Drumhead

For $r < a$

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} &= \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\end{aligned}$$

On the boundary $r = a$ we have $u(a, \theta, t) = 0$ and intial data

$$\begin{aligned}u(r, \theta, 0) &= f(r, \theta) \\ \frac{\partial u}{\partial t}(r, \theta, 0) &= g(r, \theta)\end{aligned}$$

Again assume

$$u(r, \theta, t) = T(t)R(r)\Theta(\theta)$$

Then

$$\begin{aligned}\frac{T''}{c^2 T} &= \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = -\lambda \\ \text{So} \quad r^2\frac{R''}{R} + \frac{rR'}{R} + \lambda r^2 &= -\frac{\Theta''}{\Theta} = \gamma\end{aligned}$$

So

$$\begin{aligned}T'' - c^2\lambda T &= 0 \\ \Theta'' + \gamma\Theta &= 0 \to \gamma = n^2 \\ R'' + \frac{1}{r}R' + (\lambda - \frac{\gamma}{r^2})R &= 0\end{aligned}$$

and

$$\Theta(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

We next study

$$\begin{aligned}R'' + \frac{1}{r}R' + (\lambda - \frac{n^2}{r^2})R &= 0 \\ R(0) < \infty \quad R(a) &= 0\end{aligned}$$

Let

$$\rho = \sqrt{\lambda}r$$

Then

$$R_{\rho\rho} + \frac{1}{\rho} R_\rho + (1 - \frac{n^2}{\rho^2})R = 0$$

This is a singular Sturm Liouville problem. $\rho = 0$ is a regular singular point. Assume

$$R(\rho) = \rho^\alpha \sum_{k=0}^{\infty} a_k \rho^k \quad \text{with } a_0 \neq 0$$

Then

$$\begin{array}{ll} k{=}0 & \left[\alpha(\alpha-1) + \alpha - n^2\right] a_0 = 0 \quad \Rightarrow \alpha = \pm n \\ k{=}1 & \left[(\alpha+1)\alpha + \alpha + 1 - n^2\right] a_1 = 0 \quad \Rightarrow a_1 = 0 \\ k{\geq}2 & \left[(\alpha+k)(\alpha+k-1) + \alpha + k - n^2\right] a_k + a_{k-2} = 0 \end{array}$$

By a formal Taylor series we define:

Definition 1 *Bessel function (of the first kind):*

$$\begin{aligned} J_n(\rho) &= \frac{\rho^n}{2^n n!}\left(1 - \frac{\rho^2}{2^2(n+1)} + ...\right) \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\rho}{2}\right)^{n+2j}}{j!\,(n+j)!} \end{aligned}$$

Definition 2 *Bessel function of the second kind*

$$N_s(z) = \frac{\cos(\pi s) - J_{-s}(z)}{\sin(\pi s)} \quad s \text{ not an integer}$$

Then

$$N_s(z) \sim \sqrt{\frac{2}{\pi z}} \sin(z - \frac{s\pi}{z} - \frac{\pi}{4}) + O\left(\frac{1}{z^{\frac{3}{2}}}\right) \quad z \to \infty$$

Definition 3 *Hankel functions:*

$$\begin{aligned} H_s^{\pm}(z) &= J_s(z) + iN_s(z) \\ &\sim \sqrt{\frac{2}{\pi z}} e^{\pm i\left(z - \frac{s\pi}{z} - \frac{\pi}{4}\right)} + O\left(\frac{1}{z^{\frac{3}{2}}}\right) \quad z \to \infty \end{aligned}$$

For an integer

$$\begin{aligned} N_n(z) &= \lim_{s \to n} N_s(z) \\ &= \frac{2}{\pi} J_n(z) \log(\frac{z}{2}) + \sum_{k=-n}^{\infty} a_k z^k \end{aligned}$$

So for small z

$$N_0(z) \sim \log(z)$$
$$N_n(z) \sim z^{-n}$$

So

$$R(r) = AJ_n(\sqrt{\lambda}r) + BN_n(\sqrt{\lambda}r)$$

At $r = 0$ the soltion is bounded and so $B = 0$. At $r = a$ u is zero and so

$$J_n(\sqrt{\lambda_{nm}}a) = 0$$

For every n we have an infinite number of roots (eigenvalues). So

$$u(r,\theta,t) = \sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{nm}}r)\left[C_{om}\cos(\sqrt{\lambda_{0m}}ct) + D_{0m}\sin(\sqrt{\lambda_{0m}}ct)\right]$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\left[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\right]\left[C_{nm}\cos(\sqrt{\lambda_{nm}}ct) + D_{nm}\sin(\sqrt{\lambda_{nm}}ct)\right]$$

