n$.

To show that the restriction of $>_*$ to $\mathbf{N}^n$ is $>$, suppose that X and Y are in $\mathbf{N}^n$ and $X>_* Y$. Then for some choice of strictly positive real λ_i, n_i and m_i we have

$$X-Y=\lambda_1(n_1-m_1)+\cdots+\lambda_k(n_k-m_k).$$

It suffices to show that there are positive rational μ_i with

$$X-Y=\mu_1(n_1-m_1)+\cdots+\mu_k(n_k-m_k).$$

Let $x_i=n_i-m_i$, for $1\leq i\leq k$. Let A be the matrix with columns $x_1,\ldots,x_k$. Then the equation $Ay=X-Y$ has a solution $z=(\lambda_1,\ldots,\lambda_k)^{\mathrm{T}}$ over $\mathbf{R}$. Suppose E is a rational matrix such that EA is in row echelon form. Then $EAz=E(X-Y)$ and by analysing this equation we may read off a rational vector q with $EAq=E(X-Y)$. Then the set of real solutions to $Ay=X-Y$ is of the form $H+q$, where H is a subspace of $\mathbf{R}^n$ of rank the null rank of X, and the set of rational solutions is of the form $H'+q$, where H' is a subspace of $\mathbf{Q}^n$ of rank the null rank of X, and H' is dense in H. It follows that $H'+q$ is dense in $H+q$. Now $(\mathbf{R}_{>0})^n$ is open, and $z\in(\mathbf{R}_{>0})^n\cap H+q$, so $(\mathbf{R}_{>0})^n\cap H+q$ contains a positive rational vector $(\mu_1,\ldots,\mu_k)^{\mathrm{T}}$ as required. □

Thus to study tame orderings of $\mathbf{N}^n$, which is what we are interested in when studying multisets, it is sufficient to study tame orderings on $\mathbf{R}^n$.

3.2. Cone

To study orderings in $\mathbf{R}^n$ we introduce the notion of a *cone*. (See [10] for a thorough account.) A subset of $\mathbf{R}^n$ is called a *cone* if it is closed under positive linear combinations, that is, if $x, y\in C$ then $x+y\in C$ and if $x\in C$ and $\lambda\geq 0$ then $\lambda x\in C$.

For example the set of nonnegative vectors in $\mathbf{R}^n$ is a cone, called the nonnegative orthant, and so is any subspace of $\mathbf{R}^n$. If A is an n by r matrix and $x=(x_1,\ldots,x_n)$ then it is easy to see that the solutions x to $xA\geq 0$ form a cone; for example, this cone is the nonnegative orthant when A is the identity. The solutions to a nonlinear system of inequalities can also form a cone; for example the set of triples (x, y, z) with

$$z^2\geq x^2+y^2, \qquad z\geq 0$$

is a cone in $\mathbf{R}^3$.

A vector v in a cone C is called an *extreme vector* if v cannot be written in the form $v=v_1+v_2$ where v_1 and v_2 are linearly independent vectors in C, and the set of all positive multiples of an extreme vector is called an extreme halfline. Thus each coordinate vector E_x is an extreme vector of the nonnegative orthant, but any subspace of dimension 2 or more contains no extreme vectors. A cone is called *pointed* if it has finitely many extreme halflines and is the sum of them. The following theorem [10, 2.15 and 2.16] shows how pointed cones arise in a natural way from linear inequalities.

Theorem 3.3. *Let A be an n by r matrix and $x = (x_1, \ldots, x_n)$. Let C be the cone of solutions to $xA \geq 0$. Then*

(i) *The solution b of the inequality $xA \geq 0$ is an extreme vector if and only if the set of columns A_j of A for which $bA_j = 0$ has rank $n-1$.*

(ii) *If A has rank n then C is pointed, and if further $r = n$, so that A is invertible, the extreme vectors of A are just the positive multiples of the rows of A^{-1}.*

Geometrically what is happening is this. Each column of A gives a linear inequality and the solutions to this inequality all lie on one side of a hyperplane. Taking the columns together we obtain a family of hyperplanes which together bound a cone.

