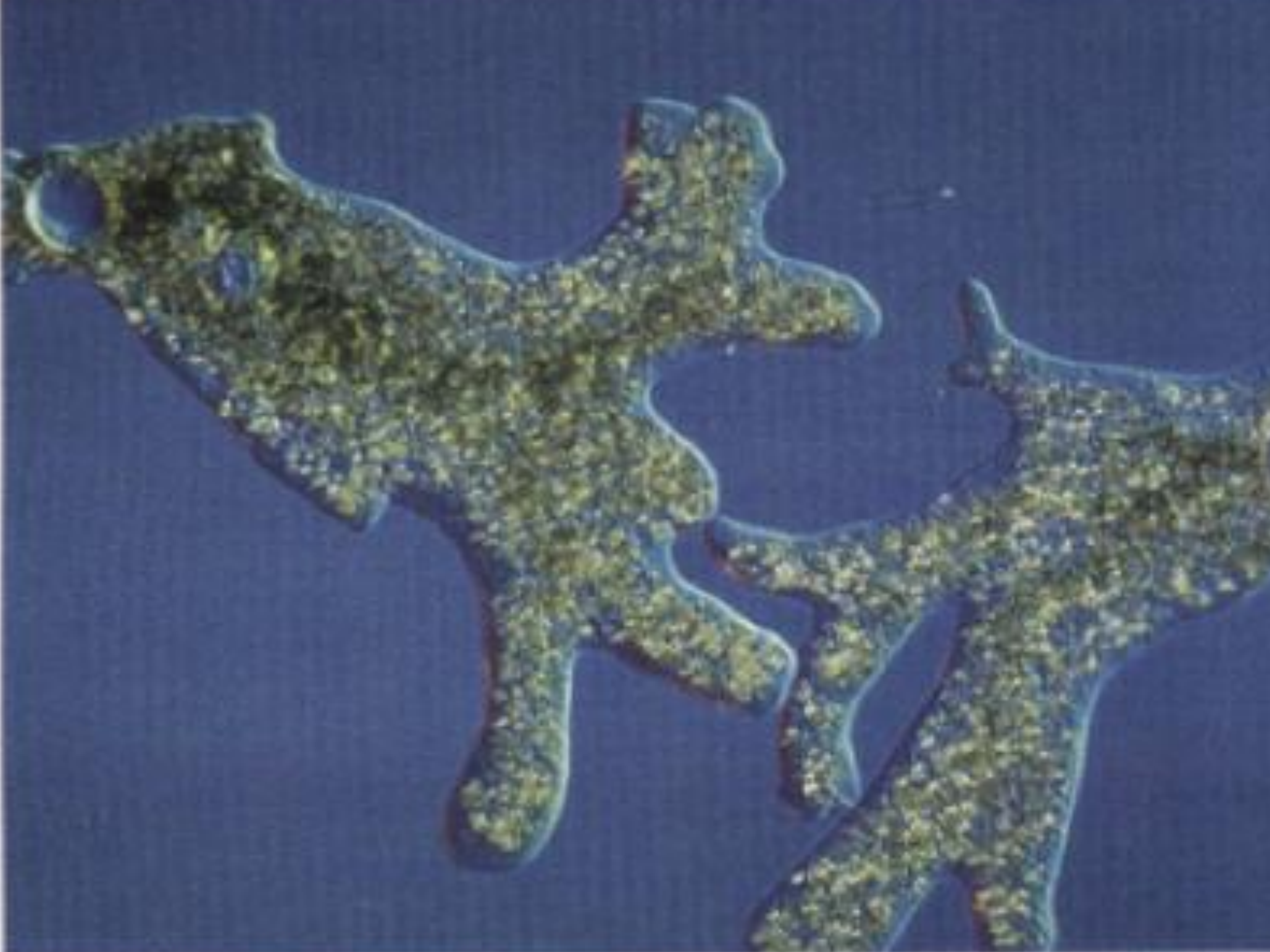
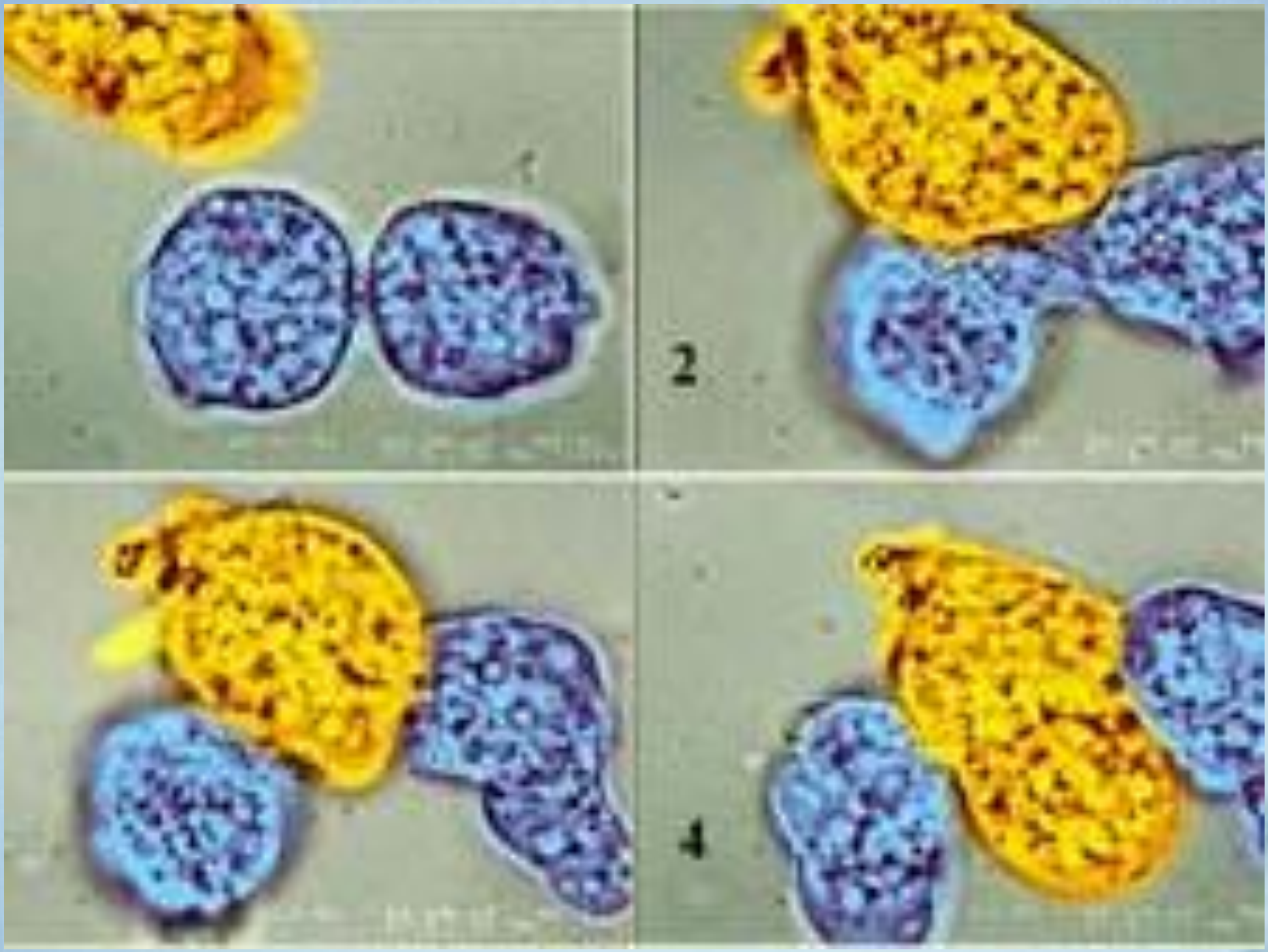
Termination

4. Well-Founded Orderings

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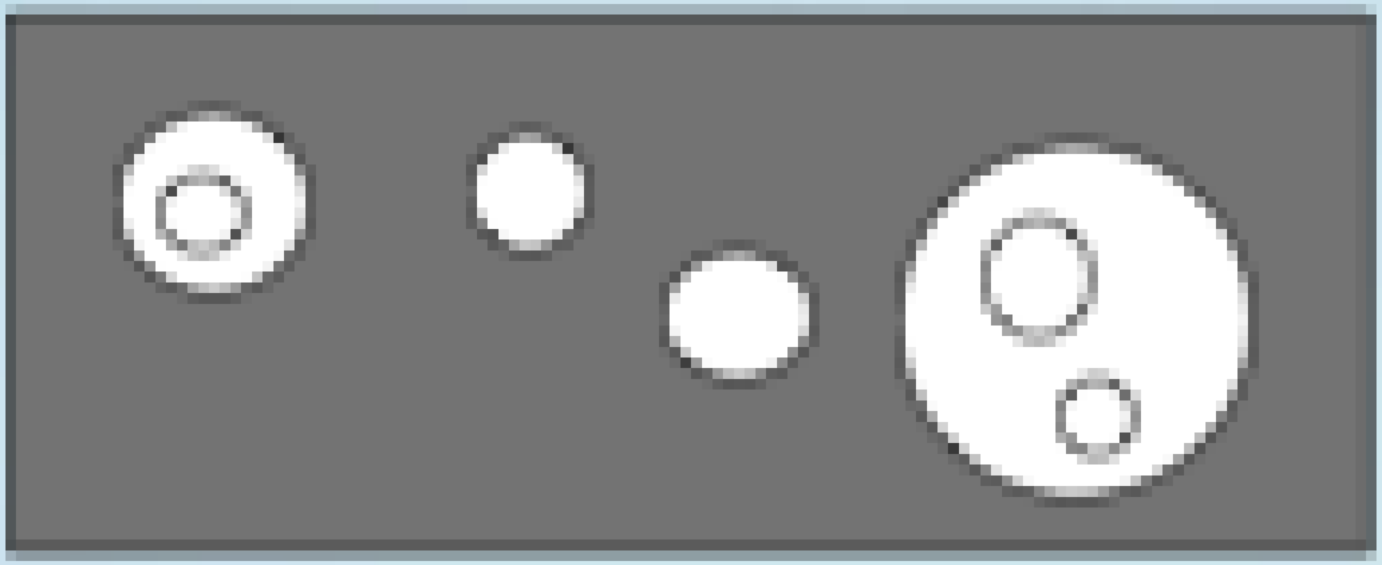
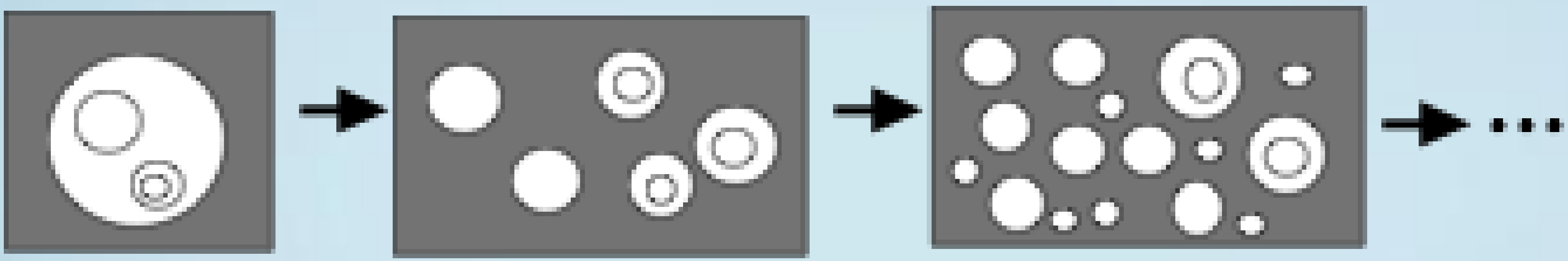
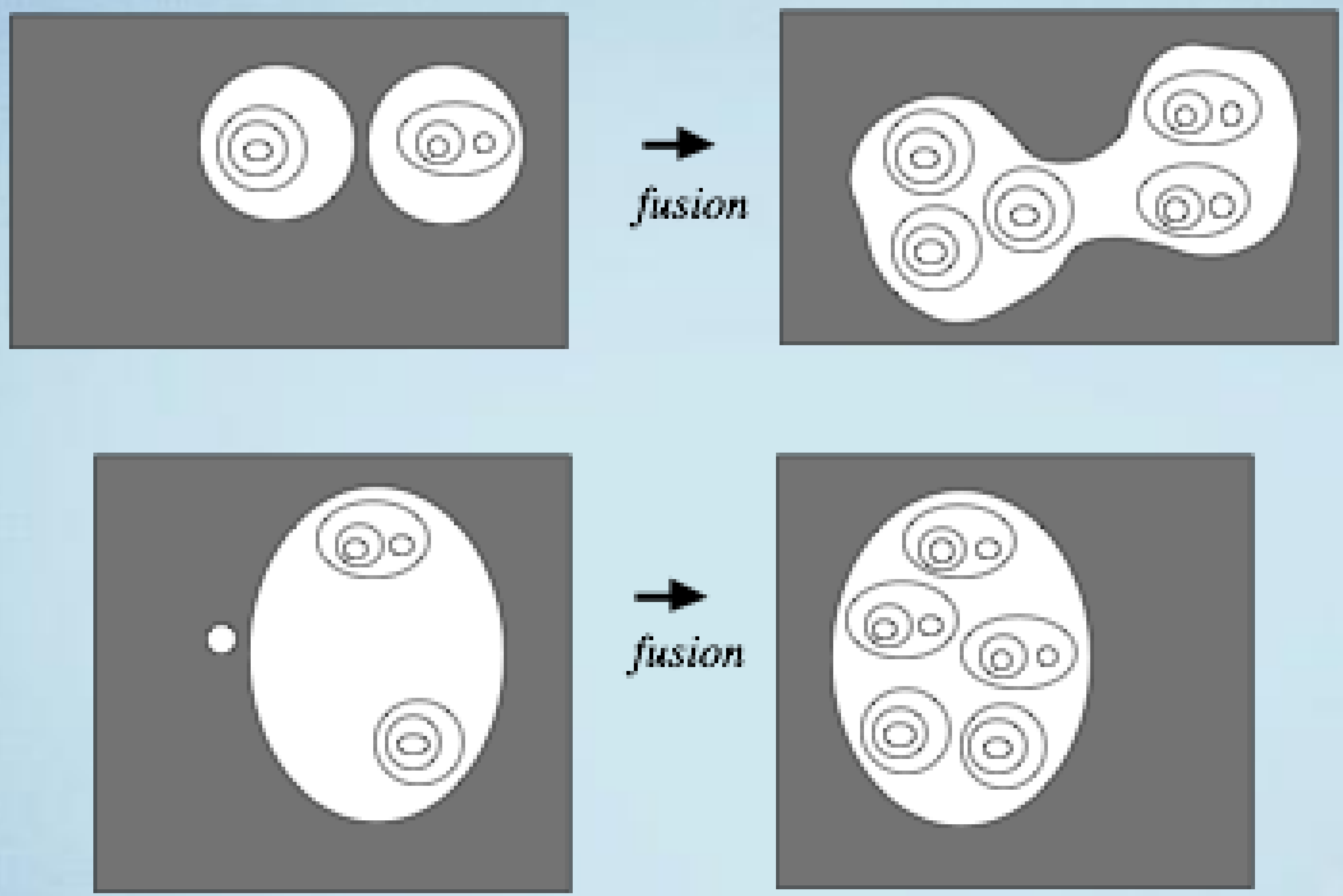
Amoebae



