semilinear equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2$$

Initial condition:

$$u(0, y) = 1$$

initial curve:

$$x = 0, \quad y = t, \quad u = 1$$

The characteristic equations are:

$$\frac{\partial x}{\partial s} = 1$$
$$\frac{\partial y}{\partial s} = 1$$
$$\frac{\partial u}{\partial s} = u^2$$

So

$$x = s + x_0 = s$$
$$y = s + y_0 = s + t$$

Also

$$\frac{du}{u^2} = ds$$
$$-\frac{1}{u} = s + c$$
$$u = -\frac{1}{s+c} = \frac{-1}{s + \frac{1}{u_0}} = \frac{u_0}{1 - su_0} = \frac{1}{1-s}$$

Inverting

$$s = x$$
$$t = y - x$$

So

$$u = \frac{1}{1-x}$$

Note at $x = 1$ we have $u = \infty$.

Homework: consider same equation with IC $\quad u = e^{-y^2}$

quasilinear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 2t$$

Initial condition:

$$u(x,0) = x = x_0$$

The characteristic equations are:

$$\begin{aligned} \frac{\partial t}{\partial s} &= 1 \\ \frac{\partial x}{\partial s} &= u \qquad \text{So } \frac{dx}{dt} = u \\ \frac{\partial u}{\partial s} &= 2t \end{aligned}$$

Note, that the characteristic curves now depend on u .
However, the first equation integrates to $t = s$ (remember t is time). So $\frac{\partial u}{\partial s} = 2t$ becomes $\frac{\partial u}{\partial t} = 2t$

$$u(x,t) = t^2 + C \qquad C = u - t^2 = u(x,0) = x_0$$

The slope of the characteristics is now given by

$$\frac{dx}{dt} = u = x_0 + t^2$$

So the characteristic curve is given by

$$\begin{aligned} x &= \frac{1}{3}t^3 + x_0(t+1) \\ \text{or} \quad x_0 &= \frac{x - \frac{1}{3}t^3}{(t+1)} \end{aligned}$$

On the characteristic

$$C = x_0 = \frac{3x - t^2}{3\,(t+1)}$$

So

$$u(x,t) = t^2 + C = t^2 + \frac{3x - t^2}{3\,(t+1)}$$

So in this case the solution is well defined for all $t > 0$

quasilinear equation

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0$$

Represents fluid flow and traffic flow.

So $\frac{du}{dt} = 0$ (i.e. u is constant) along the characteristics $\frac{dx}{dt} = a(u)$.

Since u is constant along the characteristic so is $a(u)$.

Hence $\frac{dx}{dt}$ is constant and so the characteristics are straight lines.

So

$$u(x,t) = f(x - a(u)t)$$

This an implicit solution for u. Differentiating we get

$$u_t = \frac{-a}{1 + f' a_u t} f' \quad u_x = \frac{1}{1 + f' a_u t} f'$$

Assume $a_u > 0$ In particular $a_u \neq 0$. The slope is $\frac{dt}{dx} = \frac{1}{a(u(x_0,t_0)} = \frac{1}{a(f(x_0))}$.

case I: $f' > 0$. Then u at $t = 0$ increases and the slope $\frac{1}{u}$ decreases. So the solution and its derivatives are bounded.

case II: $f' < 0$. Then u at $t = 0$ decreases and the slope $\frac{1}{u}$ increases. So the solution and its derivatives are unbounded. u_x and u_t approach infinity as $t \to \frac{-1}{u_0' a_u} = \frac{1}{|u_0'| a_u}$.

So we have two cases:

- expansion wave $a(f(z)) \geq a(f(w))$ whenever $z \leq w$. Solution and derivatives are bounded.
- compression wave = shock wave. Solution and derivatives may become unbounded.

Physically this means that later air moves faster than earlier air.

Hence, this *seems* to imply that there is no solution beyond some t_*.

However, physically the fluid continues to exist beyond the singularity!

Hence, we need to define a generalized solution that allows solutions for all time.