A cone C is called a *positive cone* if it contains each coordinate vector E_x and never contains both a non-zero vector a and its negative $-a$, that is for each x, $E_x \in C$ and if $a \in C$ and $a \neq \mathbf{0}$ then $-a \notin C$. Thus any pointed cone containing the nonnegative orthant is a positive cone, but a subspace is never a positive cone. The cone arising from a matrix A is positive if A is of rank n and has no negative entries; we call such a matrix a *positive matrix.*

Cones give rise to orderings as follows.

Definition. (i) If C is a positive cone in $\mathbf{R}^n$ define the relation $>_C$ on $\mathbf{R}^n$ by $a >_C b$ if and only if $a \neq b$ and $a - b \in C$.

(ii) If A is an n by r positive matrix of rank n define the relation $>_A$ by $a >_A b$ if and only if $a \neq b$ and $aA_i - bA_i \geq 0$ for each $1 \leq i \leq n$.

The lemma is an easy exercise.

Lemma 3.4. (i) *If C is a positive cone in $\mathbf{R}^n$ then the relation $>_C$ is an ordering on $\mathbf{R}^n$.*

(ii) *If A is an n by r positive matrix of rank n then the relation $>_A$ is an ordering on $\mathbf{R}^n$.*

If the ordering $>$ arises from a cone C (as in part (i)) then C is called the fundamental region of $>$, so that the fundamental region of the ordering in part (ii) is the pointed cone of solutions to $xA \geq 0$.

For example (see Fig. 1), in $\mathbf{R}^2$ let

$$C = \{(x, y) \mid x > 0 \text{ or } x = 0 \text{ and } y \geq 0\}$$

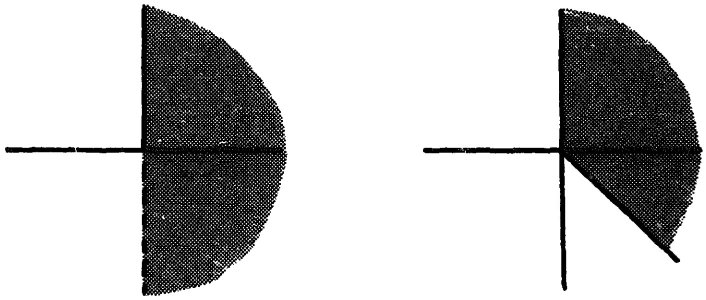

Fig. 1. The cones C and D.

and let

$$D=\{(x,y)\mid x\geqslant 0 \text{ and } x+y\geqslant 0\}.$$

Now $(x_1,y_1)>_C(x_2,y_2)$ if and only if $x_1>x_2$ or $x_1=x_2$ and $y_1>y_2$, so that $>_C$ is the lexicographic ordering. On the other hand $(x_1,y_1)\geqslant_D(x_2,y_2)$ if and only if $x_1\geqslant x_2$ and $x_1+y_1\geqslant x_2+y_2$, so that $>_D$ is $>_A$, where

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

The cone D is pointed, with extreme vectors E_x-E_y and E_y, which are the rows of

$$A^{-1}=\begin{pmatrix}1&-1\\0&1\end{pmatrix}.$$

3.3. Positive cones and tame orderings

Tameness not only captures nice properties of multiset orderings, but also characterises those orderings that arise from cones.

Theorem 3.5. *Let C be a positive cone in $\mathbf{R}^n$.*

(i) *The ordering $>_C$ is a tame ordering on $\mathbf{R}^n$. Conversely, if $>$ is a tame ordering on $\mathbf{R}^n$ then $>\,=\,>_D$ where $D=\{a\in\mathbf{R}^n\mid a\geqslant 0\}$ is a positive cone in $\mathbf{R}^n$.*

(ii) *The ordering $>_C$ inherits the ordering $>$ on S if and only if $(E_u-E_v)\in C$ whenever u and v are elements of S with $u>v$.*

(iii) *The ordering $>_C$ is a total order on $\mathbf{R}^n$ if and only if $\mathbf{R}^n=C\cup(-C)$.*