Fission




fusion
fusion

Colony Dies Out

- depth(o) = 0
- depth(a1 ... an) = 1+max{depth{ai}}
- { (depth(a),|a|) : subcolony a }
- outer fission: depth decreases
- fusion: size decreases

Colony Dies Out

- $d(a) = \text{depth}(a)$
- $\#_d(a)$ = number in a of depth d
- $\{\ (d(a), \#_{d(a)}(a), \#_{d(a)-1}(a), \ldots) : \text{colony } a\ \}$
- fission: depth decreases
- fusion: size decreases

Big Picture

- Programs are state-transition systems
- Choose a well-founded order on states
- Show that transitions are decreases

Real Picture

- Programs are state-transition systems
- Choose a function for “ranking” states
- Choose a well-founded order on ranks
- Show that transitions always decrease rank

Imaginary Picture

- Programs are state-transition systems
- Choose a function for “ranking” states
- Choose a well-founded order on ranks
- Show that transitions eventually decrease rank

Nested Loops

- r := 1
- u := 1
- loop v := u $\omega^2 (n - r) + \omega(r - s) + k$
  - until r≥n
  - s := 1
  - loop u := u+v
    - s := s+1
    - while s≤r
    - repeat
  - r := r+1
  - repeat

Per Iteration

- r := 1
- u := 1
- loop   v := u
    - until r≥n
    - s := 1
    - loop   u := u+v
        - s := s+1
        - while s≤r
        - repeat
    - r := r+1
    - repeat

ω(n-r)+r+1-s

Lexicographic

- r := 1
- u := 1
- loop v := u
  - until r≥n
  - s := 1
  - loop u := u+v
    - s := s+1
    - while s≤r
    - repeat
  - r := r+1
  - repeat

(n-r,r+1-s)

Invariants

- r := 1
- u := 1
- loop   v := u   1≤r≤n
  - until r≥n
  - s := 1
  - loop   u := u+v
    - s := s+1   1≤s≤r+1
    - while s≤r
    - repeat
  - r := r+1
  - repeat

Well-Founded Orderings

- No infinite descending sequences
- x1 > x2 > x3 > ...

Well-Founded Induction

$>$ is a w.f.o. of X

- $\underline{\forall x \in X.\ [\forall y < x.\ P(y)] \Rightarrow P(x)}$
- $\forall x \in X.\ P(x)$

Why?

Well-Founded Induction

We'll prove that if < if w.f.o. over X, then the following holds:

- **<u>$\forall x \in X. [\forall y < x. P(y)] \Rightarrow P(x)$</u>**
- **$\forall x \in X. P(x)$**

(*)

In other words, if induction scheme (*) doesn't hold – then < isn't a w.f.o. over X (meaning that correctness of induction scheme implies w.f.o.).
<u>Proof:</u> Assume that (*) isn't true, meaning that line 1 holds, but there is an element a1 in X for which P(a1) = F. Since line 1 holds,
there is a2 < a1, for which P(a2) = F (as otherwise P(a1) would be T).
For same reason, there is a3 < a2 < a1 , for which P(a3) = F and so on.
So, we got an infinite chain in X = < isn't w.f.o. ***QED***

(That's why the proof of the base case is so vital in inductive process!)

David Gries

- Under the reasonable assumption that non-determinism is bounded, the two methods are equivalent…. In this situation, we prefer using strong termination.

```
n := 0
while x > 0  do
    n := n + 1
    y := 0; while y^2 + 2y ≤ x  do y := y + 1
    if  x = y^2
    then x := y − 1
    else s := 0
        r := 0; while r^2 + 2r ≤ x − y^2  do r := r + 1
        while x > y^2 + r^2  do
            y := 0; while y^2 + 2y ≤ x  do y := y + 1
            s := s + (s + y^2 + y − x)^2
            x := x − y^2
            r := 0; while r^2 + 2r ≤ x − y^2  do r := r + 1
        for i := 1  to n  do x := r^2 + r − 1
        while s > 0  do
            r := 0; while r^2 + 2r ≤ s  do r := r + 1
            x := x + (x + r^2 + r − s)^2
            s := s − r^2
```

Contra-Gries

- To prove terminating with a natural (strong) ranking function requires Σ_0 - induction.

All-Purpose Ranks

$$0 < 1 < 2 ...$$

$$< \omega < \omega + 1 < \omega + 2 < ...$$

$$< \omega 2 < \omega 2 + 1 < ... < \omega 3 < ... < \omega 4 < ...$$

$$< \omega^2 < \omega^2 + 1 < ... < \omega^2 + \omega < \omega^2 + \omega + 1 < ...$$

$$< \omega^3 < \omega^3 + 1 < ... < \omega^4 < ... < \omega^5 < ...$$

$$< \omega^\omega < ... < \omega^{\omega^\omega} < ... < \omega^{\omega^{\omega^\omega}} < ...$$

Ordinals

$$0, 1, 2, \ldots,$$

$$\omega, \omega+1, \omega+2, \ldots,$$

$$2\omega, 2\omega+1, \ldots, 3\omega, \ldots,$$

$$\omega^2, \ldots, \omega^2+2\omega+3, \ldots, \omega^3, \ldots,$$

$$\omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots,$$

$$\varepsilon_0, \varepsilon_0+1, \ldots, 2\varepsilon_0+\omega^\omega+2\omega+3, \ldots,$$

$$\varepsilon_1, \ldots, \varepsilon_{\varepsilon_0}, \ldots$$

THE GROWTH

$0, 1, 2, 3, \ldots, \omega, \omega+1, \omega+2, \omega+3, \ldots, \omega+\omega = \omega 2, \omega 2+1, \omega 2+2, \omega 2+3, \ldots, \omega 3, \omega 3+1, \omega 3+2, \omega 3+3,$

$\omega\omega = \omega^2, \omega^2+1, \omega^2+2, \ldots, \omega^2+\omega, \ldots, \omega^2+\omega 2, \ldots, \omega^2+\omega 3, \ldots, \omega^2 2, \ldots, \omega^2 3, \ldots, \omega^3, \ldots, \ldots,$

$\ldots, \omega^\omega\omega^2, \ldots, \omega^\omega\omega^3, \ldots, \omega^\omega\omega^\omega = \omega^{\omega 2}, \ldots, \omega^{\omega 3}, \ldots, \ldots, \omega^{\omega^2}, \ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\cdot^{\cdot^{\cdot}}}} = \varepsilon_0, \varepsilon_0+1, \ldots,$

$\varepsilon_0+\omega 2, \ldots, \varepsilon_0+\omega^2, \ldots, \varepsilon_0+\omega^\omega, \ldots, \varepsilon_0+\varepsilon_0 = \varepsilon_0 2, \ldots, \varepsilon_0 3, \ldots, \varepsilon_0\omega, \ldots, \varepsilon_0\omega 2, \ldots, \varepsilon_0\omega^2, \ldots,$

$\varepsilon_0\varepsilon_0 = \varepsilon_0^2, \ldots, \varepsilon_0^3, \ldots, \varepsilon_0^\omega, \ldots, \varepsilon_0^{\omega^\omega}, \ldots, \varepsilon_0^{\varepsilon_0}, \ldots, \varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot}}}} = \varepsilon_1, \ldots, \varepsilon_2, \ldots, \varepsilon_3, \ldots, \varepsilon_\omega$

$\varepsilon_{\omega^\omega}, \ldots, \varepsilon_{\varepsilon_0}, \ldots, \varepsilon_{\varepsilon_1}, \ldots, \varepsilon_{\varepsilon_\omega}, \ldots, \varepsilon_{\varepsilon_{\omega^\omega}}, \ldots, \varepsilon_{\varepsilon_{\omega^{\omega^\omega}}}, \ldots, \varepsilon_{\varepsilon_{\varepsilon_0}}, \ldots, \varepsilon_{\varepsilon_{\varepsilon_{\cdot_{\cdot}}}} = \eta_0, \ldots, \eta_1, \ldots$

$\eta_{\varepsilon_0}, \ldots, \eta_{\varepsilon_\omega}, \ldots, \eta_{\varepsilon_{\varepsilon_0}}, \ldots, \eta_{\varepsilon_{\varepsilon_{\cdot_{\cdot}}}} = \eta_{\eta_0}, \ldots, \eta_{\eta_{\eta_{\cdot_{\cdot}}}} = \zeta_0, \ldots, \ldots\ldots\ldots\ldots\ldots\ldots$