Set $t = 0$

$$f(r,\theta) = u(r,\theta,0) = \sum_{m=1}^{\infty} C_{om}J_0(\sqrt{\lambda_{0m}}r)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}J_n(\sqrt{\lambda_{nm}}r)\left[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\right]$$

and

$$g(r,\theta) = \sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}D_{om}J_0(\sqrt{\lambda_{0m}}r)$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{nm}}D_{nm}J_n(\sqrt{\lambda_{nm}}r)\left[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)\right]$$

Define

$$j_{mn} = \int_0^r J_n^2(\sqrt{\lambda_{nm}}r)rdr = \frac{a^2}{2}\left[J_n'(\sqrt{\lambda_{nm}}a)\right]^2$$

and then using orthogonality i.e.

$$\int_{-\pi}^{\pi}\int_0^a J_n(\sqrt{\lambda_{nm}}r)J_n(\sqrt{\lambda_{np}}r)rdrd\theta = 0 \qquad \text{if } m \neq p$$

$$C_{0m} = \frac{1}{2\pi j_{0m}} \int_0^a \int_{-\pi}^{\pi} f(r,\theta) J_0(\sqrt{\lambda_{0m}} r)\, r dr d\theta$$

$$C_{mn} A_{nm} = \frac{1}{\pi j_{0m}} \int_0^a \int_{-\pi}^{\pi} f(r,\theta) J_n(\sqrt{\lambda_{nm}} r)\, \cos(n\theta) r dr d\theta$$

$$C_{mn} B_{nm} = \frac{1}{\pi j_{0m}} \int_0^a \int_{-\pi}^{\pi} f(r,\theta) J_n(\sqrt{\lambda_{nm}} r)\, \sin(n\theta) r dr d\theta$$

$$D_{mn} A_{nm} = \frac{1}{\pi j_{0m}} \int_0^a \int_{-\pi}^{\pi} g(r,\theta) J_n(\sqrt{\lambda_{nm}} r)\, \cos(n\theta) r dr d\theta$$

$$D_{mn} B_{nm} = \frac{1}{\pi j_{0m}} \int_0^a \int_{-\pi}^{\pi} g(r,\theta) J_n(\sqrt{\lambda_{nm}} r)\, \sin(n\theta) r dr d\theta$$

example

$$u(r,\theta,0) = 0$$
$$u_t(r,\theta,0) = \psi(r)$$

Then

$$u(r,\theta,t) = \sum_{m=1}^{\infty} D_{0m} J_0(\sqrt{\lambda_{nm}} r) \sin(\sqrt{\lambda_{0m}} ct)$$

$$D_{0m} = \frac{\int_0^a \int_{-\pi}^{\pi} \psi(r) J_0(\sqrt{\lambda_{0m}} r)\, r dr d\theta}{\frac{1}{2} a^2 c \sqrt{\lambda_{0m}} J_1^2(\sqrt{\lambda_{0m}} a)}$$

The fundamental frequency is

$$\sqrt{\lambda_{01}} c = z_1 \frac{c}{a} \qquad \text{where } J_0(z_1) = 0 \quad \text{smallest root}$$
$$z_1 \sim 2.405$$

Note for the one dimensional string the fundamental frequency is

$$\pi \frac{c}{a} \quad \pi = 3.14 \sim 1.3 z_1$$

Gamma function

Definition 4 *Gamma function*

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds \qquad 0 \leq x < \infty$$

Then

$$\begin{aligned} \Gamma(x+1) &= && \Gamma(x) \\ \Gamma(n+1) &= n! \qquad n \text{ an integer} \\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\ \Gamma\left(n+\frac{1}{2}\right) &= \frac{(2n)!\sqrt{\pi}}{n!2^{2n}} \end{aligned}$$

Then

$$J_s(z) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{z}{2}\right)^{s+2j}}{\Gamma(j+1)\Gamma(s+j+1)} \quad s \neq \text{ negative integer}$$

Theorem 5 *There are an infinite number of zeroes of* $J_s(z) = 0 \quad 0 < z_1 < z_2 < ...$ *each root is a simple root i.e* $J_s'(z_j) \neq 0$.
Between two zeros of $J_s(z)$ *is a zero of* $J_{s+1}(z)$ *and vice-versa.*
So J_s, J_{s+1} *separate their zeroes.*
Fot $J_0(z)$ *the first zeroes are* $z_0 \sim 2.4055.520, ...$