(iv) *The ordering $>_C$ is a weaker ordering than $>_D$ if and only if C is a subset of D.*

(v) *If C and D are positive cones then $E=C\cap D$ is a positive cone and the ordering $>_E$ is the intersection of the two orderings $>_C$ and $>_D$.*

(vi) *If A is a positive n by r matrix and C is the set of solutions to $xA\geqslant 0$ then $>_C$ inherits the ordering $>$ on S if and only if $a_{it}\geqslant a_{jt}$ for $1\leqslant t\leqslant r$ whenever $i>j$.*

Proof. Each part of the theorem follows immediately from the definitions. □

The orders $>_C$ and $>_D$ defined above on $\mathbf{R}^2$ illustrate this theorem. Part (ii) confirms that they both inherit the canonical order on the coordinates, as C and D both contain $(1, -1)$. As D arises from the matrix A given above, it also follows from part (vi) that it inherits this ordering. Since $C \cup (-C) = \mathbf{R}^2$, part (iii) shows that $>_C$ is total. Since D is a subset of C, part (iv) shows that $>_D$ is a weaker ordering than $>_C$.

For another example let $\boldsymbol{u}$ be a vector of positive real numbers. Let $C = \{\boldsymbol{x} \mid \boldsymbol{u} \cdot \boldsymbol{x} > 0\}$, where $\boldsymbol{u} \cdot \boldsymbol{v}$ denotes the scalar product of $\boldsymbol{u}$ and $\boldsymbol{v}$. Then C is a cone, but not a pointed cone. Viewed geometrically it is the interior of the halfspace with boundary the hyperplane $\boldsymbol{u} \cdot \boldsymbol{x} = 0$. The ordering $>_C$ is a tame ordering on $\mathbf{R}^n$, and its restriction to $\mathbf{N}^n$ is a tame ordering on $\mathbf{N}^n$. If the hyperplane $\boldsymbol{u} \cdot \boldsymbol{x} = 0$ contains no rational points then the ordering induced on $\mathbf{N}^n$ is total. The ordering of Knuth and Bendix [15] described in Section 2 is of this kind.

4. Examples

In this section we give three examples of the use of our techniques; the classification of all tame orderings in $\mathbf{R}^2$, the construction of a new function which assigns to any ordering on S an ordering on $\mathbf{R}^n$ which inherits it and the construction of new orders from old ones using an invertible matrix.

4.1. *Tame orderings on* $\mathbf{R}^2$

As an example of the power of this theorem, we can now describe all tame orderings on $\mathbf{R}^2$. First of all we can describe all total tame orderings. These correspond to cones C with $C \cup -C = \mathbf{R}^2$. It is not hard to see (Fig. 2) that C is one of

$$C_\alpha^+ = \{(x, y) \mid y \cos\alpha + x \sin\alpha > 0\}$$
$$\cup \{(x, y) \mid y \cos\alpha + x \sin\alpha = 0,\ x \cos\alpha \geqslant y \sin\alpha\}$$

where $0 \leqslant \alpha < \frac{1}{2}\pi$, or

$$C_\beta^- = \{(x, y) \mid y \cos\beta + x \sin\beta > 0\}$$
$$\cup \{(x, y) \mid y \cos\beta + x \sin\beta = 0,\ x \cos\beta \leqslant y \sin\beta\}$$

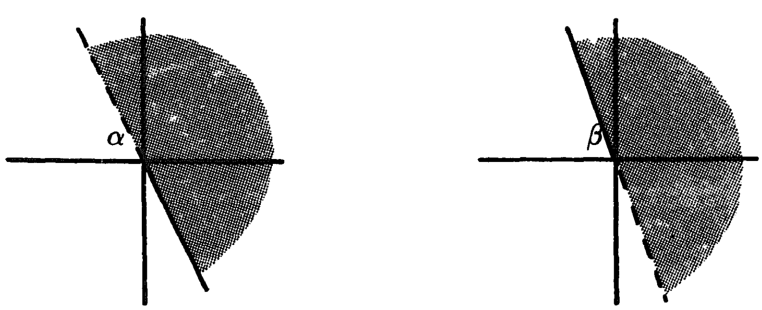

Fig. 2. The cones C_α^+ and C_β^-.

where $0<\beta\leqslant\frac{1}{2}\pi$. Thus C_0^+ gives rise to the lexicographic ordering with the y-coordinate greater than the x-coordinate, and $C_{\pi/2}^-$ to the lexicographic ordering with the coordinates ordered the other way round (the ordering C described above).