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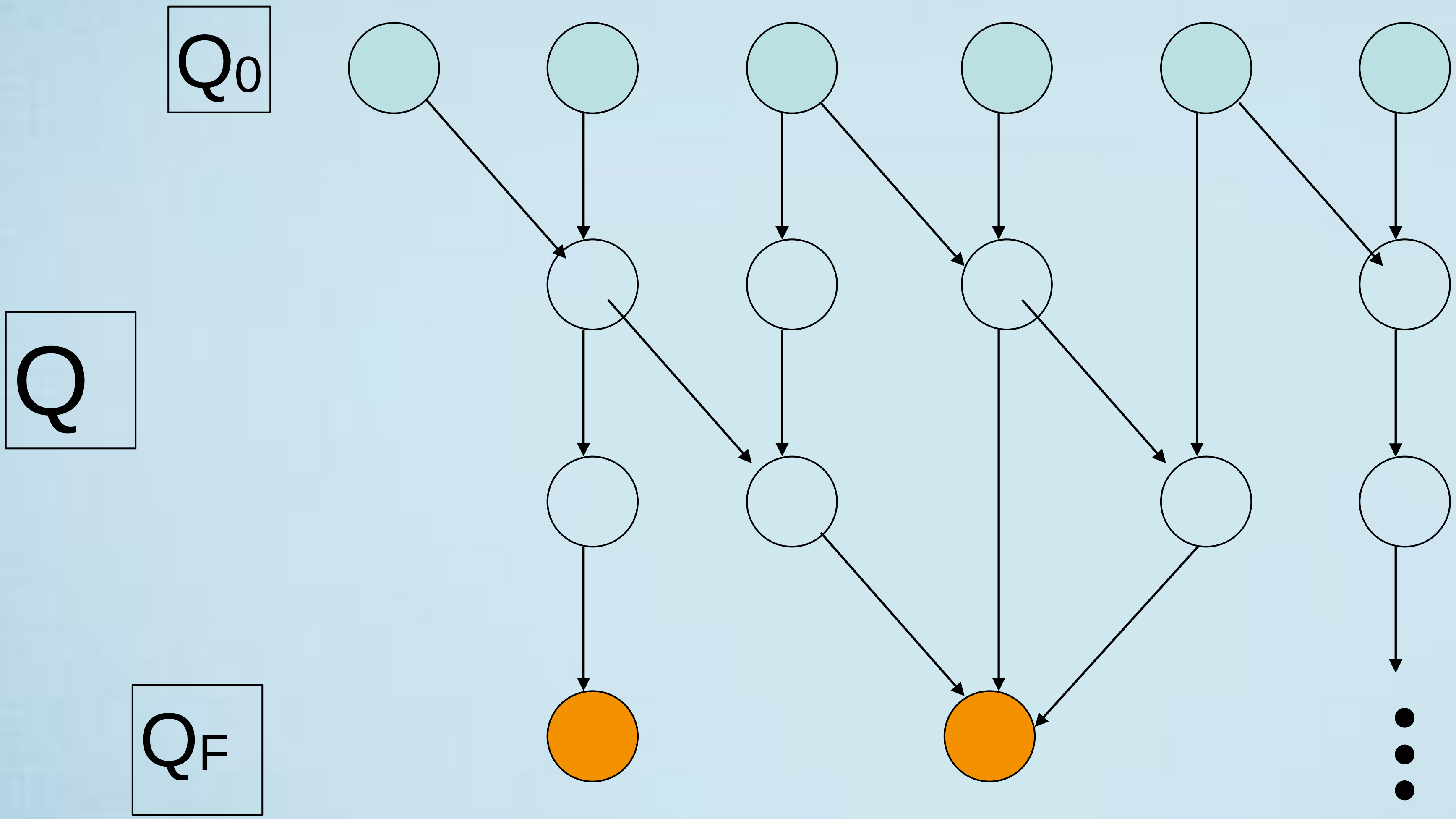
Transition System

Discrete Transition System



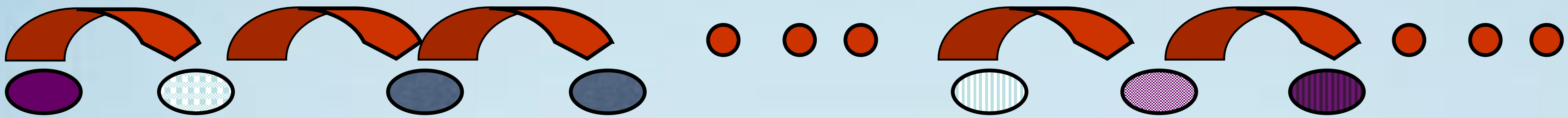
Well-Founded Method

- States Q
- Algorithm $R \subseteq Q \times Q$
- Well-founded order > on Q
- $R \subseteq >$

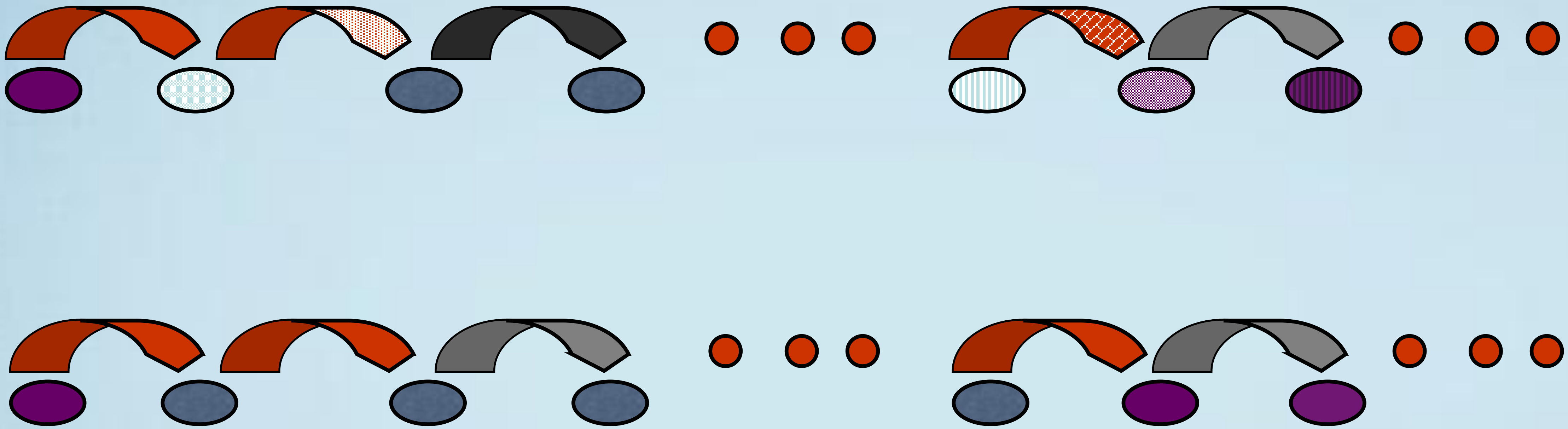
All-Purpose Ranking

- $r : Q \to Ord$
- $r(x) = \sup \{ r(y)+1 \mid x \to y \}$

Computation



Abstraction



Frank Ramsey

Frank Ramsey (1903-1930)

Frank Ramsey was British mathematician, philosopher and economist.

He had developed the “Ramsey theory”, a branch of mathematics that studies the conditions under which order must appear. Problems in “Ramsey theory” typically ask a question of the form: "how many elements of some structure must there be to guarantee that a particular property will hold?"

Ramsey's Theorem (finite case)

Before presenting Ramsey's Theorem for infinite graphs, which we would use later in proving termination, in different schemes, we start by presenting the theorem for finite graphs.

Def1: Suppose G = (V, E) is an undirected simple graph.
A c-coloring (c is a natural number) of the edges of G (not necessarily legal) is a function f: E --> {1, … , c}.

Now lets define the Ramsey Numbers R(k, s):
R(k,s) is the smallest number n, s.t. any 2-coloring (say in RED and BLUE) of K_n **(the complete graph on n vertices) either contains a monochromatic RED clique of size k, or a monochromatic BLUE clique of size s, as a sub-graph.**

Ramsey's Theorem (finite case)

Trivial Ramsey Numbers are R(1,k) = 1 (1 vertex is a 1-clique) and R(2,k) = k (as (k-1)-clique can be all RED).

It's also trivial that R(k,s) = R(s,k) (just flip the colors).

Ramsey Number are very difficult to calculate precicely, and we know very few of them.

Ramsey's Theorem states that for every k,s , R(k,s) is finite.

The theorem is easily proven by induction, after proving the following lemma: $R(k,s) \leq R(k-1,s) + R(k,s-1)$

The theorem is also generalized for any number of colors (and not just 2) and also for hyper graphs.

Ramsey's Theorem (finite case)

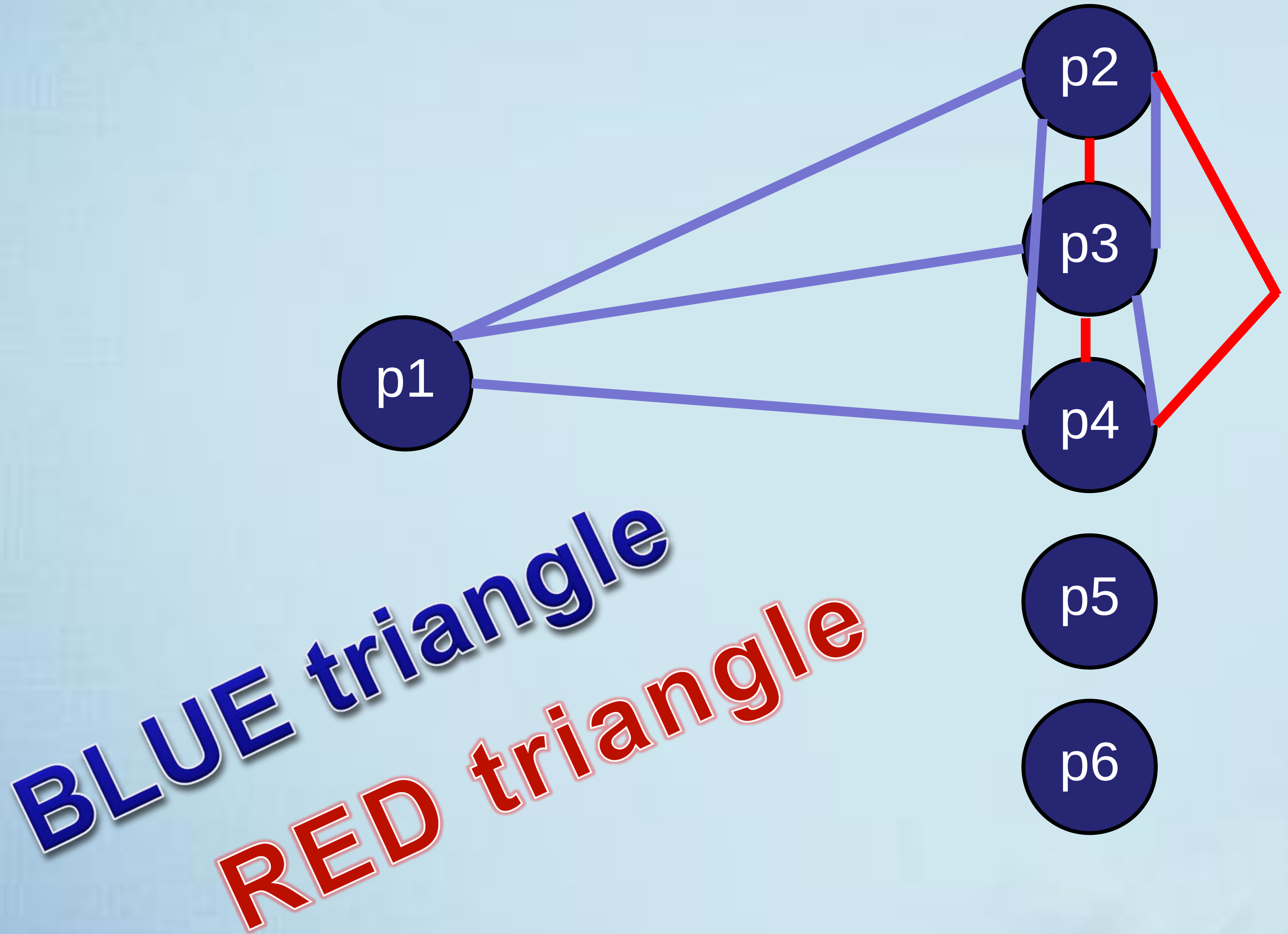
<u>"Social example":</u>

A nice "social fact", follows from Ramsey's Theorem, is that any group of 6 persons, either has 3 mutually friends, or 3 mutually strangers.

