Complexity of Boolean and Arithmetic Circuits - 236603

Exercise 2

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Please write your solution in LaTeXand submit them by email to shpilka@cs.technion.ac.il.

1. Let $f : \{0,1\}^n \to \{0,1\}$ be the majority function (i.e. $f(x) = 1$ iff $\sum_{i=1}^n x_i > n/2$). Show that $L_{U_2}(f) = \Omega(n^2)$.

2. In class we proved that for any function f, $D(f) = \Theta(\log L(f))$. Prove the same result for monotone formulas. I.e. show that for any monotone function f, $D^M(f) = \Theta(\log L^M(f))$.

3. Consider the following communication problem. Alice and Bob each holds a subset of $\{1,\ldots,n\}$. They wish to find the *median* of the multiset composed of their subsets (i.e. they want to find a number such that half of the other numbers are smaller or equal to it, and half are larger or equal to it). Give a communication protocol that solves the problem with $O(logn)$ communication (recall that we count bits and not just rounds).

4. Prove the sunflower Lemma. Recall that the Lemma states that in any collection of $t > (p-1)^l \cdot l!$ sets each of size at most l, there is a sunflower with p petals.

5. For a function $f : \{0,1\}^n \to \{0,1\}$ the *k-th slice* is the function

$$f^{(k)}(x) = \begin{cases} 0 & \sum_{i=1}^n x_i < k \\ 1 & \sum_{i=1}^n x_i > k \\ f(x) & \sum_{i=1}^n x_i = k \end{cases}$$

   Verify that each $f^{(k)}$ is a monotone function. Prove that for every f and $1 \le k \le n$ we have that

$$S^M(f^{(k)}) = S(f) + O(n).$$

   In other words, there is almost no gap between the monotone complexity and the non-monotone complexity of slice functions.

6. Let $f, g : \{0,1\}^n \to \{0,1\}$ be monotone functions. Let $h : \{0,1\}^{2n} \to \{0,1\}^2$ be

$$h(x_1, ..., x_n, y_1, ..., y_n) = f(x_1, ..., x_n), g(y_1, ..., y_n).$$

   Prove that

$$S^M(h) = S^M(f) + S^M(g).$$

   Is the result remains true if we remove the monotonicity condition? (Prove or give a counter example).