Now it is easy to see that any positive cone in $\mathbf{R}^2$ is the intersection of two of the cones above (Fig. 3), and so any tame partial ordering on $\mathbf{R}^2$ is the intersection of two distinct tame total orderings, and conversely by the lemma any such intersection is a tame ordering. (This is not true for higher dimensions!) As an example, the ordering D defined above is just $C_{\pi/2}^-\cap C_{\pi/4}^+$.

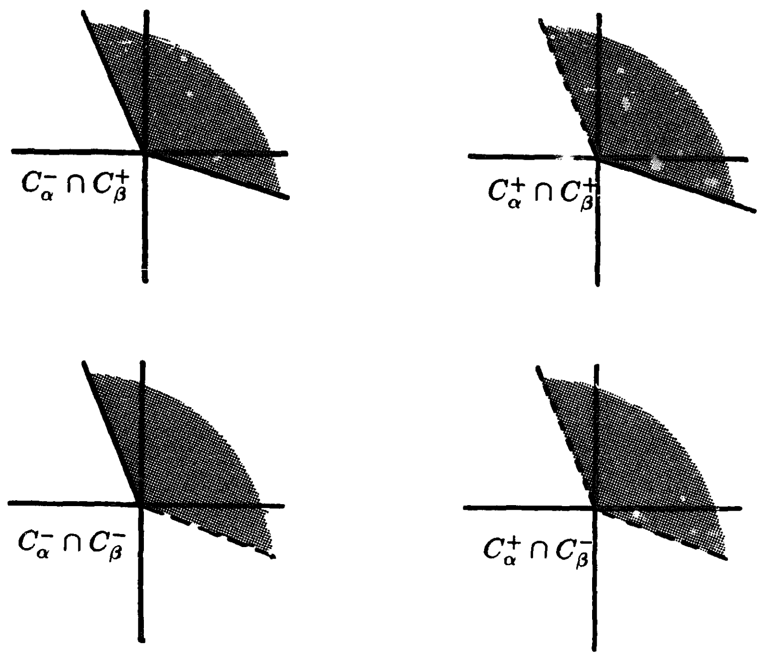

Fig. 3. Positive cones in $\mathbf{R}^2$.

Thus we have a complete classification of tame orderings on $\mathbf{R}^2$. Notice that the only tame orderings which arise from pointed cones are those of the form $C_\alpha^-\cap C_\beta^+$ with $\alpha>\beta$, such as D above, This cone is the set of solutions to $xA\geqslant 0$, where

$$A=\begin{pmatrix}\sin\alpha & \sin\beta\\ \cos\alpha & \cos\beta\end{pmatrix}.$$

4.2. A new multiset ordering

As another example of the power of Theorem 3.5 we study the ordering $\gg$ defined as follows. Let $>$ be a partial order on S, and for each $x\in S$ let $f_x(M)=\sum_{y\geqslant x}M(y)$. Then $M\gg N$ if and only if $f_x(M)\geqslant f_x(N)$ for all $x\in S$. Equivalently, let A be the n by n matrix indexed by elements of S whose i,j entry is 1 if $i\geqslant j$ in the partial order on S and 0 otherwise. Then $\gg$ is just $>_A$.

Now A is positive, and if $i>j$ then $a_{ir}\geqslant a_{jr}$ for $1\leqslant r\leqslant n$. Thus by applying Theorem 3.5 we see immediately that $\gg$ is tame and inherits $>$. Furthermore, since A is invertible we can describe the fundamental region of $\gg$, which is just the set of all semipositive combinations of the rows of A^{-1}.