<u>Proof:</u> Denote persons by p1, … , p6 – vertices of a graph. We'll connect 2 friends with BLUE edge, and 2 strangers with RED. p1 has either at least 3 BLUE or 3 RED edges from him (trivial). W.l.o.g. they'll BLUE, and to p2, p3, p4. If either of p2, p3, p4 are friends, then we have a BLUE triangle. Otherwise, they're all strangers – and we have a RED triangle.

∎

Ramsey's Theorem (finite case)



Ramsey's Theorem (infinite case)

Natural generalization of Ramsey's Theorem for infinite graphs (we'll deal just with graphs where |V| = $\aleph_0$), would be the following:

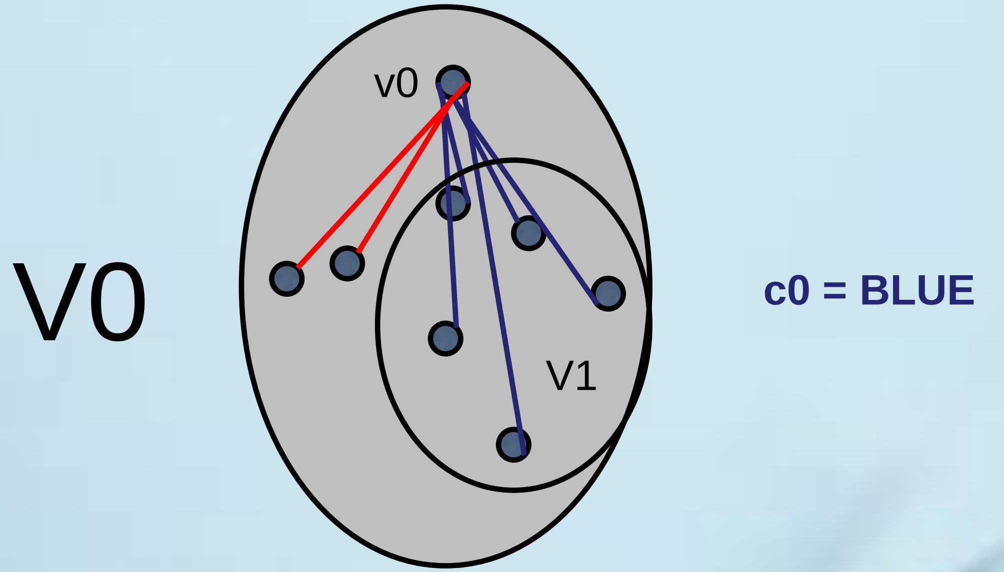
If we have an undirected simple infinite complete graph, which edges are colored by finite number of colors (mostly we'll use 2), then this graph has a monochromatic infinite clique as a sub-graph.

<u>Proof:</u> For simplicity, the set of vertices of our graph would be the natural numbers. Also, we will denote the complete graph on $V_0 = N \ as \ K_N$.
An infinite clique in this graph will be denoted with K_∞.

Ramsey's Theorem (infinite case)

<u>Proof cont.:</u> Suppose we have the edges of our K_N colored in two colors {RED, BLUE} (the proof works for any finite number of colors). Let $v_0 \in V_0$ be an arbitrary vertex. Since v0 has an infinite number of edges incident on it, and each edge has a color drawn from a finite set, some color, c0 (RED or BLUE), is the color of infinitely many of these edges.

Let V1 be the set of neighbors of v0, to which it connected with an edge colored in c0. So, $V_1 = \{x \mid COL(\{v_0, x\}) = c_0\}$. V1 is infinite, by definition.



Ramsey's Theorem (infinite case)

<u>Proof cont.:</u> Clearly, $V_1 \subset V_0$ **(v0 is in V0 but not in V1).**

As V1 is infinite, we make the same construction on it. Let $v_1 \in V_1$

be an arbitrary vertex. From v1 there is an infinite number of edges of same color, c1, <u>to vertices in V1</u>. Then, we define the infinite set V2 as previously:

$V_2 = \{x \mid COL(\{v_1, x\}) = c_1 \ and \ x \in V_1\}.$ **And also,** $V_2 \subset V_1.$

That way, we construct the infinite sequences: $\{v_i\}_{i=0}^{\infty}, \{c_i\}_{i=0}^{\infty}, \{V_i\}_{i=0}^{\infty}.$

$$V0 \xrightarrow[c0]{v0} V1 \xrightarrow[c1]{v1} V2 \cdots Vi \xrightarrow[ci]{vi} V_{i+1} \cdots$$

Ramsey's Theorem (infinite case)

<u>Proof cont.:</u> For all i, we get:

1. $v_i \in V_i$
2. $V_{i+1} \subset V_i$
3. $edge\ \{v_i, x\}$ is colored ci for every $x \in \mathrm{V}_{i+1}$

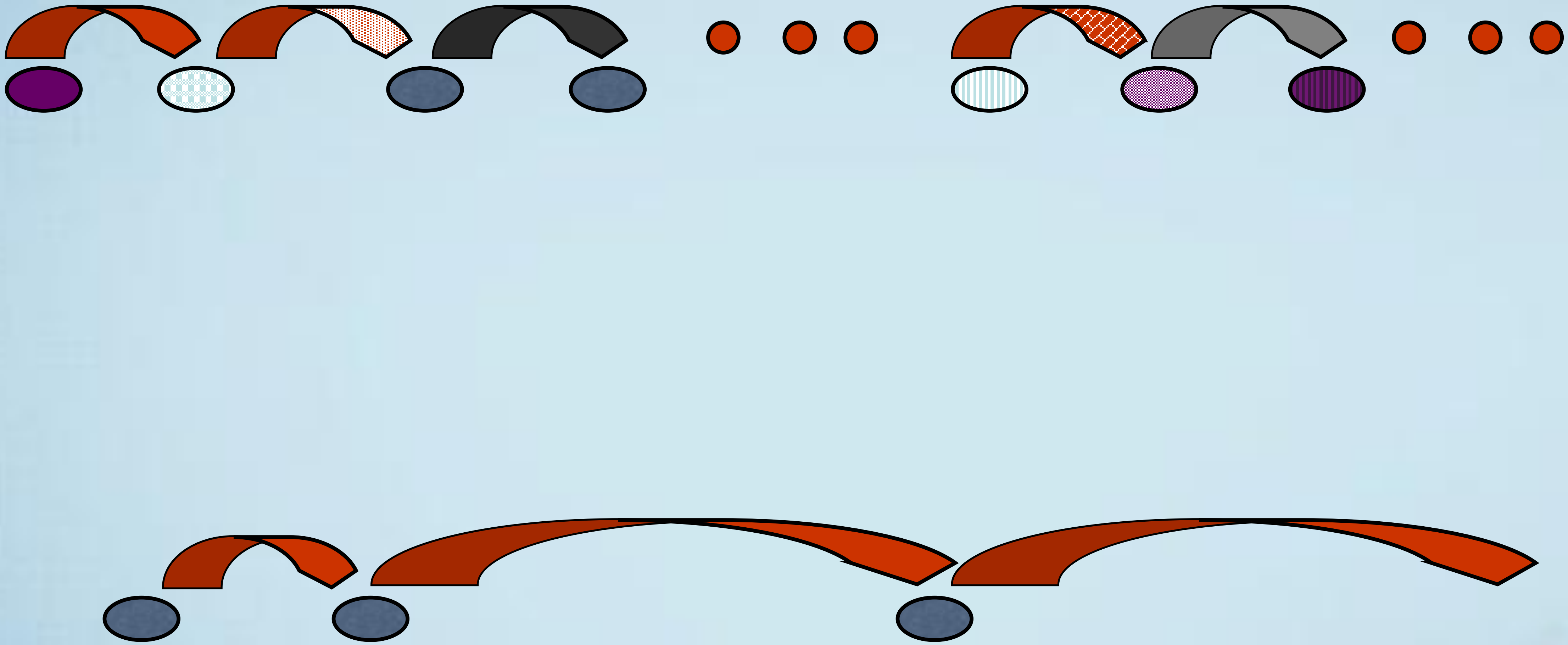
We claim that for any i, j, s.t. i < j, the edge {vi, vj} is colored ci. The proof is simple: from (1) $v_j \in V_j$, from (2) $V_j \subset V_{j-1} \subset ... \subset V_{i+1}$ and so $v_j \in V_{i+1}$. Therefore, from (3), the edge {vi, vj} is colored ci. Now, as we have only 2 colors, one of them occurs infinitely many times among c0, c1,… W.l.o.g. it'll be BLUE. Now, lets define the set: $T = \{v_i \mid c_i = BLUE\}$, and we'll show that T is a monochromatic infinite clique. Firstly, T is infinite, from previous explanation about the colors.

Ramsey's Theorem (infinite case)

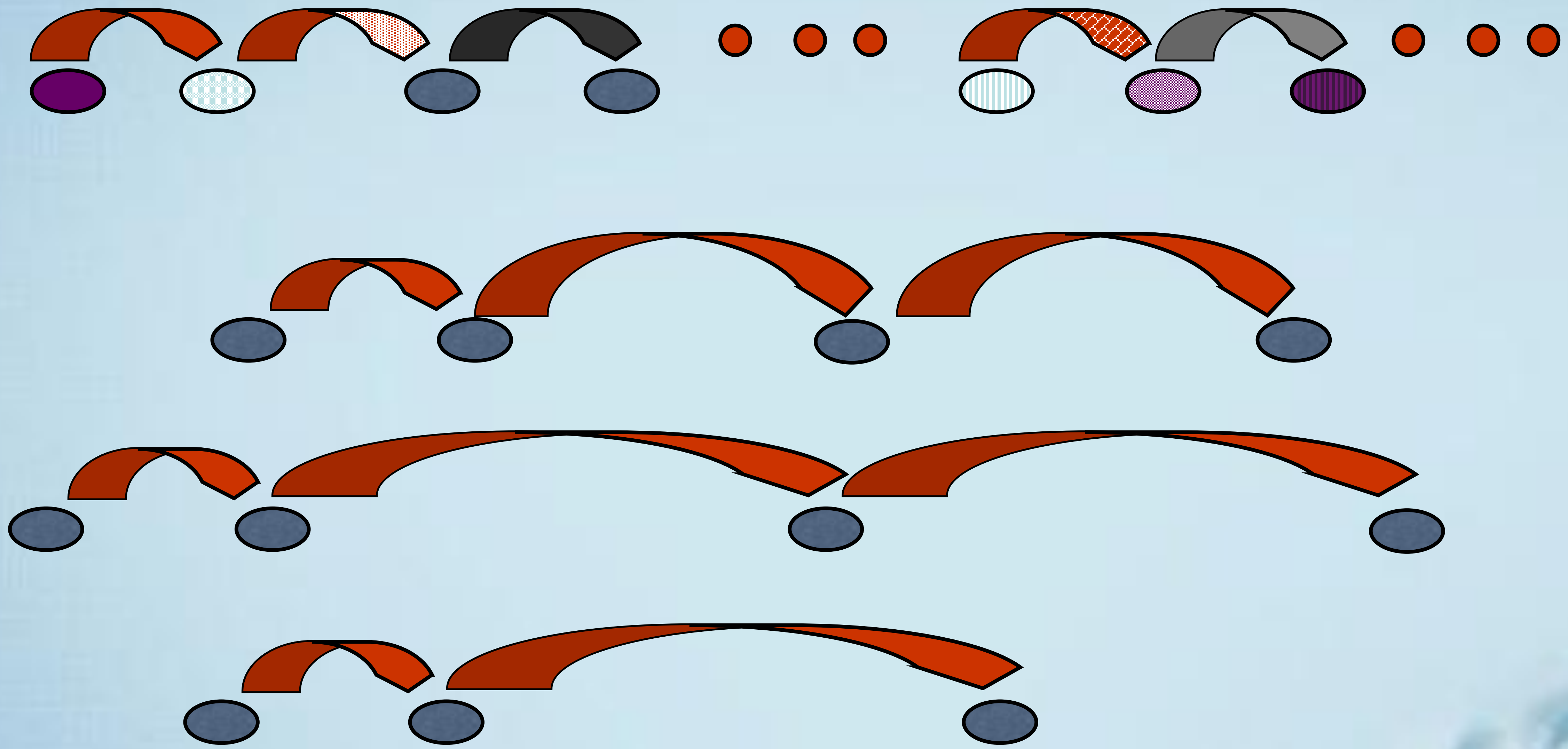
<u>Proof cont.:</u> Secondly, for all $v_i, v_j \in T$ (i < j), edge {vi, vj} is colored ci = BLUE, from previous claim. So, any edge between vertices of T is colored in BLUE ➔ T is an infinite monochromatic clique, and this finishes the proof.

