# Wave Equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{c constant}$$

Note: We can also write this as a first order system. Let $u$ be the basic unknown, e.g. the height of the string and define $\varphi = c\frac{\partial u}{\partial x}$ and $v = \frac{\partial u}{\partial t}$. Then

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= c\frac{\partial \psi}{\partial x} && \text{by definition} \\ \frac{\partial \psi}{\partial t} &= c\frac{\partial \varphi}{\partial x} && \text{by wave equation} \end{aligned}$$

or

$$\frac{\partial}{\partial t}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

*Method I:*

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right)}_{v} = 0$$

Then

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x}$$

so $\quad v = h(x + ct)$ and

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = v = h(x + ct)$$

The general solution is a particular solution by the general solution to the homogenous equation. To find a particular soltion let us guess $\quad u = f(x + ct)$. Then $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = cf' + cf' = h$. So $f' = \frac{h}{2c}$.
The general solution to the homogenous equation is $u = g(x - ct)$ .
So the general solution of the wave equations is

$$u(x,t) = f(x + ct) + g(x - ct)$$

i.e. a wave going to the right plus a wave moving to the left both at speed $c$ .

*Method II:* change of variables

$$\xi = x + ct \qquad \eta = x - ct$$
$$0 = u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}$$

So

$$u(x,t) = f(\xi) + g(\eta) = f(x+ct) + g(x-ct)$$

For now we have two arbitraty functions in $C^2$.
**Note**: If there are discontinuouities in the initial data they travel along the characteristics $x + ct = \text{constant}$ and $x - ct = \text{constant}$ . Across the characteristics the solution is smooth.

Initial data - D'Alembart

Consider the following initial value problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad -\infty < x < \infty$$
$$u(x,0) = \varphi(x) \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x)$$

From the above we know

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ \varphi(x) &= u(x,0) = f(x) + g(x) \\ \frac{\partial u}{\partial t}(x,t) &= cf'(x+ct) - cg'(x-ct) \\ \psi(x) &= \frac{\partial u}{\partial t}(x,0) = c\left[f'(x) - g'(x)\right] \end{aligned}$$

Solving

$$\begin{aligned} f'(x) &= \frac{\varphi'(x) + \frac{1}{c}\psi(x)}{2} \\ g'(x) &= \frac{\varphi'(x) - \frac{1}{c}\psi'(x)}{2} \end{aligned}$$

Integrating we get

$$\begin{aligned} f(s) &= \frac{\varphi(s)}{2} + \frac{1}{2c}\int^s \psi(z)dz + A \\ g(s) &= \frac{\varphi(s)}{2} - \frac{1}{2c}\int^s \psi(z)dz + B \end{aligned}$$

D'Alembart's formula

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z)dz$$

**Properties**:

- the solution exists and is unique
- no part of the wave goes faster than the speed $c$. In particular it takes a finite time for propagation of signals (causality).
  - In three dimensions signals move only at speed $c$ (Huygen's principle)
  - In two dimensions the entire domain between the characteristics is filled (flatland). So echoes stay forever !!
- Domain of influence: Initial conditions within a region can affect the solution only for points/time within the characteristics
- Domain of Dependence: The solution at point/time (x,t) is influence only by initial data at (x-ct, x+ct)
- Cauchy data on noncharacteristic surface determines the solution
- The solution depends continuously on the initial and boundary data.

  So the equation is well posed
- Discontinuities in second derivative propagate along characteristics
- D'Alembert's solution makes sense even if the initial data are such that $f$ and $g$ are only piecewise differentiable. Hence, we define a generalized solution as a limit of "classical solutions". So we have solutions that don't have two continuous derivatives everywhere.

Discuss vibrating string and drum

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + r\frac{\partial u}{\partial t} = 0 \qquad \text{resistance}$$
$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + ku = 0 \qquad \text{coiled spring}$$

Energy

We start with the wave equation and multiply by $u$ and integrate. Let $c^2 = \frac{T}{\rho}$

Define: Kinetic energy (KE) $= \frac{1}{2}\int \rho \left(\frac{\partial u}{\partial t}\right)^2 dx$

Potential energy (PE) $= \frac{1}{2}\int T \left(\frac{\partial u}{\partial x}\right)^2 dx$

Total energy T = KE + PE $= \frac{1}{2}\int_{-\infty}^{\infty} \left[\rho \left(\frac{\partial u}{\partial t}\right)^2 + T\left(\frac{\partial u}{\partial x}\right)^2\right] dx$

Then

$$\frac{d}{dt}(KE) = \int \rho \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dx \overset{\text{wave eq}}{=} \int T \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} dx \overset{\text{integrate by parts}}{=} -\int T \frac{\partial^2 u}{\partial x \partial t}\frac{\partial u}{\partial x} dx$$

$$\frac{d}{dt}(PE) = \int T \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t} dx$$

So we have conservation of energy

$$\frac{dE}{dt} = 0$$

Telegraph Equation

$$Lu = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + 2\beta \frac{\partial u}{\partial t} = 0 \qquad \beta, c \text{ constants}$$

$$\frac{d}{dt}(KE) = \int \rho \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} dx \overset{\text{telegraph eq}}{=} \int T \frac{\partial u}{\partial t} \left( \frac{\partial^2 u}{\partial x^2} - 2\frac{\beta}{T} \frac{\partial u}{\partial t} \right) dx \overset{\text{integrate by parts}}{=}$$

$$- \int T \frac{\partial^2 u}{\partial x \partial t} \frac{\partial u}{\partial x} dx - 2 \int \beta \left( \frac{\partial u}{\partial t} \right)^2 dx$$

$$\frac{d}{dt}(PE) = \int T \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} dx$$

So we have loss of energy

$$\frac{dE}{dt} = -2 \int \beta \left( \frac{\partial u}{\partial t} \right)^2 dx < 0$$

Define a new variable by $v(x,t) = e^{\beta t} u(x,t)$ or $u(x,t) = e^{-\beta t} v(x,t)$ . Then

$$\begin{aligned} \frac{\partial u}{\partial t} &= e^{-\beta t}(\frac{\partial v}{\partial t} - \beta v) \\ \frac{\partial^2 u}{\partial t^2} &= e^{-\beta t}(\frac{\partial^2 v}{\partial t^2} - 2\beta \frac{\partial v}{\partial t} + \beta^2 v) \end{aligned}$$

So the telegraph equation in terms of $v$ becomes

$$\begin{aligned} e^{-\beta t} \left[ \frac{\partial^2 v}{\partial t^2} - 2\beta \frac{\partial v}{\partial t} + \beta^2 v - c^2 \frac{\partial^2 v}{\partial x^2} + 2\beta \left( \frac{\partial v}{\partial t} - \beta v \right) \right] &= 0 \\ \frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} - \beta^2 v &= 0 \end{aligned}$$

Klein-Gordon Equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + m^2 u = 0$$

Similar to equation for $v$ but opposite sign of lower order term