To describe A^{-1} we need the notion of the Möbius function of a partially ordered set S (see [22]). If x and y are elements of S then $\mu(x, y)$ is defined uniquely by the equations

$$\mu(x,x)=1 \quad \text{and} \quad \sum_{y \geqslant z \geqslant x} \mu(y,z)=0 \quad \text{for } y>x,\ x \neq y.$$

When S is finite,

$$\mu(y,x)=c_0-c_1+c_2-\cdots c_n,$$

where c_i is the number of chains in S of the form $x=x_0<x_1<\cdots<x_i=y$. Thus $\mu(y,x)=0$ unless $y \geqslant x$.

Now let the matrix B be defined by $b_{ij}=\mu(i,j)$ for all i and j. Then

$$(BA)_{ij}=\sum_t b_{it}a_{tj}=\sum_{t \geqslant j} b_{it}=\sum_{i \geqslant t \geqslant j} b_{it}=\delta_{ij}$$

where δ_{ij} is 0 if $i \neq j$ and 1 if $i=j$. Thus B is A^{-1}, and we have an explicit description of the extreme vectors.

Example. As an example, let $>$ be the partially ordered set illustrated in Fig. 4. Then

$$A=\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A^{-1}=\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4.3. Matrix orderings

For another class of orderings, let A be an invertible matrix and let $>$ be any tame ordering on $\mathbf{R}^n$. Define a new ordering $>_A$ by

$$M >_A N \Leftrightarrow MA > NA.$$

We have the following theorem.

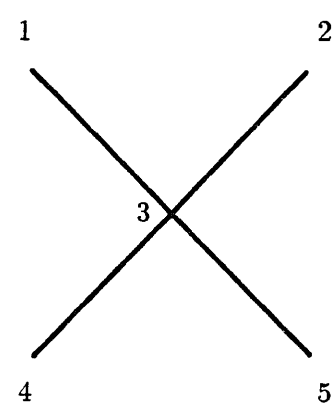

Fig. 4. A partial ordering.

Theorem 4.1. *Suppose that $>$ is tame with fundamental region H. Then $>_A$ is always additive and scalar and is tame if and only if HA^{-1} contains the nonnegative orthant, when HA^{-1} is the fundamental region of $>_A$.*

Proof. Again this follows directly from the definitions. □

As an example, the ordering $>_\alpha$ corresponding to the cone C_α^+ defined above is got from the ordering $>$ corresponding to C_0^+ by applying the matrix A which rotates the axes through the angle α, so that

$$A=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}.$$

5. The dominance ordering

In this section we use geometric techniques to generalise the dominance ordering to the case when the underlying set is only partially ordered.

If S is totally ordered and the coordinate system is canonical, the dominance ordering is defined by

$$M \gg N \Leftrightarrow \forall i\colon 1\leqslant i\leqslant n, M(1)+\cdots+M(i)\geqslant N(1)+\cdots+N(i).$$

Thus $(3,2,1)\gg(2,2,2)$ as $3\geqslant 2$, $5\geqslant 4$ and $6\geqslant 6$ but $(6,3,0,3)$ and $(5,5,1,1)$ are not comparable. This ordering is the ordering $>_A$ of the previous section, where A is the n by n matrix with 1 in all diagonal and above-diagonal entries and 0 elsewhere, and so the fundamental region is the pointed cone on the rows of A^{-1}, that is $\{E_1-E_2, E_2-E_3,\ldots, E_{n-1}-E_n, E_n\}$. This also follows from [13, Section 2.19, Lemma 2].

If we generalise this definition of the fundamental region to the case when $>$ is partial we obtain an important new ordering $\gg$, which turns out to be the minimal tame ordering which inherits $>$. We shall say that an element x *precedes* y if $x>y$ and there is no element z distinct from x and y with $x>z>y$.