■

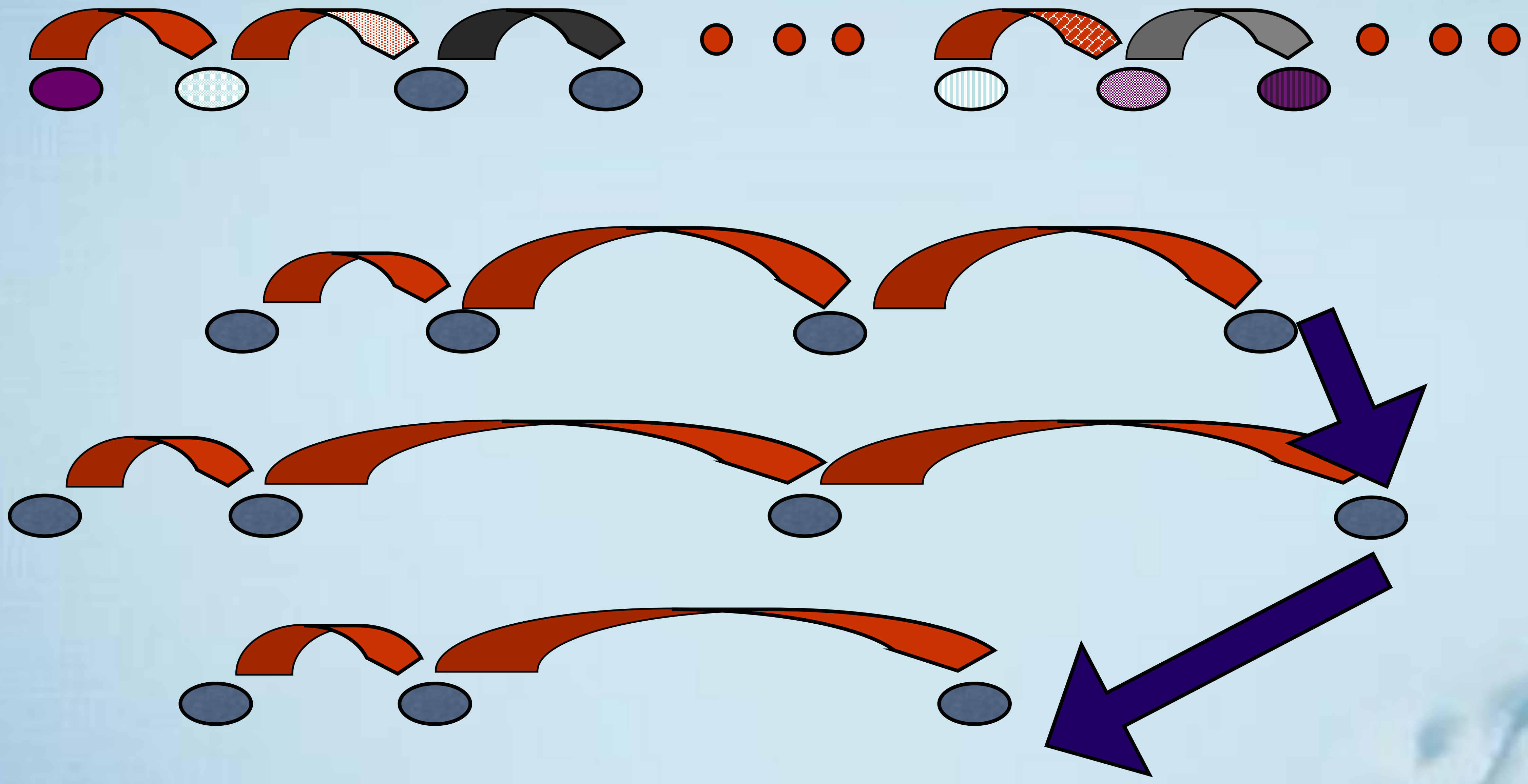
Infinite Ramsey's Theorem



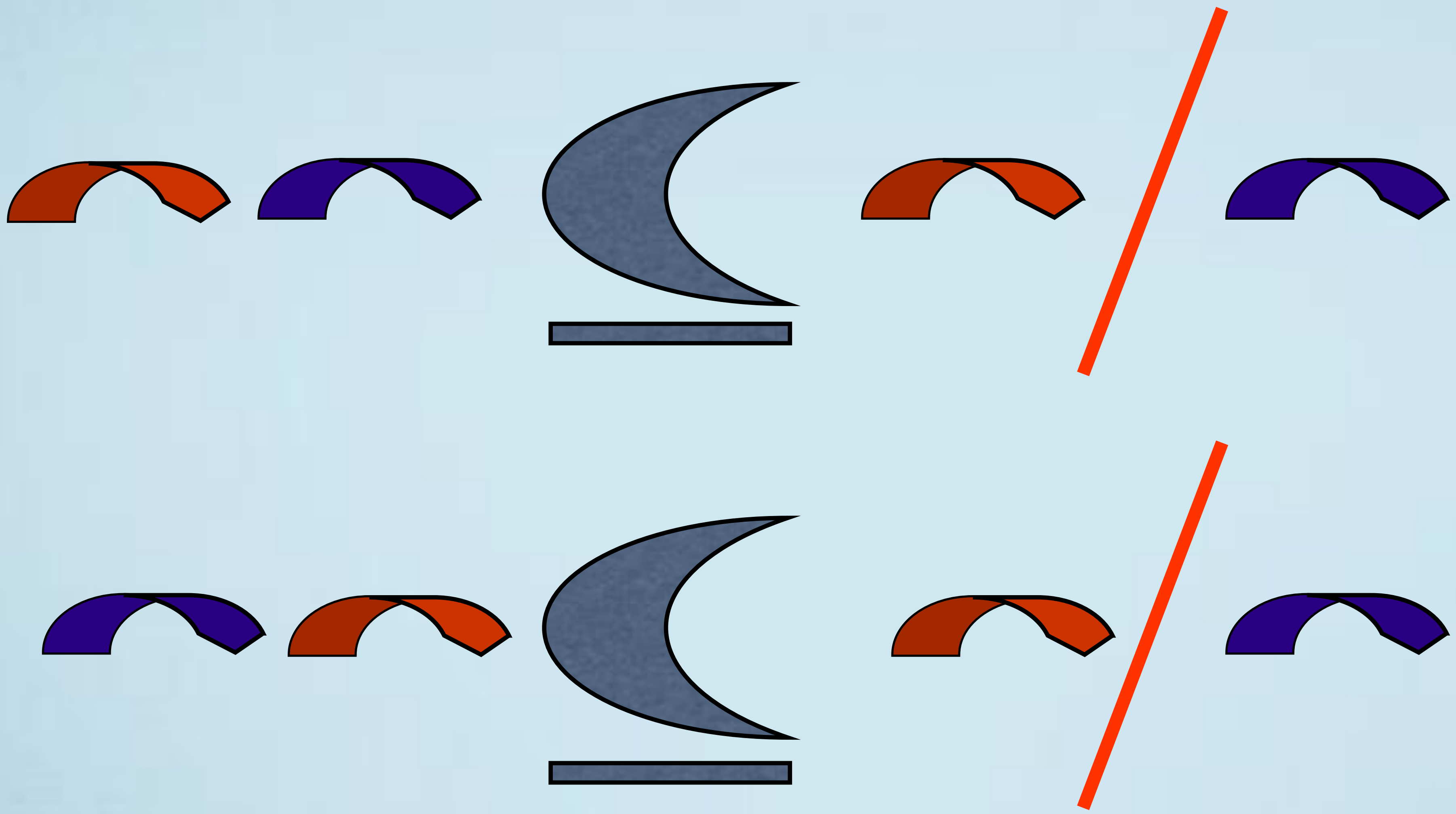
Infinite Ramsey's Theorem



Infinite Ramsey's Theorem



Closure



Proving Termination with Ramsey's Theorem

The infinite version of Ramsey's Theorem is one of the tools of proving termination of programs (together with well-founded orderings).

We'll show one example of that.

Before presenting our example program, we shell define the following:

<u>Def.:</u> if A is a set, then input(A) is user's input to program, that is taken from set A. For example: x := input(N), means that x gets a positive integer number from user's input.

Proving Termination with Ramsey's Theorem

Now, lets prove the termination of the following program, using Ramsey's Theorem:

```
(x,y,z) = (input(N), input(N), input(N))

while (x>0 and y>0 and z>0) {

        c = input({1, 2})
        if (c==1) then
                (x,y) = (x-1, input({y+1, y+2, …}))
        else
                (y,z) = (y-1, input({z+1, z+2, …}))

}
```

*** The program is taken from notes of William Gasarch, University of Maryland.**

Proving Termination with Ramsey's Theorem

If this program doesn't terminate, then there is infinite sequence

$(x1, y1, z1), (x2, y2, z2), \dots$, representing the state of the variables.

Lets look at the sub-sequence $(xi, yi, zi), \dots ,(xj, yj, zj)$.

1. If c is <u>ever</u> 1, then $xi > xj$.

2. If c is <u>never</u> 1, then $yi > yj$.

So, for all $i < j$, either $xi > xj$ or $yi > yj$.

With this fact, and with the contra-assumption that the program doesn't terminate, we'll use Ramsey's Theorem to reach a contradiction.

<u>Proof:</u> We start by defining an infinite complete graph, whose vertices would be the triplets of variables' state (xi, yi, zi).

Proving Termination with Ramsey's Theorem

Proof cont.: We then define a 2-coloring of edges of this graph:

COL(i, j) = if (xi > xj) then output BLUE

else output RED // yi > yj

From previous observation, the function is well-defined.

From Ramsey's Theorem, there is an infinite monochromatic clique in this graph. Lets denote its vertices' indexes by: $i_1 < i_2 < i_3 < \ldots$

If this clique color is BLUE, then $X_{i_1} > X_{i_2} > X_{i_3} > \ldots$

If this clique color is RED, then $Y_{i_1} > Y_{i_2} > Y_{i_3} > \ldots$

In either case, we'll eventually have a variable (x or y) ≤ 0 and hence program must terminate (while cond. is false). This is due to the fact that the variables get only integer values (and natural numbers are well-ordered). ➔ Contradiction ➔ The program terminates. ■

Disjunctive Orders

- States Q
- Algorithm $R \subseteq Q \times Q$
- Transitive closure R^+
- Well-founded orders $>$ and $\sqsupset$ on Q
- $R^+ \subseteq > \cup \sqsupset$

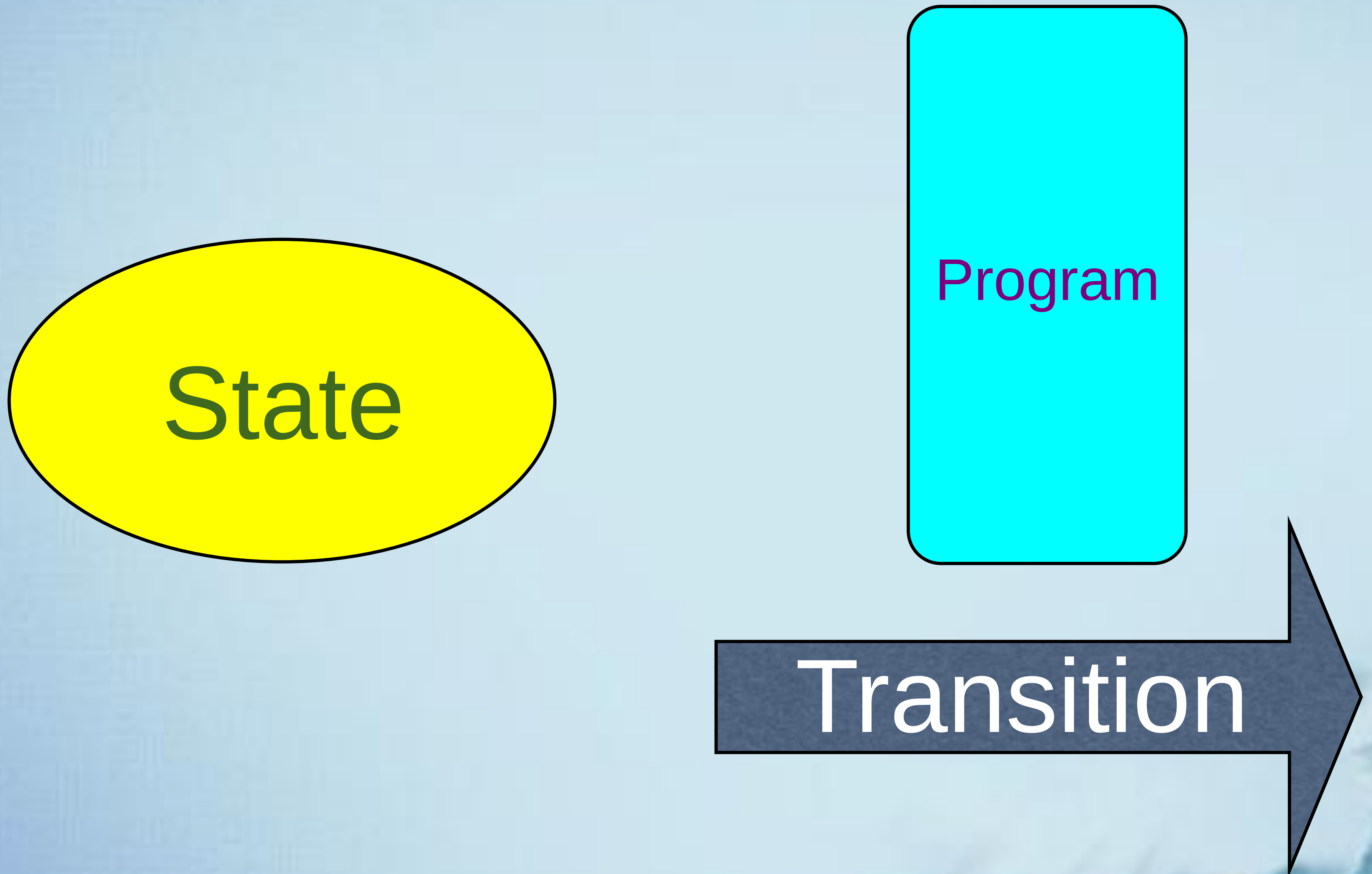
Ranking Method

- States Q
- Algorithm $R \subseteq Q \times Q$
- Well-founded order $\succ$ on W
- Ranking function $r : Q \rightarrow W$
- Define $X > Y$ if $r(X) \succ r(Y)$
- $R \subseteq >$

Invariants

- States Q
- Algorithm $R \subseteq Q \times Q$
- Well-founded order $\succ$ on W
- Ranking function $r : Q \rightarrow W$
- Define $X > Y$ if $r(X) \succ r(Y)$
- $R \subseteq >$

Algorithmic System



Classical Algorithms

- Every algorithm can be expressed precisely as a set of conditional assignments, executed in parallel repeatedly.
  - if c then f(s1,...,sn) := t
  - if c then f(s1,...,sn) := t
  - if c then f(s1,...,sn) := t

Practical Method

- States Q
- Algorithm $R \subseteq Q \times Q$
- Well-founded order $\succ$ on W
- Ranking function $r : Q \to W$
- Define $X > Y$ if $r(X) \succ r(Y)$
- $R \subseteq >$

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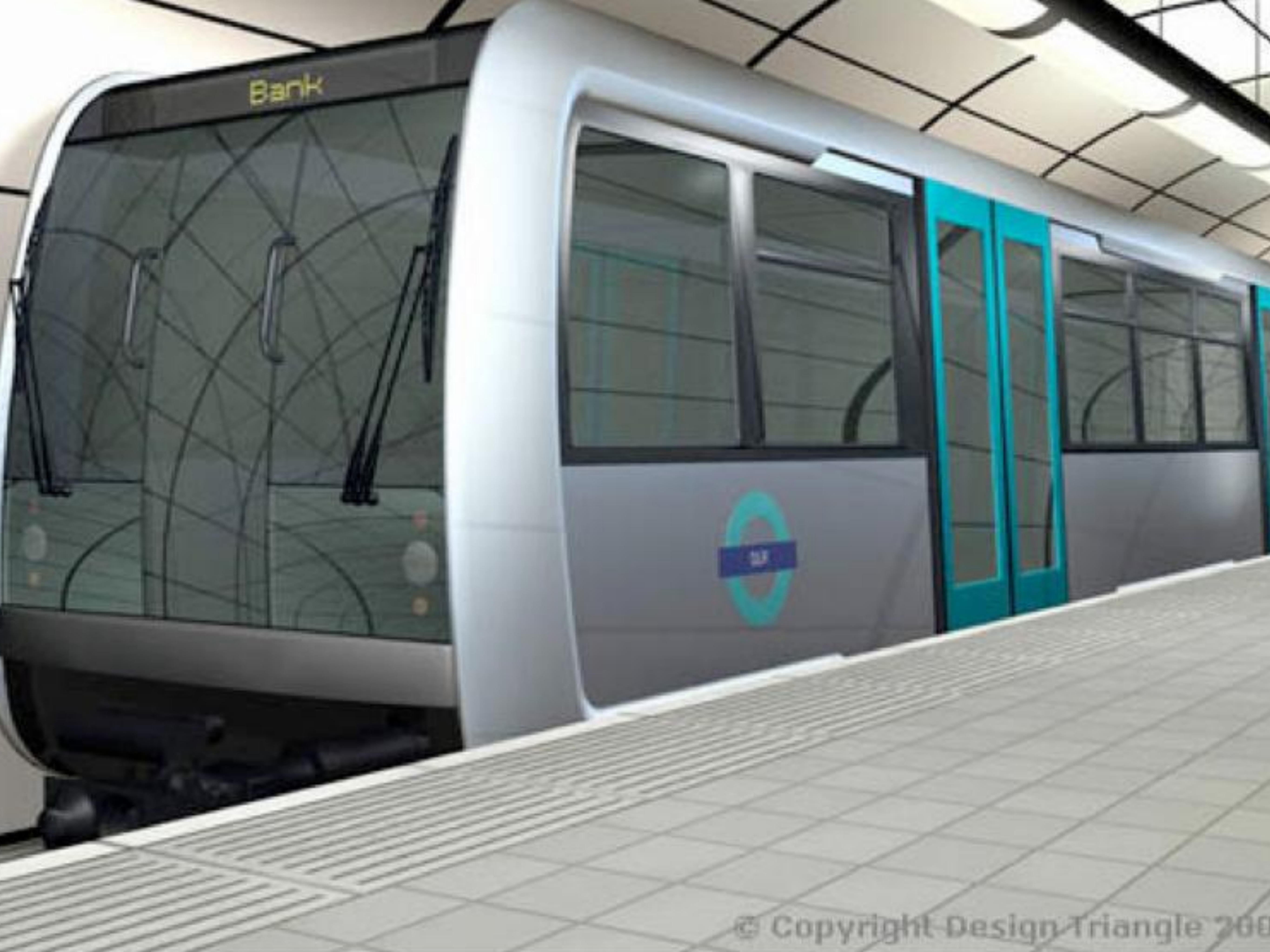
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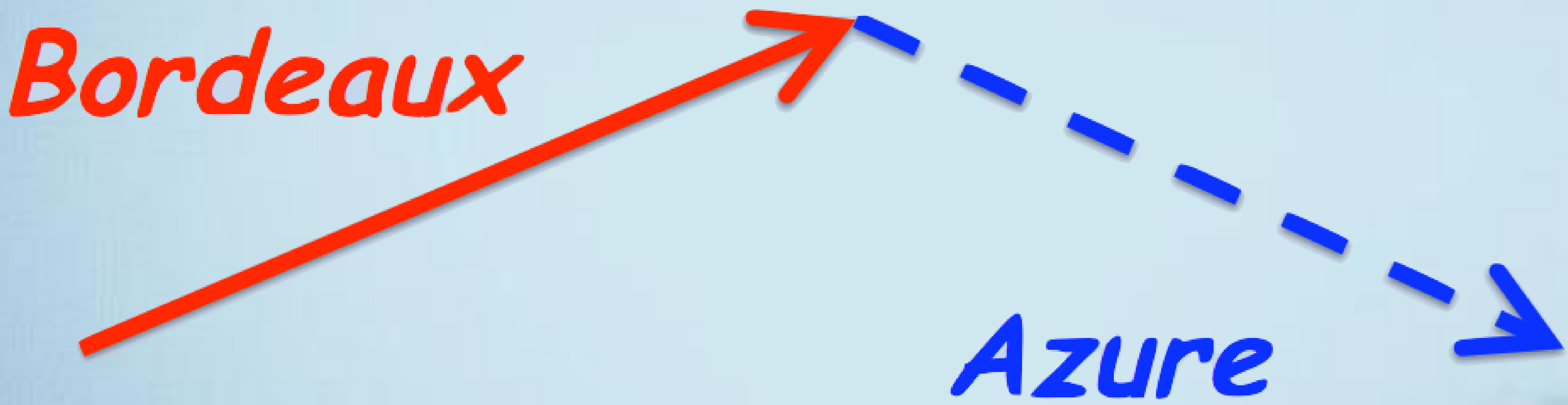
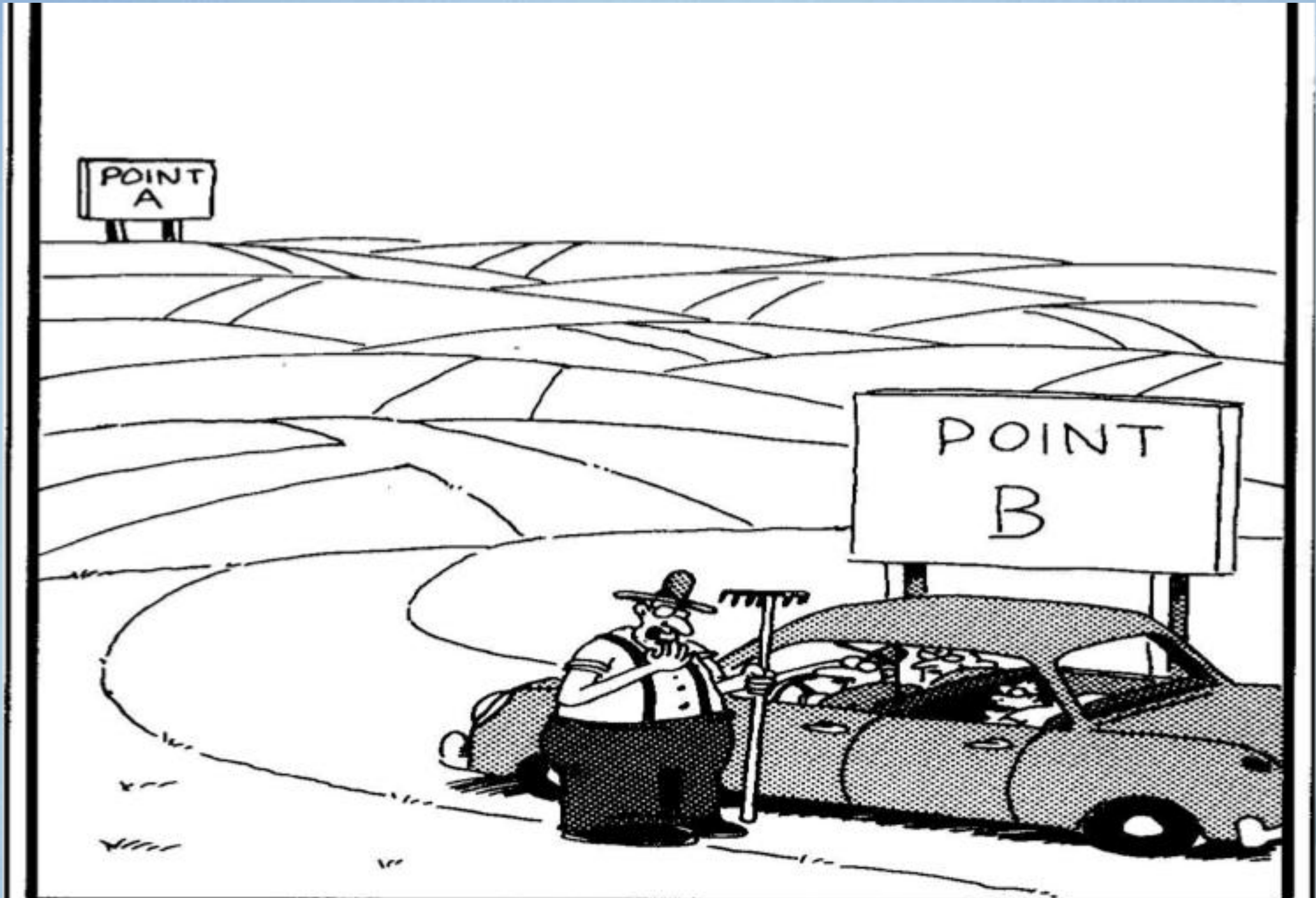
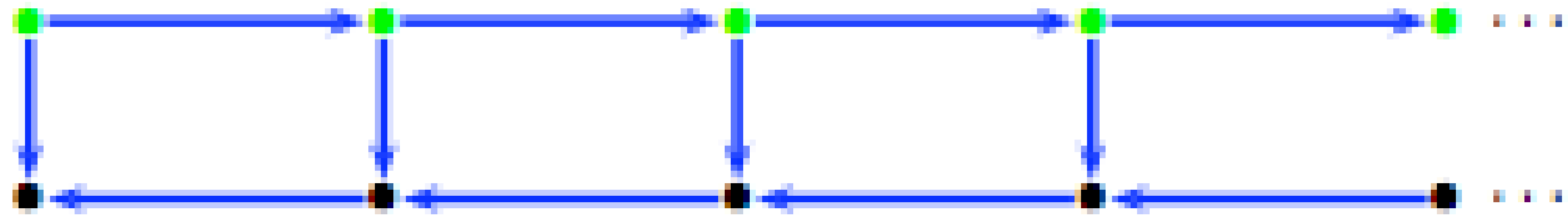