Theorem 5.1. *Let $>$ be a partial order on the finite set S and let*

$$C=\left\{\sum_{x\geqslant y} b_{x,y}(E_x-E_y)+\sum_x c_xE_x \,\middle|\, b_{x,y}\geqslant 0,\ c_x\geqslant 0\right\}.$$

Then

(i) *C is a positive cone and is pointed with extreme halflines*

$$\{(E_x-E_y)\,|\,x \text{ precedes } y\}\cup\{(E_x)\,|\,x \text{ minimal}\},$$

so that C induces a tame ordering $\gg$ which inherits $>$ on $\mathbf{R}^n$.

(ii) *Any tame order on $\mathbf{R}^n$ which inherits $>$ contains $\gg$, so that $\gg$ is the unique minimal tame order on $\mathbf{R}^n$ which inherits $>$.*

(iii) *Suppose that $>$ is total and the coordinate system is canonical. Let M^+ be the multiset M with its entries rearranged in increasing order. Then $M\gg M^+$.*

Part (ii) is a generalisation of a result first proved for total orderings by Gerstenhaber [12], and part (iii) is Lemma 1.14 in [16].

Example. As an example let $>$ be the partial ordering of Fig. 4. The cone C is just the pointed cone on $\{E_1-E_3, E_2-E_3, E_3-E_4, E_3-E_5, E_4, E_5\}$, and it turns out, applying Theorem 3.3, that this is just the solutions of

$$a_1 \geqslant 0$$

$$a_2 \geqslant 0$$

$$a_1+a_2+a_3+a_4 \geqslant 0$$

$$a_1+a_2+a_3+a_5 \geqslant 0$$

$$a_1+ \quad a_2+a_3 \geqslant 0$$

$$a_1+a_2+a_3+a_4+a_5 \geqslant 0.$$

Proof of Theorem 5.1. (i): To show that C is a positive cone we need to show that each E_x lies in C, which is clear, and that there is no non-zero element a with a and $-a$ in C, which follows if we show that whenever

$$0=\sum_{x\geqslant y} b_{x,y}(E_x-E_y)+\sum_x c_xE_x \tag{$*$}$$

with each $b_{x,y}\geqslant 0$ and each $c_x \geqslant 0$ then each $b_{x,y}=c_x=0$. To prove this, suppose that it is false and choose t to be maximal among the elements of S which appear in the sum with some $b_{t,y}>0$. The maximality condition means that all $b_{k,t}$ with $k>t$ are 0, so that equating coefficients we get $0=\sum_{y<t} b_{t,y}+c_t$, forcing each $b_{t,y}=0$, which is a contradiction.

To show that C is pointed, we need to show that each of the given vectors is extreme, and that the extreme vectors are semipositively independent, which is a special case of what we have just proved. Now suppose that u precedes v and that E_u-E_v is a sum of two linearly independent vectors in C, so that

$$E_u-E_v=\sum_{x\geqslant y} b_{x,y}(E_x-E_y)+\sum_x c_xE_x.$$

Again, equating coefficients of an element t maximal with respect to having some $b_{t,y}>0$, we find that the only such maximal element is u, and hence $b_{t,y}=0$ unless $t\leqslant u$. Equating coefficients of E_u and E_v in the expression we get

$$1=c_u+\sum_{y,u>y} b_{u,y}, \qquad -1=c_v-b_{u,v}+\sum_{y,v>y} b_{v,y}$$

and so adding the equations together

$$0=c_u+\sum_{y,u>y,y\neq v} b_{u,y}+c_v+\sum_{y,v>y} b_{v,y}$$

which means that every term on the right is zero, and so $b_{u,v}=1$, and applying $(*)$ we deduce that all the other $b_{r,s}$ and c_x are zero. We show that E_x is an extreme vector when x is minimal in exactly the same way.

(ii): Now if $>$ is a tame order on $\mathbf{R}^n$ which inherits $>$ then H, the fundamental region of $>$, contains each E_x and each $E_x - E_y$ with $x > y$, by Theorem 3.5. Thus H contains all positive linear combinations of these vectors, and hence contains the fundamental region of $\gg$. Thus $\gg$ is the unique minimal tame order on $\mathbf{R}^n$ which inherits $>$.