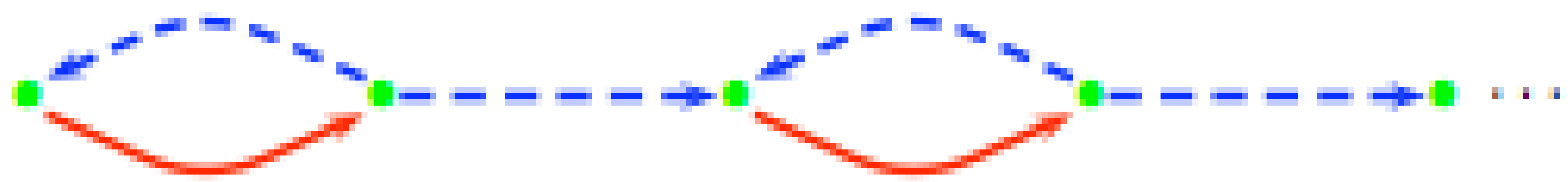


"Well, lemme think. . . . You've stumped me, son. Most folks only wanna know how to go the other way."

Mortal (black) nodes on bottom and immortal (green) nodes on top

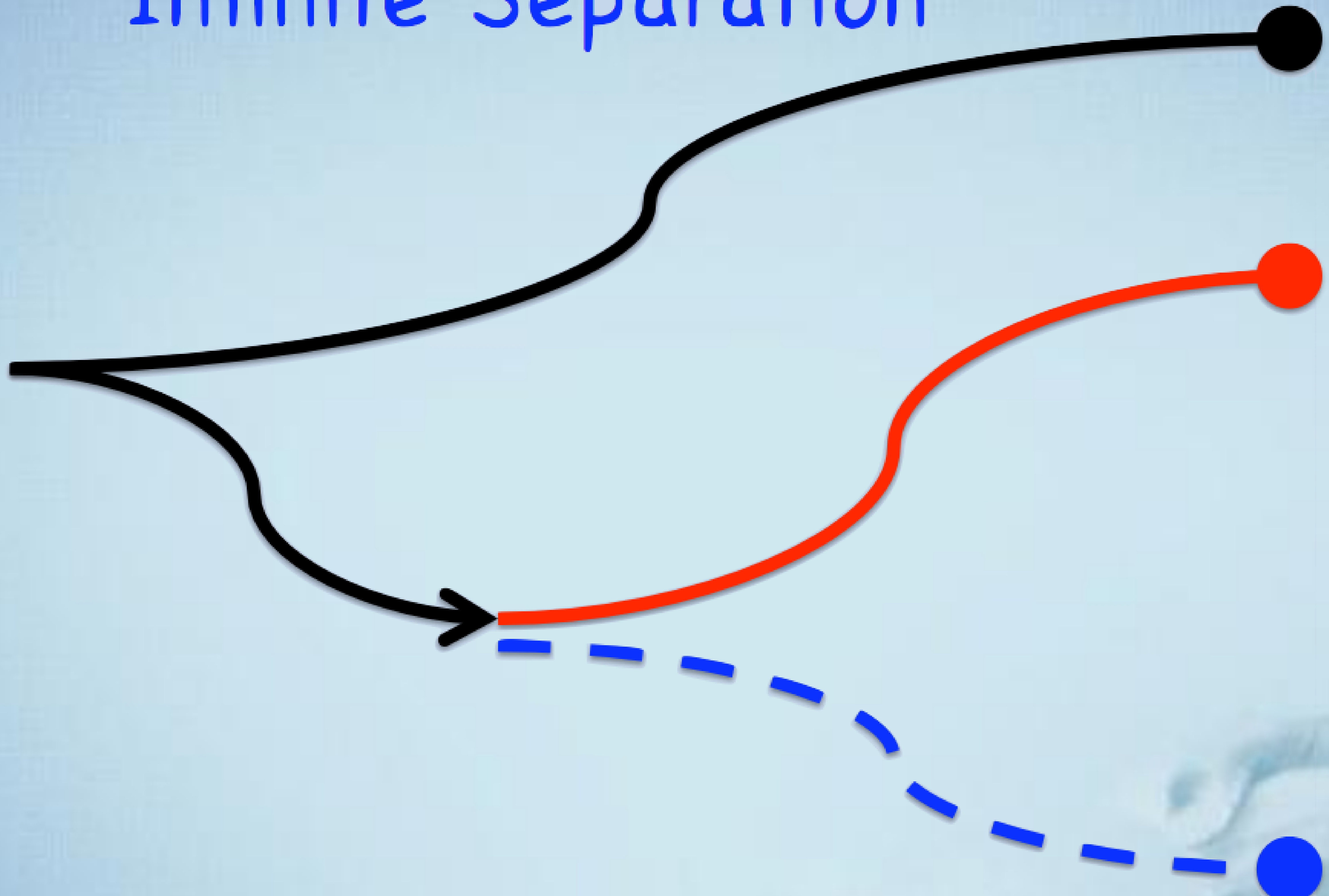


Mortal in each alone (dashed **Azure** or solid **Bordeaux**), but immortal in their union

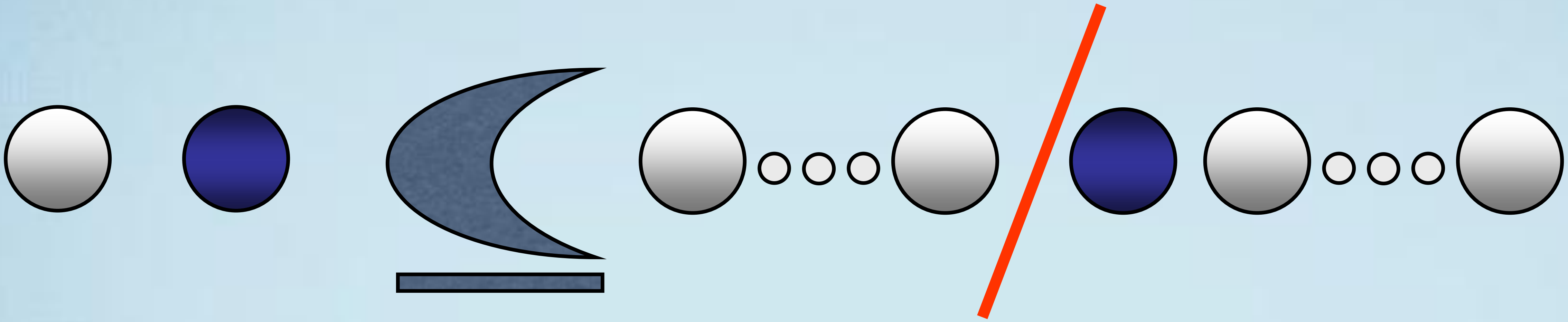


Infinite Separation

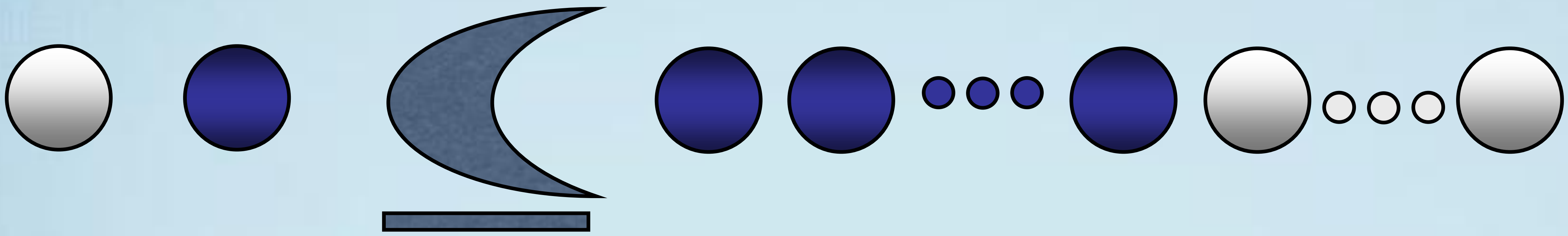
Infinite Separation



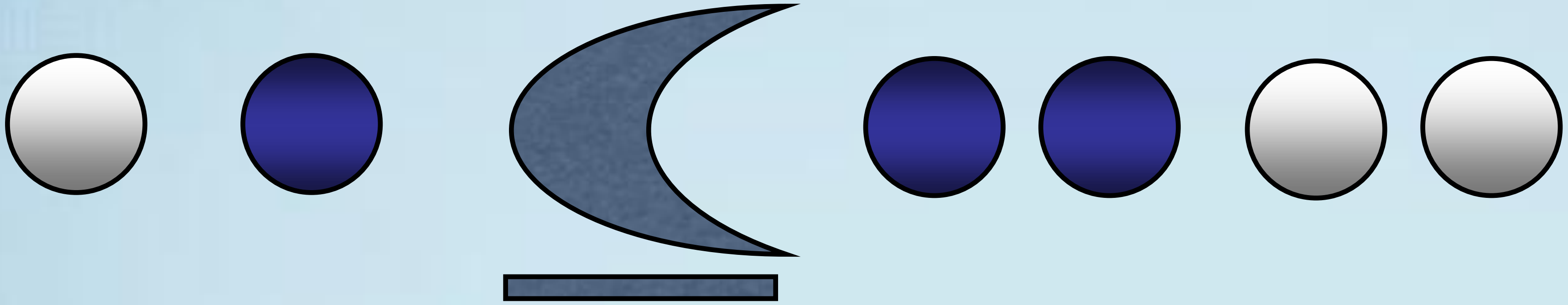
Enough?



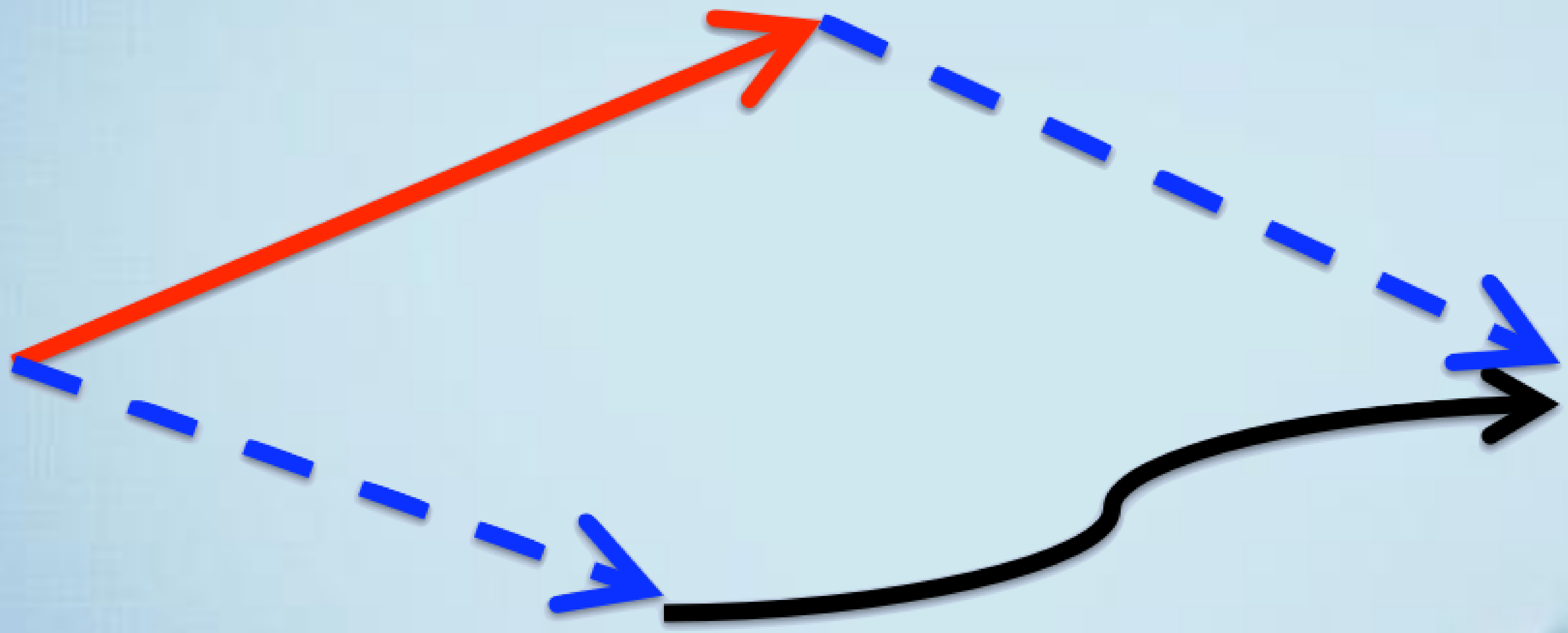
Enough?



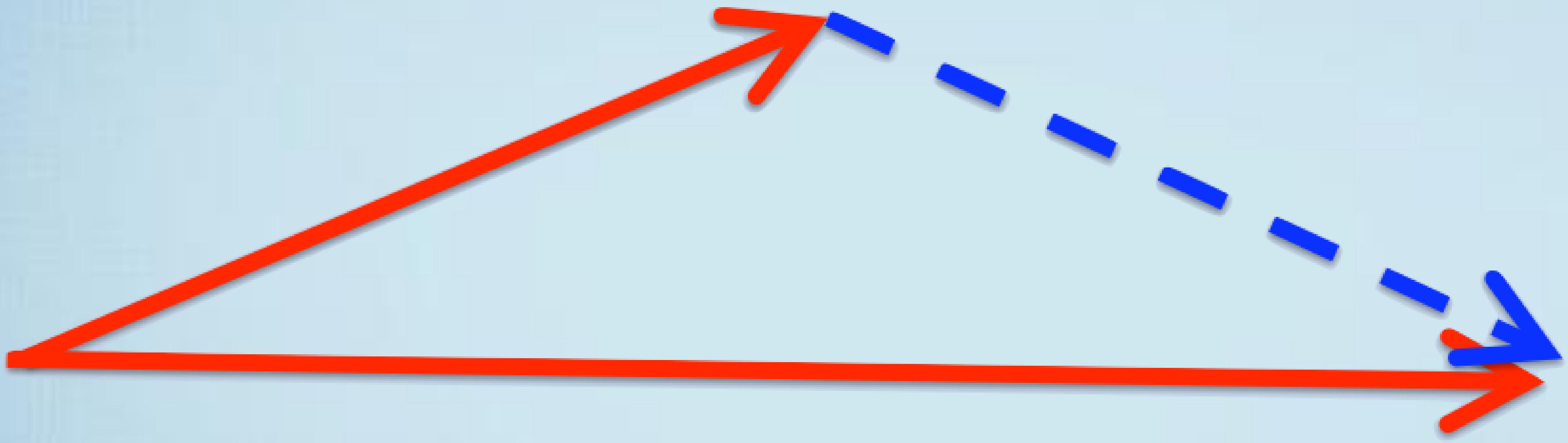
Enough?



Jumping

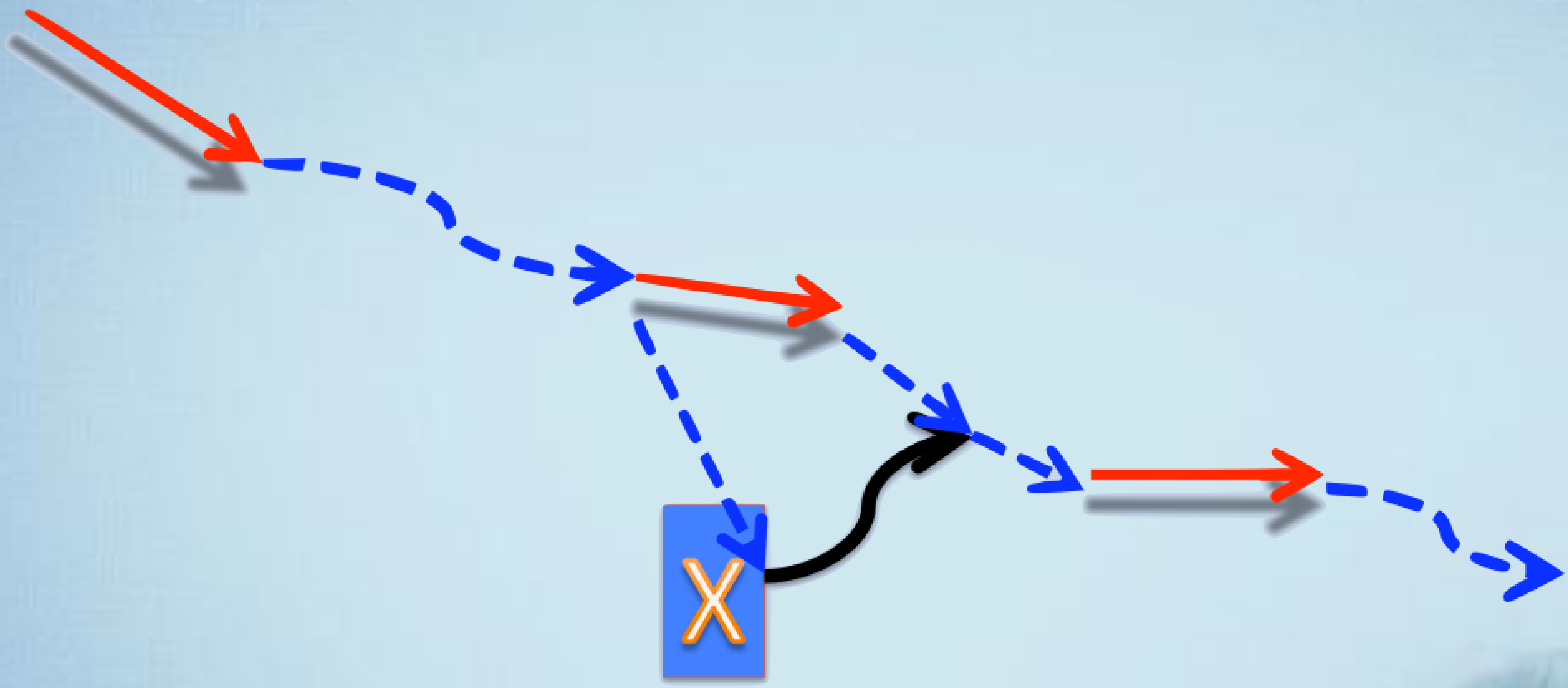


Jumping

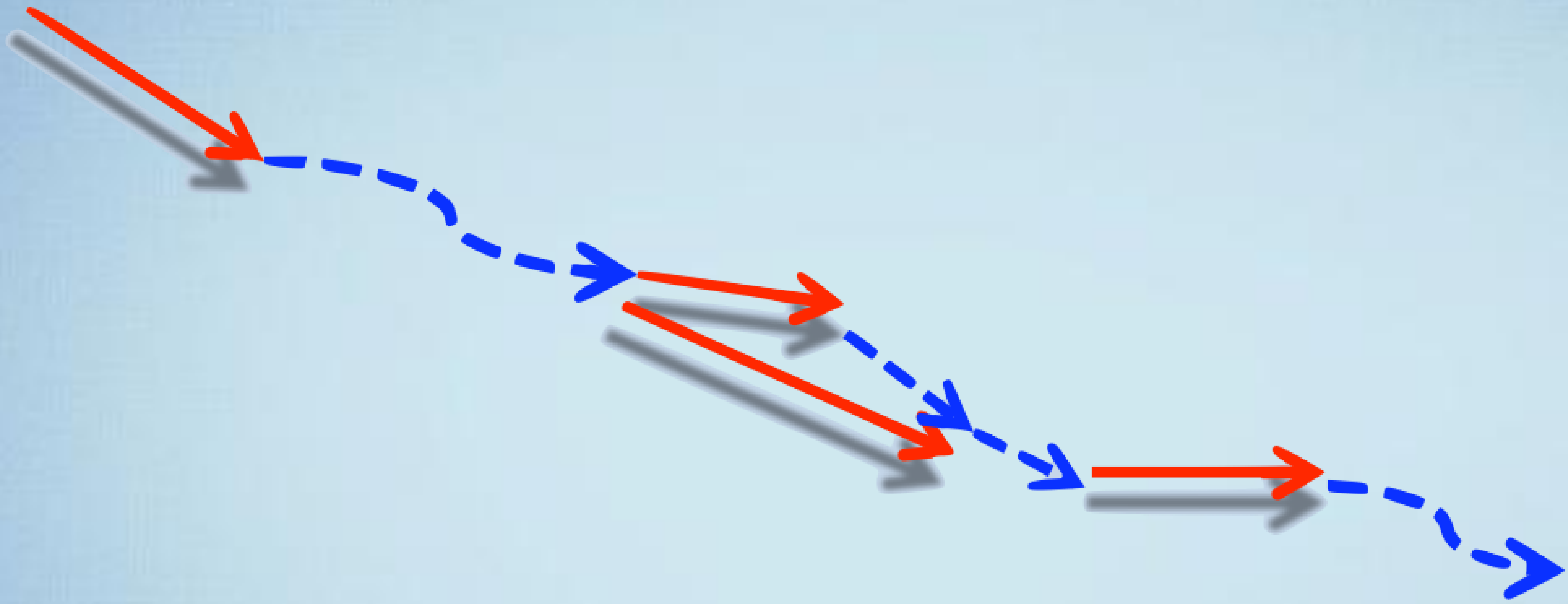



ONLY

Constriction + Jumping



Constriction + Jumping



Constriction + **Jumping**