(iii): Suppose that $M = (m_1, \ldots, m_n)$ is a sequence in a canonical coordinate system. If i is smaller than j and $m_i > m_j$ then swapping the ith and jth entries of M gives a new vector $Mc_{ij} \ll M$ by part (i). If the entries of M are not sorted in increasing order then we can find a transposition c_{ij} which transforms M to a vector which is smaller in the ordering $\gg$, and since there are only finitely many vectors which can be got by permuting the elements of M, this process must terminate, in which case M must be sorted, that is we have transformed M to M^+. Thus $M \gg M^+$. $\square$

Notice that in the proof of part (iii) we have essentially used the dominance ordering to prove the termination of a simple-minded sorting algorithm. Of course, one would normally do this using the lexicographic ordering, which is total. However, the importance of part (ii) is that it shows that the dominance ordering is the unique weakest ordering on M which will prove termination of this algorithm.

6. The standard multiset ordering

In this section we use our techniques to describe the standard multiset ordering, and show that any tame total ordering can be got from the standard multiset ordering by the action of an invertible matrix as in Theorem 4.1.

The standard multiset ordering $\gg$ is defined as follows. Let $>$ be any partial order on S, and M and N multisets. Suppose $M \neq N$. Then $M \gg N$ if and only if whenever $M(x) < N(x)$, for some $x \in S$, then there is a $y \in S$, with $y > x$ and $M(y) > N(y)$. If we restrict this ordering to N^n we obtain the multiset ordering defined by Dershowitz and Manna in [9].

As an example, if $S = \{a, b, c\}$ and $a > b$ then $\{a, a, b\} \gg \{a, b, b, b\}$, but $\{b, c\}$ and $\{a, b\}$ are not comparable.

Dershowitz showed that if the order $>$ on S is well founded then the ordering $\gg$ on N^n is also well-founded, so that in particular $\gg$ is well-founded when S is finite. It is very easy to see also that $\gg$ inherits the order $>$, and is additive, bounded and scalar. Thus the standard multiset orderings on N^n and $\mathbf{R}^n$ are tame.

If $>$ is total then $\gg$ is total and becomes the lexicographic ordering on $\mathbf{R}^n$ with respect to $>$, that is if $S = \{s_1, \ldots, s_n\}$ with $s_1 > \cdots > s_n$, then $M \gg N$ if and only if $M \neq N$ and

$$\forall i\colon 1 \leqslant i \leqslant n, \quad M(s_i) < N(s_i) \Rightarrow \exists j < i\colon M(s_j) > N(s_j).$$

We can describe the fundamental region for the multiset ordering as follows.

Theorem 6.1. *Let $\gg$ be the standard multiset ordering on* $\mathbf{R}^n$ *induced by the partial ordering $>$ on the coordinate vectors $s_1, \ldots, s_n$. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$.*

(i) *The ordering $\gg$ is a tame ordering with fundamental region*

$$F = \{\boldsymbol{a} \mid a_i < 0 \Rightarrow \exists s_j.((s_j > s_i) \text{ and } a_j > 0)\}.$$

(ii) *If $>$ is total then*

$$F = \{\boldsymbol{a} \mid a_1 > 0\} \cup \{\boldsymbol{a} \mid a_1 = 0, a_2 > 0\}$$
$$\cup \cdots \cup \{\boldsymbol{a} \mid a_1 = a_2 = \cdots = a_{n-1} = 0,\ a_n \geqslant 0\}.$$

Proof. This is just a translation of the definition into coordinate geometry. □

The fundamental region described in part (ii) of the theorem will be called the standard cone.

Our method of constructing new orderings from old is very powerful as the following result shows.

Theorem 6.2. *Let $>$ be the standard multiset ordering on* $\mathbf{R}^n$ *with respect to a given canonical basis. Let $>$ be any other tame total ordering on* $\mathbf{R}^n$*. Then there is an invertible matrix A such that $> = >_A$.*

Proof. The proof will be by induction on n. If $n = 2$ the result is clear from the classification in Section 3.

To prove the general result we need some elementary topology. If E is any subset of $\mathbf{R}^n$, let $\mathrm{cl}(E)$ denote the closure of E, that is, the set of points in $\mathbf{R}^n$ whose every neighbourhood intersects E, and $\mathrm{in}(E)$ the interior of E, that is the set of points in E which have a neighbourhood lying entirely within E. Let $\mathrm{bd}(E)$ be the boundary of E, that is $\mathrm{cl}(E) \backslash \mathrm{in}(E)$. Then $x \in \mathrm{bd}(E)$ if and only if there are sequences of points (x_i), $x_i \in E$ and (x_i'), $x_i' \notin E$, with $i \in \mathbf{N}$, both of which have limit x.

Since $>$ is total and tame, $\mathbf{R}^n = C \cup -C$, where C is a cone and the fundamental region of $>$; it follows that $-C$ is also a cone. If T is any subspace of $\mathbf{R}^n$ then $D = T \cap C$ is a cone in T, and $T = D \cup -D$ and $D \cap -D = 0$, and we can apply our result to T by induction. To prove the result we shall show that $\mathrm{bd}(C)$ is a hyperplane and $\mathrm{cl}(C)$ is a cone. If this is so then $\mathrm{cl}(C)$ is a cone containing a hyperplane so must be either $\mathbf{R}^n$, which is impossible as $\mathbf{R}^n$ contains elements which are not in $\mathrm{cl}(C)$, or the hyperplane itself which is impossible as $C \cup -C$ is the whole of $\mathbf{R}^n$, or a halfspace. Thus $\mathrm{cl}(C)$ is a halfspace, that is, the set of positive linear combinations of elements of $\mathrm{bd}(C)$ and a vector u_1 perpendicular to it. We can now apply our induction hypothesis to the cone $F = C \cap \mathrm{bd}(C)$ in $\mathrm{bd}(C)$. Let $u_2, \ldots, u_n$ be a basis for F. Then there is an invertible $n-1$ by $n-1$ matrix B_1 such that FB_1 is the standard cone on $u_2, \ldots, u_n$. Now let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & B_1 \end{pmatrix}.$$

and A_1 the linear map which sends $u_1, \ldots, u_n$ to the given canonical basis on $\mathbf{R}^n$. Then let $A = BA_1$, so that CA is the standard cone and $> = >_A$.

Now to show that $\mathrm{bd}(C)$ is a hyperplane we show first that it is a subspace, that is, if x and y are in $\mathrm{bd}(C)$ and λ is a scalar then $x+y$ and λx are in $\mathrm{bd}(C)$. Now there are sequences (x_i) and (y_i) in C with limits x and y, and so $x_i + y_i$ is a sequence in C with limit $x+y$. Similarly, there are sequences (x_i') and (y_i') of elements not in C, and therefore in $-C$, with limits x and y so that since $-C$ is a cone $x_i'+y_i'$ is a sequence in $-C$, and hence not in C, with limit $x+y$. Thus $\mathrm{bd}(C)$ is closed under addition. Similarly, by considering the sequences (λx_i) and $(\lambda x_i')$, $\mathrm{bd}(C)$ is closed under scalar multiplication, since if $\lambda > 0$ we have $(\lambda x_i) \subset C$ and $(\lambda x_i') \subset -C$, and if $\lambda < 0$ we have $(\lambda x_i) \subset -C$ and $(\lambda x_i') \subset C$. Similar arguments show that $\mathrm{cl}(C)$ is a cone.

It remains to show that $\mathrm{bd}(C)$ is a hyperplane. If it is not, then we can find a subspace T of dimension 2 which intersects $\mathrm{bd}(C)$ in 0. But then, if $D = T \cap C$, we have $T = D \cup -D$ and $D \cap D = 0$, so that $\mathrm{bd}(D)$ contains a line. But it follows from our characterisation of $\mathrm{bd}(C)$ in terms of limits of sequences that $\mathrm{bd}(D) \subset \mathrm{bd}(C) \cap T$, which contradicts our choice of C. Thus $\mathrm{bd}(C)$ is a hyperplane. $\square$

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